



MAKING MATHEMATICS VITAL

Proceedings of the
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The Australian Association
of Mathematics Teachers

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Preface

In these proceedings of the twentieth biennial conference of The Australian Association of Mathematics Teachers, *Making Mathematics Vital*, are many insights from people from all over the country and also overseas who enjoy meeting the challenge of that endeavour — making the learning and teaching of mathematics a vital enterprise. In these pages you will read about successes and also of indications where more effort and imagination is needed. You will read that the work continues and is unlikely to be any easier as we find new challenges in our paths. You will also read of the rewards for both teachers and learners when connections are made and real learning occurs.

The editors wish to thank the editorial committee for their support and hard work in bringing this proceedings to such a high standard in a short time. Members of the editorial committee are: Janette Bobis, Michael Cavanagh, Beth Southwell, Steve Thornton, and Paul White. We were greatly assisted in our task by Merrilyn Goos, who shared her experiences from the 2003 conference; and by Ann Belperio in the AAMT office. The panel of people to whom papers were sent for peer review was extensive and we wish to thank them all:

Steve Arnold	Sandra Frid	Carolyn McGinty	Howard Reeves
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Helen Forgasz	Kym Linke	Ray Peck	Leigh Wood
Pat Forster	Tom Lowrie	Thelma Perso	Robert Yen
Andrew Fraser	Karen McDaide	Cyril Quinlan	Robyn Zevenbergen

Editors: Mary Coupland, Judy Anderson, Toby Spencer.

Review process

Submissions to the conference were called for in two formats, seminar papers and workshops, with the possibility of either being subjected to peer review. These papers were reviewed blind by at least two reviewers. Papers were assessed as:

1. suitable for presentation at the conference and for publication in the proceedings, identified as ‘accepted by peer review’,
2. suitable for presentation at the conference and for publication in the proceedings, and
3. not suitable for the conference.

Papers that were designated as (1) have been identified with an asterisk (*).

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Keynote papers

Challenging mathematics and its role in the learning process

Peter Taylor

Australian Mathematics Trust

Challenge is not only an important component of the learning process but also a vital skill for life. People are confronted with challenging situations each day and need to deal with them. Fortunately the processes in solving mathematics challenges (abstract or otherwise) involve certain types of reasoning which generalise to solving challenges encountered in every day life. Mathematics has a vital role in the classroom not only because of direct application of the syllabus material but because of the reasoning processes the student can develop.

ICMI has commissioned a study to investigate the issues of challenge in the learning process. The speaker is one of the co-chairs of this study, which is scheduled to have its Study Conference in 2006 and issue its findings as a Study Volume in 2008. In this talk Peter will describe some of the attempts to define the concept of challenge itself and discuss various related issues which are being identified in the context of challenge and the learning process.

Hanna Neumann

First I would like to briefly comment on the significance of Hanna Neumann. Many of us here got to know B. H. Neumann well over the last thirty years and the pervasive positive influence he had on mathematics in this country. Hanna died much earlier and less of us who are here now knew her directly.

I did not know her personally but I was fortunate to have once attended an inspirational talk by Hanna, in Adelaide in about October 1971, only two or three months before she died. As usual, Bernhard sat proudly in the front row. My fortune to have been able to attend that Seminar helped me to understand the unique role that Hanna had in the inspiration of mathematicians and students in Australia. From the colleagues I know who were fortunate to work or study with her there is no doubt about the significant love she had for teaching and students and influence she had on the standards of teaching. It is a great honour to present this lecture named after her.

ICMI and ICMI studies

As I assume is well-known to this audience, the International Commission on Mathematics Instruction (ICMI) is the principal professional body for mathematics education and it

conducts its activities in a number of ways. It publishes bulletins, holds an international conference every four years known as ICME (the most recent one having been held in July 2004 at Copenhagen), and is the umbrella body for five affiliated study groups (with themes of history, psychology, women in mathematics, competitions and most recently the group on applications and modelling).

ICMI also administers studies, which investigate particular issues with respect to mathematics education. Each study focuses on a particular topic in mathematics education, attempts to identify issues and to address them. Each study is chaired by a single chair or two co-chairs and an international group, known as the International Program Committee (IPC) of about twelve people is appointed to control the study.

The IPC meets fairly early in the process and initiates a Study Discussion Document, which identifies the issues and announces the program. Eventually a definitive conference for the Study is held. This Study is attended by invitees after the Study Volume has published the Study Document. Usually 70 to 100 people attend, but this is by invitation after people read the Discussion Document and show how they can contribute. It is not possible to attend the Study Conference as a passenger.

Finally a definitive Study Volume appears at the culmination of the Study. The whole process is likely to take about six or seven years, but the final Study Volume becomes an authoritative document, giving the state of the art after much input and discussion.

Past studies

At this stage fourteen studies have been completed. They started in the 1980s and the completed studies essentially cover the topics of

1. The Influence of Computers and Informatics on Mathematics and its Teaching
2. School Mathematics in the 1990s
3. Mathematics as a Service Subject
4. Mathematics and Cognition
5. The Popularisation of Mathematics
6. Assessment in Mathematics Education
7. Gender in Mathematics Education
8. What is Research in Mathematics Education and What are its Results?
9. Perspectives on the Teaching of Geometry for the 21st Century
10. The Role of the History of Mathematics in the Teaching and Learning of Mathematics
11. The Teaching and Learning of Mathematics at University Level
12. The Future of the Teaching and Learning of Algebra
13. Mathematics Education in Different Cultural Traditions: A Comparative Study of East Asia and the West
14. Applications and Modeling in Mathematics Education.

Present studies

There are three studies in progress. These are

15. Teacher Education and Development
16. Challenging Mathematics in and beyond the Classroom
17. Technology Revisited.

International Program Committee (IPC)

This Study is being co-chaired by myself and Ed Barbeau, of the University of Toronto. The Study Conference will be held in Trondheim during 2006 and the Chair of the Local Organising Committee is Ingvill Stedøy. Other members of the International Program Committee are Mariolina Bartolini Bussi (Italy), Albrecht Beutelspracher (Germany), Patricia Fauring (Argentina), Derek Holton (New Zealand), Martine Janvier (France), Vladimir Protasov (Russia), Ali Rejali (Iran), Mark Saul (USA), Kenji Ueni (Japan), and Bernard Hodgson (Canada), who is Secretary-General of ICMI. In addition two members of the ICMI Executive, Maria de Losada (Colombia) and Petar Kenderov (Bulgaria) have joined the IPC.

The members of the IPC have a broad range of activities. Some are involved in competitions or related activities while others are noted for involvement in exhibitions and school based projects which provide enrichment.

Timetable of the Study

The Study and the composition of the IPC were announced in early 2003 and the IPC met formally in Modena, Italy in November 2003 in order to commence the writing of the Discussion Document and plan the Study. Further work on the Discussion Document continued until its final text was agreed at a meeting of the IPC at ICME-10. The Document outlines the remainder of the program. There will be a Study Conference in Trondheim from 28 June to 2 July 2006. Applicants to attend this conference must apply to the co-Chairs by 31 August 2005, showing how they can contribute to the Study. It will not be possible to attend without contribution and it is expected that maybe up to 100 participants will be invited. Eventually by 2008 there will be a Study Volume published. This will be the formal outcome of the Study. It might be the Proceedings of the Study Conference, or it might be articles rewritten but inspired by papers at the conference.

Discussion document

This document occupies the central core of the Study. It is available to read at the Study website at www.amt.edu.au/icmis16.html. It comprises five chapters as follows.

1. *Introduction*
This chapter basically describes what ICMI Studies do in general and the general aims of this study.
2. *Description*
This chapter asks for the definition of ‘challenge’, asks how we are providing challenge, and where, with some brief examples.
3. *Current context*
This is the longest chapter and goes much deeper into the use of challenge, listing a broad range of different types of use of challenge.
These examples are classified and range from competitions (inclusive, exclusive, team, etc.), exhibitions, problem solving in schools, research activities, etc.
4. *Questions arising*
This chapter lists many questions which arise from the previous discussion and which are asked here, prompting specific paths for the study.
5. *Call for papers*
Finally the timetable and the method of participation are outlined.

Material available and scope of the study

The World Federation of National Mathematics Competitions (WFNMC), one of the five Affiliated Study Groups of ICMI, has published a policy document on similar matters to those which might be explored by the Study. In this document, to be found on the WFNMC website www.amt.edu.au/wfnmc.html.

This policy document defines the interests of WFNMC well beyond that of competitions, including a number of areas such as enrichment course work, maths clubs, research activities, publications, etc. In the last few years support for teachers has also become an increasing theme.

There are four other ASGS, and each has been closely associated with an ICMI Study. This Study is designed to go well beyond the areas beyond the WFNMC policy document and identify all areas in which mathematics challenge applies. Many members of the IPC are from quite non-competition backgrounds.

What is challenge?

There was considerable debate about this question and there will probably be no definitive answer. However it will become in itself one of the central areas of discussion in the Study.

Probably the most common definition will be based around the idea that challenge is the experience of meeting a new, unforeseen situation and coming to grips with it. This is a critical idea which I will pursue shortly. In the real world we continually have to face new situations and deal with them. It can be argued that by learning to meet challenge in a mathematical situation students will develop the powers of abstract reasoning that will enable them eventually to be able to systematically face other situations without apparent mathematical context.

Another proposal defined challenge as a non-traditional learning experience. Yet another defined it as the process of lifting oneself from one knowledge state to a higher one.

In and beyond

The Study has the words 'in and beyond'. As such it will need to address issues in the classroom, presumably normally within the syllabus, and those many programs outside the classroom, and how each of these may affect the learning and teaching process. I will touch on some aspects.

In the classroom

To take up the first definition of 'challenge' I have posed, some will argue that in order to assist students in the learning process, the syllabus in various countries has become progressively defined to a higher level. As a result more specific outcomes are usually listed, placing time pressures on room for challenging mathematics.

There is also a question as to whether the topics which are in the syllabus lend themselves to challenge. Topics such as arithmetic and algebraic techniques, calculus and increasingly statistics, which dominate the syllabus, do lend themselves to challenge and problem solving situations.

How this is done, how it can be done better, and in fact answering the question of why it should be done, will be taken up in the Study.

Beyond the classroom

I will spend more time discussing the less familiar cases of challenge beyond the classroom. This ICMI Study acknowledges the growing demand to provide what we might call enrichment, complementing the syllabus with challenging and stimulating material, helping students to think mathematically, and from this experience develop a problem solving ability which can be helpful for broader life skills (such a demand is met by many 'suppliers' with the approval of the teacher).

Being able to do these things in a mathematical context arguably provides the ability to do so in other contexts.

This is an important aspect of mathematics. Too often in assessing potential syllabus material we can look too closely for direct application of a particular idea or skill.

All too often there are very good techniques in mathematics available to help in problem solving which are not in the syllabus, but which are nevertheless accessible to many students and improve the student's reasoning powers..

Internationally there are many examples of enrichment beyond the set syllabus, normally on a voluntary basis as some students seek to extend their knowledge and broaden their base for further study. The following topics in mathematics are often introduced to students in such enrichment situations and can lend themselves to student access quite easily:

- counting methods;
- pigeonhole principle;
- other methods of proof, e.g., contradiction, induction, invariance;
- discrete optimisation;
- geometry.

I give a few examples of where such methods have been used in problems posed to students. The problems below illustrate mathematical techniques which are used in enrichment programs are drawn from questions either set in the Australian Mathematics Competition for the Westpac Awards or the International Mathematics Tournament of Towns.

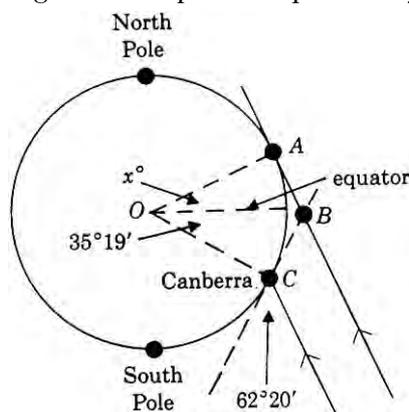
Problem 1 (geometry)

The latitude of Canberra is $35^{\circ} 19'$ S. At its highest point in the sky when viewed from Canberra the lowest star in the Southern Cross is $62^{\circ} 20'$ above the southern horizon. It can be assumed that rays of light from this star to any point on the earth are parallel. What is the northern-most latitude from which the complete Southern Cross can be seen?

Comments on solution 1

It is reasonably well-known that the Southern Cross can be seen in some northern latitudes. It was also used for navigation by the French aviators pioneering air routes across the Sahara to South America.

The solution is quite accessible to secondary students familiar with line and circle geometry and uses the following diagram. The proof requires a cyclic quadrilateral.



Problem 2 (counting)

In how many different ways can a careless office boy place four letters in four envelopes so that no one gets the right letter?

Comments on solution 2

There are two ways of doing this. The intuitive way is to systematically list all of the cases. Numbering the envelopes as 1, 2, 3 and 4 the letters can be placed in the following orders giving an answer of 9.

$$2143, 2413, 2341, 3142, 3412, 3421, 4123, 4312, 4321$$

As with other counting methods this gives the teacher the opportunity to discuss generalisations, and this generalises to the derangement formula as discussed in Niven (1965, p. 80). This idea can be further generalised as follows.

Problem 3 (counting)

In the school band, five children each own their own trumpet. In how many different ways can exactly three of the five children take home the wrong trumpet, while the other two take home the right trumpet?

Comments on Solution 3

It is not too difficult to count the cases here either on intuitive reasoning. Suppose the students taking home the wrong trumpet are called *A*, *B* and *C*. These can take the wrong trumpets in two ways: e.g., *A* takes trumpet *B*, *B* takes trumpet *C* and *C* takes trumpet *A*, or *A* takes trumpet *C*, *B* takes trumpet *A* and *C* takes trumpet *B*. We need also to know how many ways *A*, *B* and *C* can be chosen from the five. This is the same as the number of ways in which the two with the right trumpets can be chosen, this being ten; e.g., if the students are called *A*, *B*, *C*, *D* and *E* these are

- A* and *B*, *A* and *C*, *A* and *D*, *A* and *E*,
- B* and *C*, *B* and *D*, *B* and *E*,
- C* and *D*, *C* and *E*, and
- D* and *E*.

Thus the answer is $10 \times 2 = 20$.

Again this can be generalised to give a formula enabling the problem to be solved with any number of students and any fixed number to have the wrong trumpet. The formula is

$$\binom{n}{r} \times D(r)$$

For $n = 5$ the solutions can be tabulated as

r	$\binom{5}{r}$	$D(r)$	$\binom{5}{r} \times D(r)$
0	1	1	1
1	5	0	0
2	10	1	10
3	10	2	20
4	5	9	45
5	1	44	44

with n being the number of students, r being the number of wrong trumpets.

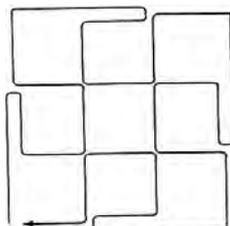
There are many other counting scenarios which can similarly be invoked in real problems. These can lead to the inclusion exclusion principle, necklace counting method of Polya, etc.

Problem 4 (discrete optimisation)

A village is constructed in the form of a square, consisting of 9 blocks, each of side length 1, in a 3x3 formation. Each block is bounded by a bitumen road. If we commence at a corner of the village, what is the smallest distance we must travel along bitumen roads, if we are to pass along each section of bitumen road at least once and finish at the same corner?

Comments on solution 4

The diagram shows a closed tour of length 28 and we claim this to be a minimum.



Each of the four corners is incident with two roads and requires at least one visit. Each of the remaining twelve intersections is incident with three or four roads and requires at least two visits. Hence the minimum is at least $4 + 12 \times 2 = 28$.

Problem 5 (proof contradiction)

There are 2000 apples, contained in several baskets. One can remove baskets and/or remove apples from the baskets. Prove that it is possible to then have an equal number of apples in each of the remaining baskets, with the total number of apples being not less than 100.

Comments on solution 5

This problem looks very difficult but by assuming the result to be false we can more easily find a contradiction as follows.

Assume the opposite: then the total number of baskets remaining is not more than 99 (otherwise we could leave 1 apple in each of 100 baskets and remove the rest).

Furthermore, the total number of baskets with at least two apples is not more than 49, the total number of baskets with at least three apples is not more than 33, etc.

So the total number of apples is not more than $99 + 49 + 33 + \dots$. This number is less than 2000. We thus have a contradiction.

Problem 6 (pigeonhole principle)

Ten friends send greeting cards to each other, each sending five cards to different people. Prove that at least two of them sent cards to each other.

Comment on solution 6

Dirichlet is reported to have first articulated this method of proof, which also bears his name, which is quite intuitive. If you have n pigeon holes and more than n pigeons to put them in, one pigeonhole must contain at least two pigeons. This proof essentially goes as follows.

We are given that each friend sends a card to 5 different of the other 9 friends. This means that there are $10 \times 9 = 90$ different routes.

By symmetry, these consist of 45 pairs (friend i to friend j and friend j to friend i).

However the number of cards sent is $10 \times 5 = 50$.

Since, each of these 50 cards is sent on a different route, by the pigeonhole principle at least $50 - 45 = 5$ cards must be sent in opposite directions along a repeated route, enough to prove what is required.

Problem 7 (invariance)

On the island of Camelot live thirteen grey, fifteen brown and seventeen crimson chameleons. If two chameleons of different colours meet, they both simultaneously change colour to the third colour (e.g., if a grey and brown chameleon meet each other they both change to crimson). Is it possible that they will eventually all be the same colour?

Comment on solution 7

Looking for an invariant is a standard method of proof. The easiest invariants to spot in a real life situation are preservation of parity. This problem is a little more difficult to solve but I have included it here because I like it so much. The situation in which two dull coloured animals can both turn to crimson after touching is interesting. When you see the solution it is not so difficult. I would strongly encourage the reader to solve this without looking at the solution. However I do give it here.

In this case the numbers of chameleons of each colour at the start have remainders of 0, 1 and 2 when divided by 3.

Each ‘meeting’ maintains such a situation (not necessarily in any order) as two of these remainders must either be reduced by 1 (or increased by two) while the other must be increased by 2 (or reduced by 1).

Thus at least two colours are present at any stage, guaranteeing the possibility of obtaining all of the three colours in fact by future meetings.

Conclusions

The above problems do not necessarily connect to the Study document. I include them to emphasise cases where some accessible techniques have been used with secondary students to provide challenging situations. They are also a collection of problems which I obviously like very much also.

The Study has wide scope and has significant implication. This interaction between school syllabus and challenge has not been looked at on this scale before.

I encourage you to take an interest and read the web site at <http://www.amt.edu.au/icmis16.html>, where the discussion document and further information can be found.

References

Niven I. (1965). *Mathematics of Choice: How to Count Without Counting*. Mathematics Association of America.

Improving statistical literacy: The respective roles of schools and the National Statistical Offices

Dennis Trewin

Australian Bureau of Statistics

Statistical Literacy is of increasing importance in all walks of life. Australians, whether as workers, citizens or social participants, are continually faced with statistical data and presentations that they should understand if they are to make sensible Informed decisions. Yet most Australians have a very limited grasp of even the simplest concepts for interpreting and using data in what is becoming an increasingly data driven society.

The ABS has a leadership role for improving statistical literacy. It can do this by being an active participant in the type of initiatives outlined above. It can also:

- provide educational programs and support materials for teachers, and
- provide relevant resources, information and events for students.

We have undertaken a number of initiatives and these will be described.

The presentation will place improvements in Statistical Literacy in the context of what we are referring to as the Australian Statistics Education System (ASES) which covers undergraduate and postgraduate training in statistics, as well as schools. Specifically our efforts to develop a National Framework for teaching of statistics in schools will be described. This work is taking place in partnership with the Curriculum Corporation and the Statistical Society of Australia.

Introduction

Statistics is not a branch of mathematics but it is a mathematical science. Mathematics teachers are probably best positioned to teach statistics at school; however, not all have had adequate training in this area.

I am sure that all of you here are well aware that a major objective for the past few years of educational bodies and Government departments alike, is to ensure that all Australians attain sound foundations in the core skills of numeracy and literacy.

The importance of these core skills stand as self-evident, not only so that school leavers can pursue further education or better position themselves to take advantage of career opportunities, but just to be equipped with basic skills to allow them to conduct everyday activities like reading newspapers, compose and submit substantial job applications and to be able to calculate grocery bills and weekly household budgets. These skills are critical for students who are beginning life in all its roles such as worker or community leader.

Another important skill for everyday life, and the skills that the Australian Bureau of Statistics (ABS) is most concerned about, is statistical literacy. The ability to understand,

interpret and evaluate statistical information is indispensable to help them understand the world around them and in making sensible, informed decisions.

Statistical understanding is becoming more important to everyday life. Australians as citizens, workers and social participants are continually faced with data that they must understand if they are to make sensible decisions. The problem is most Australians have a very limited grasp of the simplest concepts of interpreting and using data. Such limitations are likely to have significant consequences for Australia's competitiveness, and for the quality of decisions made in people's lives.

As mathematics teachers, you are all aware that mathematics subjects now incorporate teaching statistics. Therefore making statistics vital, as well as statistical literacy, is also an important issue.

Making mathematics vital in this sense, and from an ABS perspective, means furnishing students with the vital statistical skills they will need when they take their places in the workforce and community. We are not talking about turning all students into statisticians: just to develop sufficient skills to make the information age more meaningful (although hopefully some of those with a high aptitude for statistics will choose to become statisticians as a career choice).

Why the ABS and its role

Why is this important to the ABS? The role of the ABS is to 'assist and encourage informed decision making, research and discussion, within government and community, by providing a high quality, objective and responsive national statistical service'. To keep achieving this mission, ABS needs a supply of statisticians to take up positions in the ABS. ABS also want to help the Australian people understand and respond in better-informed ways to the vagaries and uncertainties of the world they live in.

ABS is involved in education to:

- ensure Australian school children acquire a sufficient understanding and appreciation of how data can be acquired and used to make informed judgments in their daily lives, as children and then as adults; and
- instil in school students sufficient interest and enthusiasm for statistics that some will seek to pursue tertiary studies in statistics.

There is a real need to encourage such interest in statistics. This need is national, as seen by the number of organisations experiencing serious difficulties recruiting an adequate supply of new statistics graduates, such as:

- universities;
- Federal and State/Territory departments and agencies;
- ABS;
- CSIRO;
- pharmaceutical companies; and
- the financial sector.

We also have altruistic motives. As the national statistical agency, we think it is important to support steps to improve statistical literacy.

Today's school students are the future users of our statistics. There are long term advantages for Australia if children know how to collect, display and use statistics. Therefore, we have made it a priority to establish a section within the ABS responsible for servicing the education sector; it is known as the National Education Services Unit (NESU). It works closely with education departments and teachers' associations. Its activities are described in the next section. We have found it particularly useful to have

teachers on six-month–twelve-month secondments to help us develop various products and services.

What the ABS is currently doing

The ABS, through NESU, is already involved with the school sector. It provides a variety of resources free through the ABS website (www.abs.gov.au). The materials are produced by teachers for teachers and are supported by the expertise of a diverse and highly qualified staff made up of statisticians, economists, mathematicians, geographers and ICT technicians.

The ABS promotes statistical literacy through:

- the production of the Statpak catalogue highlighting ABS publications with a high degree of relevance to school curricula;
- the production of special school publications such as *Statistics — A Powerful Edge* and *Measuring Australia's Economy*. Other publications such as *Measures of Australia's Progress* and *Australian Social Trends* are also relevant to schools. Lesson plans are prepared to help the learning process;
- responding to curriculum reviews highlighting the importance of statistical literacy skills;
- continuing to develop STATSERCISE and exSTATIC (and other mathematics resources);
- preparing data sets, including bivariate data sets, to support the mathematics classroom;
- implementing a new site map to make it easier for teachers to find relevant material on the NESU webpages;
- continuing to respond to the needs of teachers by including useful content on the education pages of the ABS website;
- development of interactive games on statistical concepts, including *A Tale of Two Worlds*;
- promoting ABS education materials at conferences and through education media;
- assisting text book publishers to include ABS data in their publications;
- developing a professional development resource to assist teachers to develop statistical literacy;
- supporting other organisations using or promoting statistics in schools.

Other national statistical agencies are undertaking similar activities. We are keeping in touch with them so that we can learn off each other's ideas. The International Statistical Institute has a special education section which is now putting extra effort into improving statistical literacy and exchanging knowhow between countries.

We believe that the best way to encourage students to take an interest in statistics is to introduce statistical concepts in fun and engaging ways as early as possible. There should be no attempt to teach statistical theory at schools.

The NESU is already providing census lesson plans for Prep to Year 10 and worksheets like exSTATIC and STATSERCISE, which are designed for and used in upper primary classes. These resources introduce simple statistical concepts that build on the established principles taught to them by their teachers. By entering primary schools with our resources, we hope to attract the interests of students and for students to carry this interest with them to secondary education and beyond. These resources have proven to be highly popular with students and mathematics teachers.

To assist teachers and follow up the interest generated in primary school, the NESU

has provided resources for secondary schools including:

- lesson plans on the Australian economy, society, environment and progress; and
- mathematics-dedicated datasets, including bivariate datasets, both based on current real data and ready to download to computer or calculator.

These resources are designed for students, are ready for classroom use and require very little, if any, modification by teachers. Each resource comes with extensive explanations, incorporating the latest techniques in teaching practices and are designed to support the material in any text. Many of the resources and teaching tools facilitate integrated learning, recognising the importance of students learning mathematics and statistics in more than just the mathematics classroom.

Census at school

The *Census at school* site will provide an on-line data collection project designed for upper primary to middle secondary students, where students collect information about themselves using questions that reflect their own interests. Students will fill in their census forms as part of a whole class activity. Data samples can then be used for teaching and learning across a range of Key Learning Areas.

The program is based on a similar program developed by the Royal Statistical Society of the United Kingdom, under the guidance of Neville Davies (which has property rights to the brand). It has been extremely successful in improving statistical literacy and has been extended to many other countries, including several in Africa. It is also being built in collaboration with ‘early adaptors’ of the program in Australia, such as the Noel Baker Centre for School Mathematics in South Australia and the Office of Education and Statistical Research in Queensland.

The primary aim of *Census at school* is to encourage the development of statistical literacy in students and a learning experience for teachers. *Census at school* will be conducted in the lead up to the 2006 Census of Population and Housing in August. The census will be topical and so of interest to students and teachers. It will be an engaging, educationally focussed, on-line learning experience that assists both students and teachers in building skills in creating, using and interpreting data and understanding statistical concepts.

This initiative will have a direct impact on improving statistical literacy among Australian school students and teachers by:

- creating a better understanding of statistics and increasing their use in the classroom;
- encouraging student interest in statistics by involving them in data about themselves;
- demonstrating practical use of statistics in everyday life through their application in cross-curricula lesson plans;
- fostering a good understanding of statistical collections, particularly population censuses;
- meeting a major ICT requirement for Key Learning Areas through the use of the Internet for educational purposes.

The ABS intends to provide a range of high quality lesson plans aimed at supporting classroom teachers across a range of subjects and Australia wide. These will be developed with the support of teachers. In addition, the website will provide contextual support materials of interest to students and teachers.

A national framework for teaching statistics

Each State and Territory has elements of statistics and probability taught in its school curriculum (the term *chance and data* is in common parlance) developed with varying degrees of input from statistical scientists, however a coordinated national framework is required if all students are to develop statistical literacy.

This section describes a development known as the Australian Statistics Education System (ASES) being led by Dr Nick Fisher (development started when he was President of the Statistical Society of Australia (SSAI) and myself (as head of the Australian Bureau of Statistics, the largest employer of statistics). It covers primary, secondary and tertiary levels. The Curriculum Corporation has been assisting with the schools component.

The commencement of this development was on 10 July 2002, when the ABS and the SSAI hosted a meeting of the leaders of the statistical profession in Australia, including a high proportion of professors of statistics, to discuss a serious emerging issue: the increasing shortage of professional statisticians in Australia. Attendees also included the President of the Australian Association of Mathematics Teachers, and the Science Advisor to the Federal Minister for Education, Training and Science.

Table 1. A possible high-level framework covering all levels of education.

Stage	Enabling process	Outcomes
Schools 1. Develop familiarity with basic concepts of statistical science 2. Stimulate some to take more serious courses in latter part of high school	<ul style="list-style-type: none"> • Educational (training) programs & support materials for teachers • Resources, information & events for students • Awareness and support for materials for careers advisers 	<ul style="list-style-type: none"> • Australians more statistically literate, better skilled to deal with pressures of data-driven society • Students interested in pursuing further study of statistics
Undergraduate 1. Service courses and statistics majors at universities 2. Bachelor of Statistical Science at selected universities	<ul style="list-style-type: none"> • Degree program framework • Resources, information & events for students • Training programs and support for materials for careers advisers • Degree program accreditation • Industry support program 	<ul style="list-style-type: none"> • Graduates bring more statistical understanding to their work • Graduates able to pursue statistics as a profession
Graduate programs 1. Dip.Stats. (Diploma of Statistics) 2. M.Stats (Master of Statistics) 3. D.Stats (Professional doctorate) 4. PhD (Normal doctorate)	<ul style="list-style-type: none"> • Degree program framework • Statistics research centres • Information & events for students • Industry support program 	<ul style="list-style-type: none"> • Graduates for jobs requiring more advanced statistics • Research statisticians working in industry, business and government • Academic statisticians — teaching and research at universities

The major employers of statisticians are experiencing serious difficulties recruiting an adequate supply of new statistics graduates. The universities in particular are deeply concerned: the demographic profile of their statistical staff is such that many will be retiring over the next few years yet younger statisticians are simply not emerging from graduate programs to replace them. This is creating problems for both teaching and research.

This is coming at a time when the demand for statisticians is increasing. Information is becoming an increasingly important asset in many industries, where statisticians are essential for proper design, collection, analysis and interpretation of statistical information.

Objectives of the framework

There are a number of objectives that this framework attempts to fulfil:

1. to ensure that Australian school children acquire a sufficient understanding and appreciation of how data can be acquired and used to make decisions and informed judgements in their daily lives, as children and then as adults;
2. to instil in more statistically-able school students sufficient interest and enthusiasm for statistics that they will seek to pursue tertiary studies in statistics with a view of making a career in the area;
3. to support Australian science and scientific research, and business and industry innovation through the availability of high quality and appropriately trained statisticians and a statistically informed population.

Table 1 sets out a possible high level framework, covering all levels of education. This is just a quick overview of a framework that begins with schools, moves through undergraduate and postgraduate programs and ends with career advice and industry support. For now, let us concentrate on schools.

Schools

Our objectives for the Schools component are:

- (a) Help school children develop familiarity with the basic ideas of extracting information from data as this relates to their lives
- (b) Stimulate some of the more statistically able to undertake further studies in Statistics

Two basic stages of implementation

Stage 1: Development of an appropriate framework for a K–12 statistics program that addresses both objective (a) and (b) above. It will be essential to consult closely with school education bodies, key professional organisations, schools and community stakeholders.

Stage 2: Development of appropriate educational and ongoing programs for teachers, and suitable supporting resources for teaching the subject.

The success of this proposal will be dependent on:

- (i) National agreement about the curriculum basis for the study of statistics
- (ii) A supply of excellent resource materials for teachers to use that develop in students a "yearning for learning" about statistics
- (iii) Appropriate professional development programs that provide teachers with:
 - the skills and knowledge to use the resource material; and
 - a suitable level of current knowledge and understanding about statistics and a means of maintaining their knowledge according to their teaching needs.

We have been working with the Curriculum Corporation to develop this. It has the support of their Board and we are awaiting funding before taking it to the next stage.

Not everyone agrees that statistical concepts are best taught by mathematics teachers. The Past President of the Royal Statistical Society, Adrian Smith, prepared a report *Making Mathematics Count* for the UK government. In this report, he states:

Twenty-five percent of the timetable for GCSE is now statistics and data handling. That wasn't there 20 years ago. We should take the statistics out of GCSE maths. The addition of it has led to this loss of time for practice and fluency and absorption, so mathematical core skills have gone down.

He says statistics should be taught generally through the curriculum, not just in maths. But there is agreement that the focus should be on statistical concepts not theory and that any emphasis on "theory" should be on mathematical concepts. These are a fundamental underpinning for tertiary studies of statistical theory.

How does Australia compare with the rest of the world?

In the mid 1990s, the OECD countries decided to include direct measures of student learning through what has become known as the Programme for International Student Assessment (PISA). All thirty OECD countries participate and a growing number of others are becoming involved: twenty-eight additional countries have signed on for PISA 2006.

PISA is aimed at fifteen year-olds and assesses their literacy, numeracy and science skills. It is designed to be neutral to cultural influences. In the 2003 study, there was special emphasis on assessing numeracy skills including one statistics component (probability).

The 2003 results will be available in December 2004 and will be presented to the Conference. In past PISA studies, Australia has fared reasonably well. It will be interesting to see if that performance is maintained. Another interesting aspect of the PISA study was that, in Australia, the correlation between PISA scores and socio-economic status (the so-called socio-economic gradient) was higher than in many other countries, such as the Scandinavian countries.

Outcomes

At the end of the day what are the outcomes for the students? What should we be attempting to achieve for our young people? As has been suggested:

- today's students need to develop an ability to make informed decisions and judgements in regard to both their student lives now and for their adult lives in the future;
- statistically-able students who want to pursue a career in statistics are appropriately supported, and have increased awareness of statistics as a career choice.

For those with good numerical skills, being a statistician can be a very rewarding career. This is a strong view held by those who have pursued such a career. 'Finding work has not been difficult (there are not too many unemployed statisticians) and the pay is good, although not too many statisticians make a fortune from their career. Nevertheless, some of the pharmaceutical companies are offering six-figure salaries to good quality recent statistics graduates...'

As part of the national framework, highlighted above, processes would be put in place to provide a coordinated careers advisory process, where links would be set-up between secondary and tertiary educational bodies. This process should also link into industry bodies and large private and public sector organisations. In this case, education would not stop at the learning process, it would help students take those first steps after school, into tertiary education and employment. A key element has to be to convince the students of the importance and relevance of a statistics career. Many do not currently understand the opportunities or the nature of work undertaken by statisticians. The same might also be said for teachers.

In today's climate, careers are based less and less on remaining with one company, moving through departments and positions to build a career. Today, careers are built through moving from business to business and industry to industry, taking opportunity of vacancies and building experience. Statistical skills that are very transferable between industries and organisations. Having good qualifications in statistics means being able to apply those skills to any area; in other words, to apply statistical techniques, you do not have to be an expert in a particular field.

The national framework has real and important outcomes for students — for all students, who will become more statistically literate and for statistically-able students who can benefit from good career prospects. As soon as we get the green light to progress with the framework, we will begin consultations with representatives of mathematics teachers' associations.

Summary

In closing, let me say that one major factor for success, be it for a community, an economy, an organisation or an individual, is how information is used, particularly today with the amount, different sources and different formats, of this information. Australia needs a society of individuals able to source, compile and analyse this information to make informed judgements leading to sensible decision making. In today's information society, information is becoming more readily available particularly through the Internet.

Many teachers are already teaching statistics in the classroom and the ABS, through the NESU, has already begun to put in place systems, networks and resources to assist teachers to help students develop essential statistical skills. These resources include: worksheets, lesson plans, datasets, webquests and other curriculum based activities at both primary and secondary levels. In addition, the NESU is embarking on its most ambitious initiative to date, the *Census at school* program, a program aimed at increasing the level of understanding, interest and profile of statistics among both students and teachers.

These efforts are a start, however to take things further and fully develop statistical literacy in Australian students, more coordination is needed. The ABS has, in partnership with SSAI and Curriculum Corporation, outlined a national plan for a coordinated approach to teaching statistics in schools at the primary, secondary and tertiary levels and a careers advisory service to facilitate progress through the educational stages. Vital to the success of this approach in schools is the development of:

- a statistics program for Years K–12;
- appropriate education and ongoing programs for teachers;
- suitable supporting resources for teaching the subject; and
- a career advisory process for the students.

Mathematics teachers will play a critical role in such a coordinated, national approach to teaching statistics. The skills learned in the mathematics classroom are essential to

understand and make sense of information. It is this link between the importance of statistics as an essential skill and the role that mathematics plays in the development of statistical skills, that is another reason why mathematics is vital. Statistics is an essential skill for our young people. It is vital for their future.

We hope we are taking the right steps to support mathematics teachers, particularly in their teaching of statistical concepts. We welcome any feedback.

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Keeping learning on track: Formative assessment and the regulation of learning

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Introduction

'I'd love to teach for deep understanding, but I have to raise my students' test scores.' I have heard this sentiment from hundreds of teachers in many countries. Implicit in this statement is the notion that raising test scores is not compatible with teaching for deep understanding. As pressures for teachers to be accountable for the performance of their students increase, does this mean that there is no room for teaching for deep understanding? Or is there a way to achieve both?

Over the course of a ten-year study, Paul Black and I sought to find out if using assessment to support learning, rather than just to measure its results, can improve students' achievement, even when such achievement is measured in the form of state-mandated tests. In reviewing 250 studies from around the world, published between 1987 and 1998, we found that a focus by teachers on assessment for learning, as opposed to assessment of learning, produced a substantial increase in students' achievement (Black & Wiliam, 1998a). Since the studies also revealed that day-to-day classroom assessment was relatively rare, we felt that considerable improvements would result from supporting teachers in developing this aspect of their practice (Black & Wiliam, 1998b). The studies did not reveal, however, how this could be achieved and whether such gains would be sustained over an extended period of time.

Since 1999, we have worked with many groups of teachers, from both primary and secondary schools, in the United Kingdom and in the United States. and these collaborations have shown that our initial optimism was justified. In a variety of settings teachers have found that by teaching for deep understanding has resulted in an increase in student performance on externally-set tests and examinations (Wiliam et al., 2004).

The details of how we put these ideas into practice can be found elsewhere (Black et al., 2002; Black et al., 2003). In this paper, I want to describe the key ingredients of formative assessment: effective questioning, feedback, ensuring learners understand the criteria for success, and peer- and self-assessment, and then to show how they fit together within the general idea of the 'regulation of learning'.

What makes a good question?

Two items used in the Third International Mathematics and Science Study (TIMSS) are shown in Figure 1 below. Although apparently quite similar, the success rates on the two

items were very different. For example, in Israel, 88% of the students answered the first items correctly, while only 46% answered the second correctly, with 39% choosing response (b). The reason for this is that many students, in learning about fractions, develop the naive conception that the largest fraction is the one with the smallest denominator, and the smallest fraction is the one with the largest denominator. This approach leads to the correct answer for the first item, but leads to an incorrect response to the second. Furthermore, if we note that 46% plus 39% is very close to 88%, this provides strong evidence that many students who answered the first item correctly, did so with an incorrect strategy. In this sense, the first item is a much weaker item than the second, because many students can get it right for the wrong reasons.

This illustrates a very general principle in teachers' classroom questioning. By asking questions of students, teachers try to establish whether students have understood what they are meant to be learning, and if students answer the questions correctly, it is tempting to assume that the students' conceptions match those of the teacher. However, all that has really been established is that the students' conceptions fit, within the limitations of the questions. Unless the questions used are very rich, there will be a number of students who manage to give all the right responses, while having very different conceptions from those intended.

Item 1 (success rate 88%)			
Which fraction is the smallest?			
a) $\frac{1}{6}$	b) $\frac{2}{3}$	c) $\frac{1}{3}$	d) $\frac{1}{2}$
Item 2 (success rate 46%)			
Which fraction is the largest?			
a) $\frac{4}{5}$	b) $\frac{3}{4}$	c) $\frac{5}{8}$	d) $\frac{7}{10}$

Figure 1. Two items from the Third International Mathematics and Science Study.

A particularly stark example of this is the following pair of simultaneous equations:

$$\begin{aligned} 3a &= 24 \\ a + b &= 16 \end{aligned}$$

Many students find this difficult, saying that it cannot be done. The teacher might conclude that they need some more help with equations of this sort, but the most likely reason for the difficulties with this item is not to with mathematical skills but with their beliefs. If the students are encouraged to talk about their difficulty, they often say things like, 'I keep on getting b is 8, but it can't be because a is.' The reason that many students have developed such a belief is, of course, that before they were introduced to solving equations, they were will probably have been practising substitution of numbers into algebraic formulas, where each letter stood for a different number. Although the students will not have been taught that each letter must stand for a different number, they have generalised implicit rules from their previous experience, just as because we always show them triangles where the lowest side is horizontal, they talk of 'upside-down triangles' (Askew & Wiliam, 1995).

The important point here is that we would not have known about these unintended conceptions if the second equation had been $a + b = 17$ instead of $a + b = 16$. Items that reveal unintended conceptions — in other words that provide a ‘window into thinking’ — are difficult to generate, but they are crucially important if we are to improve the quality of students’ mathematical learning.

Some people have argued that these unintended conceptions are the result of poor teaching. If only the teacher had phrased their explanation more carefully, had ensured that no unintended features were learnt alongside the intended features, then these misconceptions would not arise.

This argument fails to acknowledge two important points. The first is that this kind of over-generalisation is a fundamental feature of human thinking. When young children say things like ‘I spented all my money’, they are demonstrating a remarkable feat of generalisation. From the huge messiness of the language that they hear around them, they have learnt that to create the past tense of a verb, one adds ‘d’ or ‘ed’. In the same way, if one asks young children what causes the wind, the most common answer is ‘trees’. They have not been taught this, but have observed that trees are swaying when the wind is blowing and (like many politicians) have inferred a causation from a correlation.

The second point is that even if we wanted to, we are unable to control the student’s environment to the extent necessary for unintended conceptions not to arise. For example, it is well known that many students believe that the result of multiplying 2.3 by 10 is 2.30. It is highly unlikely that they have been taught this. Rather this belief arises as a result of observing regularities in what they see around them. The result of multiplying whole-numbers by 10 is just to add a zero, so why should not that work for all numbers? The only way to prevent students from acquiring this ‘misconception’ would be to introduce decimals before one introduces multiplying single-digit numbers by 10, which is clearly absurd. The important point is that we must acknowledge that what students learn is not necessarily what the teacher intended, and it is essential that teachers explore students’ thinking before assuming that students have ‘understood’ something. In this sense assessment is the bridge between teaching and learning.

Questions that give us this ‘window into thinking’ are hard to find, but within any school there will be good selection of rich questions in use; the trouble is that each teacher will have her or his stock of good questions, but these questions do not get shared within the school, and are certainly not seen as central to good teaching.

In most Anglophone countries, teachers spend the majority of their lesson preparation time in marking books, almost invariably doing so alone. In some other countries, the majority of lesson preparation time is spent planning how new topics can be introduced, which contexts and examples will be used, and so on. This is sometimes done individually or with groups of teachers working together. In Japan, however, teachers spend a substantial proportion of their lesson preparation time working together to devise questions to use in order to find out whether their teaching has been successful, in particular through the process known as ‘lesson study’ (Fernandez & Makoto, 2004).

Now in thinking up good questions, it is important not to allow the traditional concerns of reliability and validity to determine what makes a good question. For example, many teachers think that the following question, taken from the *Chelsea Diagnostic Test* for algebra, is ‘unfair’:

Simplify (if possible): $2a + 5b$

This item is felt to be unfair because students ‘know’ that in answering test questions, you have to do some work, so it must be possible to simplify this expression, otherwise the

teacher would not have asked the question. I would agree that to use this item in a test or an examination where the goal is to determine a student's achievement would probably not be a good idea. However, for the purpose of finding out whether students understand algebra, it is a very good item indeed. If, in the context of classroom work, rather than a formal test or exam, a student can be tempted to 'simplify' $2a + 5b$ then I want to know that, because it means that I have not managed to develop in the student a real sense of what algebra is about.

Similar issues are raised by asking students which of the following two fractions is the larger:

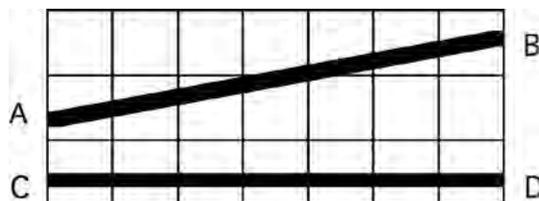
$$\frac{3}{7} \quad \frac{3}{11}$$

Now in some senses this is a 'trick question'. There is no doubt that this is a very hard item, with typically only around one fourteen-year old in six able to give the correct answer (compared with around three-quarters of fourteen-year-olds being able to select correctly the larger of two 'ordinary' fractions). It may not, therefore, be a very good item to use in a test of students' achievement; but as a teacher, I think it is very important for me to know if my students think that three-elevenths is larger than three-sevenths. The fact that this item is seen as a 'trick question' shows how deeply ingrained into our practice the summative function of assessment is.

A third example, that caused considerable disquiet among teachers when it was used in a national test, is based on the following item, again taken from one of the *Chelsea Diagnostic Tests*:

Which of the following statements is true:

1. AB is longer than CD
2. AB is shorter than CD
3. AB and CD are the same length



Again, viewed in terms of formal tests and examinations, this may be an unfair item, but in terms of a teacher's need to establish secure foundations for future learning, I would argue that this is entirely appropriate.

Rich questions, of the kind described above, provide teachers not just with evidence about what their students can do, but also what the teacher needs to do next, in order to broaden or deepen understanding.

Classroom questioning

There is also a substantial body of evidence about the most effective ways to use classroom questions. In many schools, teachers tend to use questions as a way of directing the attention of the class, and keeping students 'on task', by scattering questions all around the classroom. This probably does keep the majority of students 'on their toes' but makes only a limited contribution to supporting learning. What is far less frequent is to see a teacher, in a whole-class lesson, have an extended exchange with a single student, involving a

second, third, fourth or even fifth follow-up question to the student's initial answer. With such questions, the level of classroom dialogue can be built up to quite a sophisticated level, with consequent positive effects on learning. Of course, changing one's questioning style is very difficult where students are used to a particular set of practices (and may even regard asking supplementary questions as 'unfair'). It may even be that other students see extended exchanges between the teacher and another student as a chance to relax and go 'off task', but as soon as students understand that the teacher may well be asking them what they have learned from a particular exchange between another student and the teacher, their concentration is likely to be quite high.

How much time a teacher allows a student to respond before evaluating the response is also important. It is well known that teachers do not allow students much time to answer questions, and, if they do not receive a response quickly, they will 'help' the student by providing a clue or weakening the question in some way, or even moving on to another student. However, what is not widely appreciated is that the amount of time between the student providing an answer and the teacher's evaluation of that answer is just as important, if not more so. Of course, where the question is a simple matter of fact recollection, then allowing a student time to reflect and expand upon the answer is unlikely to help much. However, where the question requires thought, then increasing the time between the end of the student's answer and the teacher's evaluation from the average 'wait time' of less than a second to three seconds, produces measurable increases in learning (although increases beyond three seconds have little effect, and may cause lessons to lose pace).

In fact, questions need not always come from the teacher. There is substantial evidence that students' learning is enhanced by getting them to generate their own questions (Foos et al., 1994). If instead of writing an end-of-topic test herself, the teacher asks the students to write a test that tests the work the class has been doing, the teacher can gather useful evidence about what the students think they have been learning, which is often very different from what the teacher thinks the class has been learning. This can be a particularly effective strategy with disaffected older students, who often feel threatened by tests. Asking them to write a test for the topic they have completed, and making clear that the teacher is going to mark the question rather than the answers, can be a hugely liberating experience for many students.

Some researchers have gone even further, and shown that questions can limit classroom discourse, since they tend to demand a simple answer. There is a substantial body of evidence the classroom learning is enhanced considerably by shifting from asking questions to making statements (Dillon, 1988). For example, instead of asking, 'Are all squares rectangles?', which seems to require a 'simple' yes/no answer, the level of classroom discourse (and student learning) is improved considerably by framing the same question as a statement — 'All squares are rectangles,' — and asking students to discuss this in small groups before presenting a reasoned conclusion to the class.

The quality of feedback

Ruth Butler (1998) investigated the effectiveness of different kinds of feedback on 132 Year 7 students in twelve classes in four Israeli schools. For the first lesson, the students in each class were given a booklet containing a range of divergent thinking tasks. At the end of the lesson, their work was collected in. This work was then marked by independent markers. At the beginning of the next lesson, two days later, the students were given feedback on the work they had done in the first lesson. In four of the classes students were

given marks (which were scaled so as to range from 40 to 99) while in another four of the classes, students were given comments, such as, 'You thought of quite a few interesting ideas; maybe you could think of more ideas.' In the other four classes, the students were given both marks and comments.

Then, the students were asked to attempt some similar tasks, and told that they would get the same sort of feedback as they had received for the first lesson's work. Again, the work was collected in and marked.

Those given only marks made no gain from the first lesson to the second. Those who had received high marks in the tests were interested in the work, but those who had received low marks were not. The students given only comments scored, on average, 30% more on the work done in the second lesson than on the first, and the interest of all the students in the work was high. However, those given both marks and comments made no gain from the first lesson to the second, and those who had received high marks showed high interest while those who received low marks did not.

In other words, far from producing the best effects of both kinds of feedback, giving marks alongside the comments completely washed out the beneficial effects of the comments. The use of both marks and comments is probably the most widespread form of feedback used in the Anglophone world, and yet this study (and others like it — see below) show that it is no more effective than marks alone. In other words, if you write careful diagnostic comments on a student's work, and then put a score or grade on it, you are wasting your time. The students who get the high scores do not need to read the comments and the students who get the low scores do not want to. You would be better off just giving a score. The students will not learn anything from this but you will save yourself a great deal of time.

A clear indication of the role that ego plays in learning is given by another study by Ruth Butler (1987). In this study, two hundred Year 6 and 7 students spent a lesson working on a variety of divergent thinking tasks. Again, the work was collected in and the students were given one of four kinds of feedback on this work at the beginning of the second lesson (again two days later):

- a quarter of the students were given comments;
- a quarter were given grades;
- a quarter were given written praise; and
- a quarter were given no feedback at all.

The quality of the work done in the second lesson was compared to that done in the first. The quality of work of those given comments had improved substantially compared to the first lesson, but those given grades and praise had made no more progress than those given absolutely no feedback throughout their learning of this topic.

At the end of the second lesson, the students were given a questionnaire about what factors influenced their work. In particular the questionnaire sought to establish whether the students attributed successes and failures to themselves (called ego-involvement) or to the work they were doing (task-involvement). Examples of ego- and task-involving attributions are shown in Table 1.

Those students given comments during their work on the topic had high levels of task-involvement, but their levels of ego-involvement were the same as those given no feedback. However, those given praise and those given grades had comparable levels of task-involvement to the control group, but their levels of ego-involvement were substantially higher. The only effect of the grades and the praise, therefore, was to increase the sense of ego-involvement without increasing achievement.

This should not surprise us. In pastoral work, we have known for many years that one should criticise the behaviour, not the child, thus focussing on task-involving rather than

Table 1: Ego- and task-related attributions.

Attribution of	Ego	Task
Effort	To do better than others	Interest
	To avoid doing worse than others	To improve performance
Success	Ability	Interest
	Performance of others	Effort
		Experience of previous learning

ego-involving feedback. These findings are also consistent with the research on praise carried out in the 1970s which showed clearly that praise was not necessarily ‘a good thing’ — in fact the best teachers appear to praise slightly less than average (Good & Grouws, 1975). It is the quality, rather than the quantity of praise that is important and in particular, teacher praise is far more effective if it is infrequent, credible, contingent, specific and genuine (Brophy, 1981). It is also essential that praise is related to factors within an individual’s control, so that praising a gifted student just for being gifted is likely to lead to negative consequences in the long term.

The timing of feedback is also crucial. If it is given too early, before students have had a chance to work on a problem, then they will learn less. Most of this research has been done in the United States, where it goes under the name of ‘peekability research’, because the important question is whether students are able to ‘peek’ at the answers before they have tried to answer the question. However, a British study, undertaken by Simmonds and Cope (1993) found similar results. Pairs of students aged between 9 and 11 worked on angle and rotation problems. Some of these worked on the problems using *Logo* and some worked on the problems using pencil and paper. The students working in *Logo* were able to use a ‘trial and improvement’ strategy that enabled them to get a solution with little mental effort. However, for those working with pencil and paper, working out the effect of a single rotation was much more time consuming, and thus the students had an incentive to think carefully, and this greater ‘mindfulness’ led to more learning.

The effects of feedback highlighted above might suggest that the more feedback, the better, but this is not necessarily the case. Day and Cordon (1993) looked at the learning of a group of sixty-four Year 4 students on reasoning tasks. Half of the students were given a ‘scaffolded’ response when they got stuck — in other words, they were given only as much help as they needed to make progress, while the other half were given a complete solution as soon as they got stuck, and then given a new problem to work on. Those given the ‘scaffolded’ response learnt more, and retained their learning longer than those given full solutions.

In a sense, this is hardly surprising, since those given the complete solutions had the opportunity for learning taken away from them. As well as saving time, therefore, developing skills of ‘minimal intervention’ promote better learning.

Sometimes, the help need not even be related to the subject matter. Often, when a student is given a new task, the student asks for help immediately. When the teacher asks, ‘What can’t you do?’ it is common to hear the reply, ‘I can’t do any of it’. In such circumstances, the student’s reaction may be caused by anxiety about the unfamiliar nature of the task, and it is frequently possible to support the student by saying something like, ‘Copy out that table, and I’ll be back in five minutes to help you fill it in’. This is often all the support the student needs. Copying out the table forces the student to look in detail at how the table is laid out, and this ‘busy work’ can provide time for students to make sense of the task themselves.

The consistency of these messages from research on the effects of feedback extends well beyond school and other educational settings. A review of 131 well-designed studies in educational and workplace settings found that, on average, feedback did improve performance, but this average effect disguised substantial differences between studies. Perhaps most surprisingly, in 40% of the studies, giving feedback had a negative impact on performance. In other words, in two out of every five carefully-controlled scientific studies, giving people feedback on their performance made their performance worse than if they were given no feedback on their performance at all! On further investigation, the researchers found that feedback makes performance worse when it is focussed on the self-esteem or self-image (as is the case with grades and praise). The use of praise can increase motivation, but then it becomes necessary to use praise all the time to maintain the motivation. In this situation, it is very difficult to maintain praise as genuine and sincere. In contrast, the use of feedback improves performance when it is focussed on what needs to be done to improve, and particularly when it gives specific details about how to improve.

This suggests that feedback is not the same as formative assessment. Feedback is a necessary first step, but feedback is formative only if the information fed back to the learner is used by the learner in improving performance. If the information fed back to the learner is intended to be helpful, but cannot be used by the learner in improving her own performance it is not formative. It is rather like telling an unsuccessful comedian to ‘be funnier’.

As noted above, the quality of feedback is a powerful influence on the way that learners attribute their successes and failures. A series of research studies, carried out by Carol Dweck over twenty years (see Dweck, 2000 for a summary), has shown that different students differ in the whether they regard their success and failures as:

- being due to ‘internal’ factors (such as one’s own performance) or ‘external’ factors (such as getting a lenient or a severe marker);
- being due to ‘stable’ factors (such as one’s ability) or ‘unstable’ factors (such as effort or luck); and
- applying globally to everything one undertakes, or related only to the specific activity on which one succeeded or failed.

Table 2 gives some examples of attributions of success and failure.

Table 2. Dimensions of attributions of success and failure.

Attribution	Success	Failure
locus	<i>internal</i> : ‘I got a good mark because it was a good piece of work’ <i>external</i> : ‘I got a good mark because the teacher likes me’	<i>internal</i> : ‘I got a low mark because it wasn’t a very good piece of work’ <i>external</i> : ‘I got a low mark because the teacher doesn’t like me’
stability	<i>stable</i> : ‘I got a good exam-mark because I’m good at that subject’ <i>unstable</i> : ‘I got a good exam-mark because I was lucky in the questions that came up’	<i>stable</i> : ‘I got a bad exam-mark because I’m no good at that subject’ <i>unstable</i> : ‘I got a bad exam-mark because I hadn’t done any revision’
specificity	<i>specific</i> : ‘I’m good at that but that’s the only thing I’m good at’ <i>global</i> : ‘I’m good at that means I’ll be good at everything’	<i>specific</i> : ‘I’m no good at that but I’m good at everything else’ <i>global</i> : ‘I’m useless at everything’

Dweck and others have found that boys are more likely to attribute their successes to stable causes (such as ability), and their failures to unstable causes (such as lack of effort and bad luck). This would certainly explain the high degree of confidence with which boys approach tests or examinations for which they are completely unprepared. More controversially, the same research suggests that girls attribute their successes to unstable causes (such as effort) and their failures to stable causes (such as lack of ability), leading to what has been termed ‘learned helplessness’.

More recent work in this area suggests that what matters more, in terms of motivation, is whether students see ability as fixed or incremental. Students who believe that ability is fixed will see any piece of work that they are given as a chance either to re-affirm their ability, or to be ‘shown-up’. If they are confident in their ability to achieve what is asked of them, then they will attempt the task. However, if their confidence in their ability to carry out their task is low, then, unless the task is so hard that no-one is expected to succeed, they will avoid the challenge, and this can be seen in mathematics classrooms all over the world every day. Taking all things into account, a large number of students decide that they would rather be thought lazy than stupid, and refuse to engage with the task, and this is a direct consequence of the belief that ability is fixed. In contrast, those who see ability as incremental see all challenges as chances to learn — to become more clever — and will therefore try harder in the face of failure. What is perhaps most important here is that these views of ability are generally not global: the same students often believe that ability in schoolwork is fixed, while at the same time believe that ability in athletics is incremental, in that the more one trains, the more one’s ability increases. What we therefore need to do is to ensure that the feedback we give students supports a view of ability as incremental rather than fixed.

Perhaps surprisingly for educational research, the research on feedback paints a remarkably coherent picture. Feedback to learners should focus on what they need to do to improve, rather than on how well they have done, and should avoid comparison with others. Students who are used to having every piece of work scored or graded will resist this, wanting to know whether a particular piece of work is good or not, and in some cases, depending on the situation, the teacher may need to go along with this. In the long term, however, we should aim to reduce the amount of ego-involving feedback we give to learners (and with new entrants to the school, not begin the process at all), and focus on the student’s learning needs. Furthermore, feedback should not just tell students to work harder or be ‘more systematic’, the feedback should contain a recipe for future action, otherwise it is not formative. Finally, feedback should be designed so as to lead all students to believe that ability — even in mathematics — is incremental; in other words, the more we ‘train’ at mathematics, the more clever we become.

Although there is a clear set of priorities for the development of feedback, there is no ‘one right way’ to do this. The feedback routines in each class will need to be thoroughly integrated into the daily work of the class, and so it will look slightly different in every classroom. This means that no-one can tell teachers how this should be done: it will be a matter for each teacher to work out a way of incorporating some of these ideas into her or his own practice.

Sharing criteria with learners

Frederiksen and White (1997) undertook a study of three teachers, each of whom taught four parallel Year 8 classes in two US schools. The average size of the classes was 31. In order to assess the representativeness of the sample, all the students in the study were

given a basic skills test, and their scores were close to the national average. All twelve classes followed a novel curriculum (called *ThinkerTools*) for a term. The curriculum had been designed to promote thinking in the science classroom through a focus on a series of seven scientific investigations (approximately two weeks each). Each investigation incorporated a series of evaluation activities. In two of each teacher’s four classes these evaluation episodes took the form of a discussion about what they liked and disliked about the topic. For the other two classes they engaged in a process of ‘reflective assessment’. Through a series of small-group and individual activities, the students were introduced to the nine assessment criteria (each of which was assessed on a 5-point scale) that the teacher would use in evaluating their work. At the end of each episode within an investigation, the students were asked to assess their performance against two of the criteria, and at the end of the investigation, students had to assess their performance against all nine. Whenever they assessed themselves, they had to write a brief statement showing which aspects of their work formed the basis for their rating. At the end of each investigation, students presented their work to the class, and the students used the criteria to give each other feedback.

As well as the students’ self-evaluations, the teachers also assessed each investigation, scoring both the quality of the presentation and the quality of the written report, each being scored on a 1 to 5 scale. The possible score on each of the seven investigations therefore ranged from 2 to 10.

The mean project scores achieved by the students in the two groups over the seven investigations are summarised in Table 3, classified according to their score on the basic skills test.

Table 3. Mean project scores for students.

Group	Score on basic skills test		
	Low	Intermediate	High
Likes and dislikes	4.6	5.9	6.6
Reflective assessment	6.7	7.2	7.4

Note: the 95% confidence interval for each of these means is approximately 0.5 either side of the mean

Two features are immediately apparent in these data. The first is that the mean scores are higher for the students doing ‘reflective assessment’, when compared with the control group; in other words, all students improved their scores when they thought about what it was that was to count as good work. However, much more significantly, the difference between the ‘likes and dislikes’ group and the ‘assessment’ group was much greater for students with weak basic skills. This suggests that, at least in part, low achievement in schools is exacerbated by students’ not understanding what it is they are meant to be doing — an interpretation borne out by the work of Eddie Gray and David Tall (1994), who have shown that ‘low-attainers’ often struggle because what they are trying to do is actually much harder than what the ‘high-attainers’ are doing. This study, and others like it, shows how important it is to ensure that students understand the criteria against which their work will be assessed. Otherwise we are in danger of producing students who do not understand what is important and what is not. As the old joke about project work has it: ‘four weeks on the cover and two on the contents’.

Now although it is clear that students need to understand the standards against which their work will be assessed, the study by Frederiksen and White shows that the criteria themselves are only the starting point. At the beginning, the words do not have the meaning for the student that they have for the teacher. Just giving ‘quality criteria’ or

‘success criteria’ to students will not work, unless students have a chance to see what this might mean in the context of their own work.

Since we understand the meanings of the criteria that we work with, it is tempting to think of them as definitions of quality, but in truth, they are more like labels we use to talk about ideas in our heads. For example, ‘being systematic’ in an investigation is not something we can define explicitly, but we can help students develop what Guy Claxton calls a ‘nose for quality’.

One of the easiest ways of doing this is to do what Frederiksen and White did: marking schemes are shared with students, but they are given time to think through, in discussion with others, what this might mean in practice, applied to their own work. We should not assume that the students will understand these right away, but the criteria will provide a focus for negotiating with students about what counts as quality in the mathematics classroom.

Another way of helping students understand the criteria for success is, before asking the students to embark on (say) an investigation, to get them to look at the work of other students (suitably anonymised) on similar (although not, of course the same) investigations. In small groups, they can then be asked to decide which of pieces of students’ work are good investigations, and why. It is not necessary, or even desirable, for the students to come to firm conclusions and a definition of quality; what is crucial is that they have an opportunity to explore notions of ‘quality’ for themselves. Spending time looking at other students’ work, rather than producing their own work, may seem like ‘time off-task’, but the evidence is that it is a considerable benefit, particularly for ‘low-attainers’.

Student peer- and self-assessment

Whether students can really assess their own performance objectively is a matter of heated debate, but very often the debate takes place at cross-purposes. Opponents of self-assessment say that students cannot possibly assess their own performance objectively, but this is an argument about summative self-assessment; no-one is seriously suggesting that students ought to be able to write their own school-leaving certificates. What really matters is whether self-assessment can enhance learning, and in this regard, accuracy is a secondary concern.

The power of student self-assessment is shown very clearly in an experiment by Fontana and Fernandez (1994). A group of twenty-five Portuguese primary school teachers met for two hours each week over a twenty-week period during which they were trained in the use of a structured approach to student self-assessment. The approach to self-assessment involved an exploratory component and a prescriptive component. In the exploratory component, each day, at a set time, students organised and carried out individual plans of work, choosing tasks from a range offered to them by the teacher, and had to evaluate their performance against their plans once each week. The progression within the exploratory component had two strands: over the twenty weeks, the tasks and areas in which the students worked were to take on the student’s own ideas more and more, and secondly, the criteria that the students used to assess themselves were to become more objective and precise.

The prescriptive component took the form of a series of activities, organised hierarchically, with the choice of activity made by the teacher on the basis of diagnostic assessments of the students. During the first two weeks, children chose from a set of carefully structured tasks, and were then asked to assess themselves. For the next four weeks, students constructed their own mathematical problems following the patterns of those used in

weeks 1 and 2, and evaluated them as before, but were required to identify any problems they had, and whether they had sought appropriate help from the teacher.

Over the next four weeks, students were given further sets of learning objectives by the teacher, and again had to devise problems, but now, they were not given examples by the teacher. Finally, in the last ten weeks, students were allowed to set their own learning objectives, to construct relevant mathematical problems, to select appropriate apparatus, and to identify suitable self-assessments.

Another twenty teachers, matched in terms of age, qualifications, experience, using the same curriculum scheme, for the same amount of time, and doing the same amount of inservice training, acted as a control group. The 354 students being taught by the twenty-five teachers using self-assessment, and the 313 students being taught by the twenty teachers acting as a control group were each given the same mathematics test at the beginning of the project, and again at the end of the project. Over the course of the experiment, the marks of the students taught by the control-group teachers improved by 7.8 marks. The marks of the students taught by the teachers developing self-assessment improved by 15 marks — almost twice as big an improvement.

Now the details of the particular approach to self-assessment are not given in the paper, and are in any case not that important — Portuguese primary schools are, after all, very different from those in other countries. However this is just one of a huge range of studies, in different countries, and looking at students of different ages, that have found a similar pattern. Involving students in assessing their own learning improves that learning.

The regulation of learning

Although at first sight quite different, the four elements of effective formative assessment outlined above form a coherent set of strategies for raising achievement. The coherence of these ideas can be seen more clearly by considering three crucial processes in learning:

- where the learners are in their learning;
- where they are going;
- how to get there;

and the role of the learner, her or his peers, and the teacher in these processes. The result of crossing these two dimensions is shown in Table 4.

Table 4. Aspects of formative assessment.

	Where the learner is	Where they are going	How to get there
Teacher	Evoking information	Establishing goals	Feedback
Peer	Peer-assessment	Sharing success criteria	Peer-tutoring
Student	Self-assessment	Sharing success criteria	Self-directed learning

Rich questioning and effective feedback focus on the teacher’s role: first being clear about where we want students to get to (curricular goals), asking appropriate questions to find out where they are, and feeding back to students in ways that the students can use in improving their own performance. Sharing criteria with learners and student self-assessment focus on the learner’s role: first being clear about where they want to get to, and then monitoring their own progress towards that goal.

The elements in Table 4 can be integrated within a more general theoretical framework of the regulation of learning processes as suggested Perrenoud (1991; 1998). Within such a framework, the actions of the teacher, the learners, and the context of the classroom are all evaluated with respect to the extent to which they contribute to guiding the

learning towards the intended goal.

From this perspective, the task of the teacher is not necessarily to teach, but to create situations in which students learn. This focus emphasises what it is that students learn, rather than what teachers do. Most teachers appear to be quite skilled at regulating or controlling the activities in which students engage, but have only a hazy idea of the learning that results. This is especially evident in interviews before lessons where teachers focus much more on the planned activities than on the resulting learning (e.g., ‘I’m going to have them do X’). In a way, this is inevitable, since only the activities can be manipulated directly. Nevertheless, it is clear that in teachers who have developed their formative assessment practices, there is a strong shift in emphasis away from regulating the activities in which students engage, and towards the learning that results (Black et al., 2003). Indeed, from such a perspective, even to describe the task of the teacher as teaching is misleading, since it is rather to ‘engineer’ situations in which student learn.

However, in this context, it is important to note that the ‘engineering of learning environments’ does not guarantee that the learning is proceeds in fruitful ways. Many visual arts classroom are productive, in that they do lead to significant learning on the part of students, but what any given student might learn is impossible to predict. An emphasis on the regulation of learning processes entails ensuring that the learning that is taking place is as intended.

When the learning environment is well-regulated, much of the regulation is pro-active, through the setting up of didactical situations. The regulation can be unmediated within such didactical situations, when, for example, a teacher ‘does not intervene in person, but puts in place a “metacognitive culture”, mutual forms of teaching and the organisation of regulation of learning processes run by technologies or incorporated into classroom organisation and management’ (Perrenoud, 1998, p. 100). For example, a teacher’s decision to use realistic contexts in the mathematics classroom can provide a source of proactive regulation, because then students can determine the reasonableness of their answers. If students calculate that the average cost per slice of pizza (say) is \$200, provided they are genuinely engaged in the activity, they will know that this solution is unreasonable, and so the use of realistic settings provides a ‘self-checking’ mechanism.

On the other hand, the didactical situation may be set up so that the regulation is achieved through the mediation of the teacher, when the teacher, in planning the lesson, creates questions, prompts or activities that evoke responses from the students that the teacher can use to determine the progress of the learning, and if necessary, to make adjustments. Examples of such questions are, ‘Is calculus exact or approximate?’ or ‘Would your mass be the same on the moon?’. (In this context it is worth noting that each of these questions is ‘closed’ in that there is only one correct response; their value is that although they are closed, each question is focused on a specific misconception.)

The ‘upstream’ planning therefore creates, ‘downstream’, the possibility that the learning activities may change course in the light of the students’ responses. These ‘moments of contingency’ — points in the instructional sequence when the instruction can proceed in different directions according to the responses of the student — are at the heart of the regulation of learning.

These moments arise continuously in whole-class teaching, where teachers are constantly having to make sense of students’ responses, interpreting them in terms of learning needs, and making appropriate responses. However, they also arise when the teacher circulates around the classroom, looking at individual students’ work, observing the extent to which the students are ‘on track’. In most teaching of mathematics, the regulation of learning will be relatively tight, so that the teacher will attempt to ‘bring into line’ all learners who are not heading towards the particular goal sought by the teacher;

in these subjects, the goal of learning is generally both highly specific and common to all the students in a class. In contrast, when the class is doing an investigation, the regulation will be much looser. Rather than a single goal, there is likely to be a broad horizon of appropriate goals, all of which are acceptable, and the teacher will intervene to bring the learners ‘into line’ only when the trajectory of the learner is radically different from that intended by the teacher. In this context, it is worth noting that there are significant cultural differences in how to use this information. In the United States or the United Kingdom, the teacher will typically intervene with individual students where they appear not to be ‘on track’ whereas in Japan, the teacher is far more likely to observe all the students carefully, while walking round the class, and then will select some major issues for discussion with the whole class.

One of the features that makes a lesson ‘formative’, then, is that the lesson can change course in the light of evidence about the progress of learning. This is in stark contrast to the ‘traditional’ pattern of classroom interaction, exemplified by the following extract:

‘Yesterday we talked about triangles, and we had a special name for triangles with three sides the same. Anyone remember what it was? ... Begins with E... Equi...’

In terms of formative assessment, there are two salient points about such an exchange. First, little is contingent on the responses of the students, except how long it takes to get on to the next part of the teacher’s ‘script’, so there is little scope for ‘downstream’ regulation. The teacher is interested only in getting to the word ‘equilateral’ in order that she can move on, and so all incorrect answers are treated as equivalent. The only information that the teacher extracts from the students’ responses is whether they can recall the word ‘equilateral’ or not.

The second point is that the situation that the teacher set up in the first place — the question she chose to ask — has little potential for providing the teacher with useful information about the students’ thinking, except, possibly, whether the students can recall the word ‘equilateral’. This is typical in situations where the questions that the teacher uses in whole-class interaction have not been prepared in advance (in other words, when there is little or no pro-active or ‘upstream’ regulation).

Similar considerations apply when the teacher collects in the students’ notebooks and attempts to give helpful feedback to the students in the form of comments on how to improve rather than grades or percentage scores. If sufficient attention has not been given ‘upstream’ to the design of the tasks given to the students, then the teacher may find that she has nothing useful to say to the students. Ideally, from examining the students’ responses to the task, the teacher would be able to judge how to (a) help the learners learn better and (b) what she might do to improve the teaching of this topic. In this way, the assessment could be formative for the students, through the feedback she provides, and formative for the teacher herself, in that appropriate analysis of the students responses might suggest how the lesson could be improved.

Summary

In this paper, I have outlined some of the research that suggests that focusing on the use of day-to-day formative assessment is one of the most powerful ways of improving learning in the mathematics classroom. In other words, even if teachers do not care about deep understanding, and instead wish only to increase their students’ test scores, then attention to formative assessment appears to be one of, if not the, most powerful way to do this.

To be effective, these strategies must be embedded into the day-to-day life of the classroom, and must be integrated into whatever curriculum scheme is being used. That is why

there can be no recipe that will work for everyone. Each teacher will have to find a way of incorporating these ideas into their own practice, and effective formative assessment will look very different in different classrooms. It will, however, have some distinguishing features. Students will be thinking more often than they are trying to remember something, they will believe that by working hard, they get cleverer, they will understand what they are working towards, and will know how they are progressing.

In some ways, this is an old-fashioned message; indeed, none of the strategies that teachers have used to put these principles into practice in their classrooms is new. What is new is that we now have hard empirical evidence that quality learning does lead to higher achievement, even when performance is measured through externally-mandated tests. What is also new is the broad theoretical framework of the regulation of learning, which may help teachers to understand how these ideas can be implemented effectively, so that teachers and students can, together, keep the learning of mathematics ‘on track’.

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Oversights and insights: Mathematics teaching and learning

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To suggest that children often have different interpretations of mathematical experiences than those intended by their teachers almost goes without saying — almost, but not quite. During the extensive research and development phase of the West Australian First Steps in Mathematics (K–7) program, we continued to be surprised by children’s responses to tasks and, more to the point, to be surprised at how often the problem lay, not with them, but with us. In this paper, I draw on the insights produced through this program to hypothesise about some oversights in our pedagogy — the things we do not notice and we forget to say — and the implications for teaching and learning mathematics.

Almost a decade ago, colleagues and I began work on a research and development project funded by the Education Department of Western Australia (EDWA) intended to ‘improve the mathematics outcomes of primary school students, particularly students at risk of not achieving their educational potential, by improving primary teachers’ understanding of teaching and learning in mathematics within a developmental framework.’

After considerable consultation, it seemed to us that three things were needed:

- broad explanatory frameworks to assist teachers to interpret children’s responses to mathematical tasks;
- an articulation of key understandings that underpin children’s learning;
- questions/activities which teachers can use to get children to tell them what they (the children) are thinking and what they do and do not know.

We started work on measurement. We were provoked to begin there by the experience of collecting student work and visiting classrooms across Australia when developing the *National Mathematics Profiles*. That work suggested that teachers’ views about what students did and did not understand about measurement varied widely. There also appeared to be surprisingly little development of underlying ideas about measurement even when practical measurement activities occurred. We had teachers of young primary children insisting that their students could use units to measure, but teachers of older students insisting they could not. They insisted children understood how to use a balance beam to decide which had the greater mass (or at least which was heavier), whereas teachers of older children said they did not. Unless we were prepared to accept that the children’s understanding actually did deteriorate during the primary years, they could not all be correct!

Is counting units measuring?

Our observations of children in classrooms persuaded us that indeed children in the pre-school and early primary years were counting ‘units’ to match physical quantities; they counted the number of cupfuls of rice it took to fill the jug, the number of rods that fit across the table and some even counted the blocks needed to make the balance ‘balance’. They were, of course, a little inaccurate: they spilt some rice, had gaps between the rods and overlapped them somewhat, and they got distracted when balancing the beam. However, these inaccuracies may have simply been in their execution, their dexterity, their concentration. The question is: what was going on in their heads? We knew that they were counting; what we did not know was whether they were measuring.

We trawled the research literature for work on young children’s understanding of measurement ideas and, to our considerable surprise, came up with very little that addressed our questions. So we turned to the mathematics itself: if early primary teachers think children are measuring and later primary and secondary teachers think they are not, does the difference lie with what they (the teachers) mean by ‘measuring’? What does it mean to measure something? What is its purpose?

Although it may seem obvious, it took us a while to state to ourselves in simple — rather than technical or abstract — terms that the essential purpose of measurement is ‘to compare’ indirectly; a measurement is always a comparison of one thing with another. We assign numbers to an attribute of things and then we compare the numbers to decide which is bigger/greater without directly matching or comparing.

Only then were we able to develop the kinds of questions that helped us to work out what young children understood themselves to be doing when they ‘measured’. Only then were we in a position to find ways to ask children what they did and did not ‘know’.

We started with length. Children aged between five and eight years old were (individually) given rods and asked, ‘We want to know how wide (long) the table is. Can you use these rods to see how wide it is?’. If children were unable to begin, we prompted: ‘Can you use the rods to measure the table?’; ‘Can you use the rods to see how many fit across the table?’. The majority were able to do a reasonable job of this although their rods were not always, or even often, laid out with great precision. They were, nevertheless, generally quite confident of their ‘measurement’. We then told the children that we would like to move the table into the next room, but we were not sure if it would fit through the doorway. It was heavy, so: ‘Could we work out whether the table would fit through the doorway without having to try it first?’

Children in Years 1 and 2, and even in Year 3, did not use the information from the rods at all. Some were unwilling even to accept the agenda. They tried, if possible, to pull the table over to the doorway. ‘You can’t tell,’ said one, ‘the table’s too heavy.’ Many resisted all prompting, even when we resorted to pointing to the rods and suggesting their use. One older child who did try to use the rods placed them vertically up and around the door frame.

When we changed direction and offered a piece of string and asked whether they could use the string to decide if the table would go through the doorway, one child tied the string to the table and tried to pull it to the doorway. Another, simply threw the string across and over the table. It did occur to us that the problem may have been that the children had to measure the empty space between the vertical frames of the doorway. When we asked children to decide whether a cupboard would fit on a wall, however, the situation did not improve.

We ask children ‘to measure’ and they come to know what it is that we expect them to do in response to requests such as: ‘Use the rods to measure how wide the table is’. We

see children laying tiles along a line or curve, filling containers, balancing beams, covering shapes. What do they think they are doing? We also see them directly comparing things. Do they see any connection between directly comparing things and using a unit? When do they see the connection?

Based on a range of tasks involving length, mass and capacity, we concluded that young children can indeed 'count units' and call it 'measuring' but they may not realise the significance of the count as an indication of the size of the object and so not think to use unit information to answer such questions as, 'Which frog is heavier?' or 'Will the table slide through the door?'.

When later we worked on Number, we learned that many young children who are very good 'counters' of collections may nevertheless not understand the significance of the count as an indicator of quantity (Nunes and Bryant 1996). Thus children who can count when they are asked to find 'how many' or if the word 'count' is mentioned may not trust the count to help them decide if there are enough drinks for the children (Willis, 2002). They may hand out the drinks or put a name to each drink or guess. Is it any wonder that they do not see the significance of counting units for deciding whether a table will go through the doorway, which is a much more complex task? If they do not see this, in what sense are they 'measuring'?

Among the children we interviewed, we found that it was not until about Year 3 that the idea of what you are doing when you measure began to emerge. Young children may correctly respond to a request to, for example, 'count how many pens fit across the table' and may have learned to call this 'measuring' but for them the task is one of counting to see 'how many fit' in much the same way as we might ask how many people fit in the car or how many times did you turn before the music finished. Due to the fact that they see the task literally as counting, children may be casual in their use of instruments and not really understand why it matters if they, for example, spill part of their spoonfuls. It is not surprising then that, even prompted, they do not use the information they have collected to decide if the table will slide through the door.

At about seven or eight years old, children when prompted begin to use a unit to decide which of two things is bigger. Nevertheless, they are often still tricked by conflicting information, for example, believing that the size can change when a different unit is used; and even at eight or nine years old, the social meaning of fit dominates. They obey the exhortation to cover a region without gaps or overlaps but they do not really see the significance of 'filling'. This is hardly surprising. For many practical purposes, 'fitting' is much more relevant and is only partly related to overall size; consider, for example, how many people 'fit' in the elevator or around the table. In many situations, dimensions are more important than, say, area or volume: a pipe could have a volume of a cubic metre and not a single decimetre cube fit in it!

However, if children are to understand measurement, they have to see measuring as a process of using units as a replacement for direct comparison, they have to trust the unit count and see its significance as a substitute for direct comparison. And this is one of the things that we often forget 'to say' or to get children 'to say'. That is, the two ideas of comparing the size of things and of deciding 'how many fit' must become connected in children's minds so that they understand unit information as giving an indication of size and enabling two things to be compared without directly matching them.

Many children, possibly most children, do come to understand this and because many do, we assume that we have 'taught it'. Too many, however, take too long to reach this point so that activities which are ostensibly extending their understanding of measurement are actually just extending their practice of counting. For some, this connection is never really made; which brings me to a second thing we sometimes forget to say.

What is the problem with gaps and overlaps?

At around 8 or 9 years of age, many children have begun to understand why it matters to be careful in their use of units. They will repeat uniform units of length and capacity reasonably carefully although they may not understand what this has to do with, for example, lining up the zero on a ruler. This is not surprising, since it is not altogether easy. If you do think it is easy, let me challenge you with the following question.

When we start to teach children about area, we tell them to cover the surface ‘without gaps or overlaps’. Why? What happens if there are gaps? What happens if there are overlaps? One produces an overestimate and one produces an underestimate. Do we remember to draw this out or to point this out? Do we leave it unsaid and possibly — probably — unthought? Which is which?

Think about using a ruler. We urge children to match the zero point rather than the edge of the ruler to the starting point of the ‘length’ to be measured? Why? Is it the equivalent of a gap or an overlap? Does it give an overestimate or an underestimate? Do we overestimate or underestimate when we use a tape measure that has stretched over time? Is it analogous to a gap or an overlap?

When using spoons to measure the capacity of a container, is over heaping the spoons analogous a gap or an overlap? What if we spill some?

How often do we ‘teach’ length and capacity and mass and area as though they were unconnected to each other, each to be understood differently, each with their own ‘rules’ of procedure. How often do we forget to help students sense, see and say the connections between the careful use of units in each of these contexts or even to articulate that if we are to use a measurement to make a comparison then we must be able to rely on the count being the same each time? This is the reason why we squeeze as many units in as we possibly can, but no more.

When we directly measure things, we repeat units either directly or using a calibrated scale, to make as close a match as possible with the thing being measured. Of course, making a close match is easier for some attributes than others. The idea of repeating a unit to match a region or object may be conceptually and practically more difficult than repeating a unit to match a line segment or balance a brick. Why do we have calibrated scales to help us measure length and mass and capacity, but not area? This leads me to a third thing we sometimes forget to say.

What is a unit — and an instrument?

Most of us understand that a degree celsius is a unit and a thermometer is an instrument, a centimetre is a unit and a tape measure an instrument, a minute is a unit and a clock an instrument. We need to learn, first, to choose appropriate units and, secondly, to choose appropriate instruments and use them appropriately. The first is probably the more abstract and the second the more practical but both are important and both can be conceptually challenging. Nevertheless, the distinction between units and instruments as used in the context of measurement, while perhaps not always articulated, is clear — is it not?

The very fact that this question occurred to us at all, spoke to the unease of our team, and we resorted to my dictionary where, reassuringly, we found that an instrument is ‘a device’ (*The Concise Macquarie Dictionary*, 1986, p. 643) whereas a unit is a ‘magnitude’ or ‘a specified amount’ (p. 1418), consistent with the distinctions made above. So what was the origin of our unease?

Around the middle primary years, comparing directly and ‘how many units fit...?’ gen-

erally have come together in children's minds and they have an understanding of what it means 'to measure'. Unprompted, they will use a measurement to decide whether one thing is bigger or smaller than another, understand why it helps to use the same size unit repeatedly to measure a thing and why it is necessary to use the same unit for each quantity when comparisons are to be made. The conscientious among them will use uniform units consistently and carefully to measure quantities that are uni-dimensional, such as length, capacity and mass, as well as angle and time. They also use uniform units of area although they may struggle with what to do along the edges when covering regions.

However, children can use uni-dimensional units before they can use multi-dimensional units such as for area. This will be no surprise to anyone. When children use a unit to measure length, generally they can match and find that six units 'fit' but seven do not, and so can see and say that 'it is between six and seven units long'. It is a different matter for area: when children use a tile and find that six tiles 'fit' within a region but seven do not, what are they to conclude? Giving a more extreme example, the following region has an area of between six and seven square centimetres but no 'centimetre squares' fit in.



Many primary and secondary teachers know only too well that students, asked to find things with an area of more than a square metre, will expect to be able to fit a square with sides of a metre into the region.

What does this say about their understanding of units? Is it that they understand what a unit is, but have difficulty in applying this understanding in the practically more difficult case of area, or is it that they do not really understand units? Our tentative conclusion is the latter: that many students persist even into the secondary years with a practical notion of measuring as finding out 'how many fit' or 'match'. This will usually give the right answer for uni-dimensional measures. It is a practical, sensible everyday use of mathematics. It makes sense in context. Unfortunately, students who persist with only the practical notion of fit will struggle when they try to deal with area. They will be the students who misunderstand what a 'square metre' is.

In this same regard we might ask why capacity seems easy compared with volume, even though mathematically speaking they are (more or less) the same. Again our tentative hypothesis is a simple one. In practical everyday contexts, 'units' of capacity are fluid and so they naturally flow and spill to fill, and this is typically not the case with the volume. This is obviously only part of the story but it is an important part, in our opinion.

So what is the critical thing here? What have we forgotten to say? A unit is not an object, rather a unit is an amount. Indeed, not only do we forget to say this, we may actually and explicitly teach just the opposite. How often have we said to the students in our classrooms (and as teacher educators to the student teachers in our classrooms) that they should choose a unit that tiles without gaps or overlaps? Is it surprising that they think of the unit as an object that has shape, when units are really sizes?

Of course, units do not tile, instruments do! The rod that we lay along the table and the small square that we carefully fit within a border are each instruments. The length of the rod and the area of the square are the units — a subtle distinction, yes, but an important subtle distinction. This distinction is needed to see that when we fill a container with sand to find its capacity, the materials used to represent the unit flow and spill, but the unit does not change. It is needed in order to understand that the object used to represent our unit can be transformed in any way to improve the fit or match, so long as the unit quantity does not change. It is needed in order to understand why the rectangle above has an area of more than six square centimetres when no centimetre squares 'fit in'.

One of our tasks for the *First Steps* program was to identify key understandings that

underpin children's use of units and direct measurement. In early versions we included the following:

- units should relate well to the attribute to be measured and be easy to repeat in order to match the objects to be measured;
- to measure consistently we need to use our unit in a way that ensures a good match with the object to be measured.

Our final version includes the following three key understandings, which are more complex but we think more correct and closer to what needs to be understood:

- the instrument we choose to represent our unit should relate well to the attribute to be measured and be easy to repeat to match the thing to be measured;
- to measure consistently we need to use our instrument in a way that ensures a good match of the unit with the object to be measured;
- units are quantities and so we can use different representations of the same unit so long as we do not change the quantity (Willis et al., in press).

In short, and to return to the *Macquarie Dictionary*, a unit is a 'magnitude' or 'a specified amount' and an instrument (whether a calibrated scale or a floor tile) which is 'a device' for representing the unit in a suitable form to enable measurement.

Conclusion

When I talk about some of these insights into children's thinking about measurement, there tend to be three kinds of reaction: the first is, 'Well of course,' the second is disagreement, 'No that isn't right,' and the third is, 'Oh dear, now I feel dreadfully insecure about my teaching. I do not know enough.'

With respect to the first, I agree. There is an 'of course'-ness about each of the understandings referred to above, but I do wonder if it is precisely because these ideas seem so obvious that we may forget to articulate them and to ensure that our students have 'got the point'. With respect to the second, some suggest that the point (for example, that the reason we measure the table is so that we can answer questions like 'will it go through the door') is, in fact, obvious to children or is well articulated in our curricula and teaching. To this I can only say that our work has suggested that too many points are not at all obvious to children and that when we find the right questions to ask of them, their answers are often disconcerting and distressing to teachers and teacher educators alike. When this happens, we often get the third reaction. To this I have two responses: first, we never know enough, but it is never too late to learn — our students are forgiving, and we can try again, so long as we are open to finding out what it is they are thinking. Secondly, we almost certainly all do know all of the mathematics above implicitly; the difficulty is that we may have not have made it sufficiently explicit even to ourselves to systematically attend to it in our teaching. We need to get into the habit of reminding ourselves: I know what I am doing and what the point is of that task, but what do they think they are doing and what do they think is the point?

Acknowledgement

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Addendum

One of the aims of the First Step Program was to develop broad explanatory frameworks to assist teachers to interpret children's responses to mathematical tasks. As a result of analyses such as that above, based on both the research literature available and our own observations, we identified a number of broad phases through which students' thinking develops.

During the *emergent* phase, students initially attend to overall appearance of size, recognising one thing as perceptually bigger than another and using comparative language in a fairly undifferentiated and absolute way (big/small) rather than as describing comparative size (bigger/smaller). Over time, they note that their communities distinguish between different forms of 'bigness' (or size) and make relative judgments of size. As a result, they begin to understand and use the everyday language of attributes and comparison used within their home and school environment, differentiating between attributes that are obviously perceptually different.

During the *matching and comparing* phase, students match in a conscious way in order to decide which is bigger by familiar readily-perceived and distinguished attributes such as length, mass, capacity and time. They also repeat copies of objects, amounts and actions to decide 'how many fit (balance or match) a provided object or event'. As a result, they learn to directly compare things to decide which is longer, fatter, heavier, holds more or took longer. They also learn what people expect them to do in response to questions such as, 'How long (tall, wide or heavy) is it?' or when explicitly asked to measure something.

During the *quantifying* phase, students connect the two ideas of directly comparing the size of things and of deciding 'how many fit' and so come to understand that the count of actual or imagined repetitions of units gives an indication of size and enables two things to be compared without directly matching them. As a result, they trust information about repetitions of units as an indicator of size and are prepared use this in making comparisons of objects.

During the *measuring* phase, students come to understand the unit as an amount (rather than an object or a mark on a scale) and to see the process of matching a unit with an object as equivalent to subdividing the object into bits of the same size as the unit and counting the bits. As a result, they see that part units can be combined to form whole units and they understand and trust the measurement as a property or description of the object being measured which does not change as a result of the choice or placement of units.

During the *relating* phase, students come to trust measurement information, including things they cannot see or handle, and to understand measurement relationships, both those between attributes and those between units. As a result, they work with measurement information itself and can use measurements to compare things, including those they have not directly experienced and to indirectly measure things.

Seminar papers

Students' conceptual understanding and critical thinking? A case for concept maps and vee diagrams in mathematics problem solving*

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Assessing students' conceptual understanding and critical thinking is a real problem for student teachers. The paper presents applications of Novak-type concept maps and Gowin's vee diagrams as means of depicting students' understanding of the knowledge structure of mathematics topics and problems and promoting critical thinking whilst solving problems. Although completed concept maps and vee diagrams illustrate connections between concepts and procedures, their construction is dependent on the extent of reflective and critical thinking students invest in the process. The tools may be used to support students' reasoning while and thinking and working mathematically in primary and secondary mathematics classrooms.

Introduction

Student teachers' usual feedback after teaching practicum revolves around issues of what to teach, and how to teach it. While the former emphasises knowledge and skills to be developed, the latter is a pedagogical issue of teaching strategies and student activities. Of equal importance are assessment questions which typically follow such class discussions such as, 'How do you assess what you teach?' and, 'How would you know students are really getting it?' often asked with an air of perplexity especially when describing classroom experiences in which previous lessons were perceived to be 'reasonably well-taught'. Later they find, much to their disbelief, that some students are still not 'getting it'. Subsequent discussions eventually converge onto another equally relevant question about lesson objectives. For example, if one of the objectives is developing students' understanding of concepts, how do you assess this conceptual understanding? Certainly, there are well-established ways that experienced teachers have refined over the years (Ollerton, 2003; Zevenbergen, Dole & Wright, 2004). One example is through appropriately designed problems and investigative activities that challenge students' knowledge of the conceptual ideas (conceptual understanding) as well as test their ability to critically analyse and apply ideas in the context of problems (critical thinking).

Traditionally, to assess students' understanding of concepts (language), students express the meanings in their own words perhaps with some illustrative examples or initiate a class discussion around the concepts. Relevant terminology for topics are usually lists in syllabus documents such as those in the New South Wales Board of Studies syllabus

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(2002). In this paper, I present an argument for the potential use of two meta-cognitive tools, concept maps and vee diagrams, as viable means of assessing students' conceptual understanding, fluency with the language of mathematics and critical thinking in problem solving. While students' understanding of the topic's mathematical language may be illustrated on a concept map in which nodes are concept names with linking words describing interconnections, student's higher-order problem solving skills such as critical thinking may be assessed using vee diagrams. Supplementary to established methods (Ollerton, 2003; Zevenbergen, Dole & Wright, 2004), I propose that vee diagrams provide a systematic guide to scaffold students' reasoning and conjecturing as they contemplate ways of solving a problem. However, before presenting examples from research conducted with mathematics students and teachers, I will describe the two meta-cognitive tools.

Concept maps and vee diagrams

The literature refers to different types and uses of concept maps (Liyanage & Thomas, 2002; Williams, 1998; Ruiz-Primo & Shavelson, 1996), however this paper focusses on the Novak-type concept maps in which concepts are arranged in a hierarchical order of generality with respect to the main topic and including linking words (Novak, 1998; 2002; 2004; Novak & Gowin, 1984; Mintzes, Wandersee & Novak, 1998; 2000). Novak defines a proposition as a statement formed by a (node – linking words – node) triad or strings of triads. By joining selected nodes, and describing links, students demonstrate their knowledge and understanding of concepts embedded hierarchically within the network of interconnecting concepts. This constructive activity provides the teacher with an idea of the state and level of students' understanding of the mathematics involved.

A vee diagram, on the other hand, is a heuristic for analysing the knowledge structure of a problem (adapted from Gowin's epistemological vee (Novak & Gowin, 1984); Novak, 2002, 1998) in terms of its conceptual framework (left-hand side LHS) and methodological information (right-hand side RHS). Figure 1 shows the vee diagram structure with its telling questions to guide the reasoning and thinking process as students analyse a mathematics problem (Afamasaga-Fuata'i, 1998; 2004). The curved arrow indicates the constant interplay between the two sides as students reflect upon given information and critically analyse the knowledge structure of the problem and relevant mathematics whilst simultaneously searching for suitable mathematical principles that suggest methods of transforming given information to generate potential solutions. If, instead, students have already obtained a solution then the challenge is to think in the reverse direction, in identifying principles underpinning their methods. Specifically, students should be encouraged to provide conceptual justifications for their solution's main steps to overtly make connections between procedures and concepts. This is similar to asking students to explain 'why' a problem is solved a particular way as '[l]earning to ask why is discovering that there are reasons not just facts, that statements can be justified, not just asserted loudly and slowly in order to persuade through intimidation' (Mason, 2001, p. 8). For that reason, establishing classroom practices of students justifying solution steps in terms of mathematical principles and displaying the conceptual and methodological information side by side on a vee diagram overtly focus students' attention on the dynamic interplay between concepts and procedures. Mason (2002) further argues that, '[b]y supporting learners in developing and refining their powers to think mathematically it is possible to go some way to, if not guarantee, at least make more likely that learners will construe through doing, know through construing, and know to act (to do) through

knowing to use their developing powers to think mathematically (p. 6).’ If sustained over time, students can begin to raise their awareness of their own powers to reason, make connections, and think analytically and mathematically (Mason, 2002). Spending sufficient time on a problem to ensure that students are not only learning about methods of solving problems but are routinely providing justifications, posing and solving challenging problems, have been identified as significant features of Japanese mathematics classrooms (Hollingsworth, 2003; Stigler & Hiebert, 1999, as cited in Anderson, 2003).

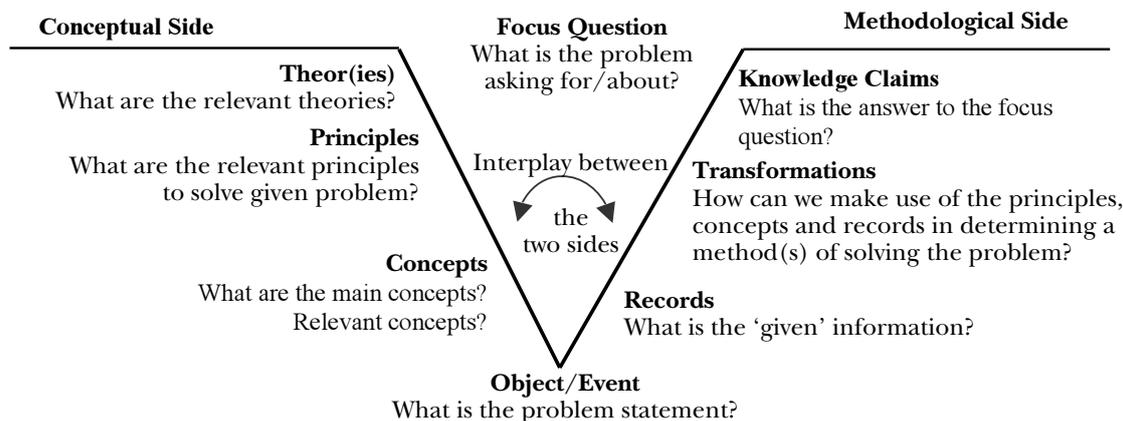


Figure 1. Problem solving vee diagram of mathematics problems (Afamasaga-Fuatai, 2004, 1998).

In the following sections, examples of concept maps and vee diagrams are presented to illustrate their potential as tools to assess, monitor, teach and develop students’ conceptual understanding, fluency with the language of mathematics and scaffold their critical thinking and reasoning in problem solving.

Concept map examples

Dora, a mathematics teacher who participated in the study, was asked to construct a concept map to illustrate functions. The recommended procedure is compiling an initial list of 8 to 10 relevant concepts, ranking concepts from most general to most specific and then arranging them in a meaningful hierarchy. After ranking and positioning her concepts hierarchically, interconnecting nodes and describing links, Dora produced her first concept map as redrawn in Figure 2. Choosing to place the main concept ‘Functions or Mappings’ at Level 1, the rest were strategically placed to facilitate valid interconnections distributed over five more levels. Reading vertically from top to bottom, it is possible to identify about 22 meaningful, complete propositions as listed in Table 1. The first column (Table 1) indicates level connections. For example, Proposition 7 is formed from relevant nodes at Levels 1, 3, 4, and 5 whilst Proposition 22 is a crosslink from Level 6 back to Level 1. By inspection, Table 1 indicates that the propositions are typical statements students make when articulating their understanding of functions.

Pedagogically, a concept map such as in Figure 2 can be used as a focus for a discussion or a means of implementing the *working mathematically* process strand particularly the processes of questioning, applying strategies, communicating, reasoning and reflecting (NSW, 2002). If propositions are mathematically incorrect or vague then it provides an opportunity for the teacher and student to negotiate for an acceptable re-statement of linking words and/or possible re-organisation of the hierarchy. This teacher-student

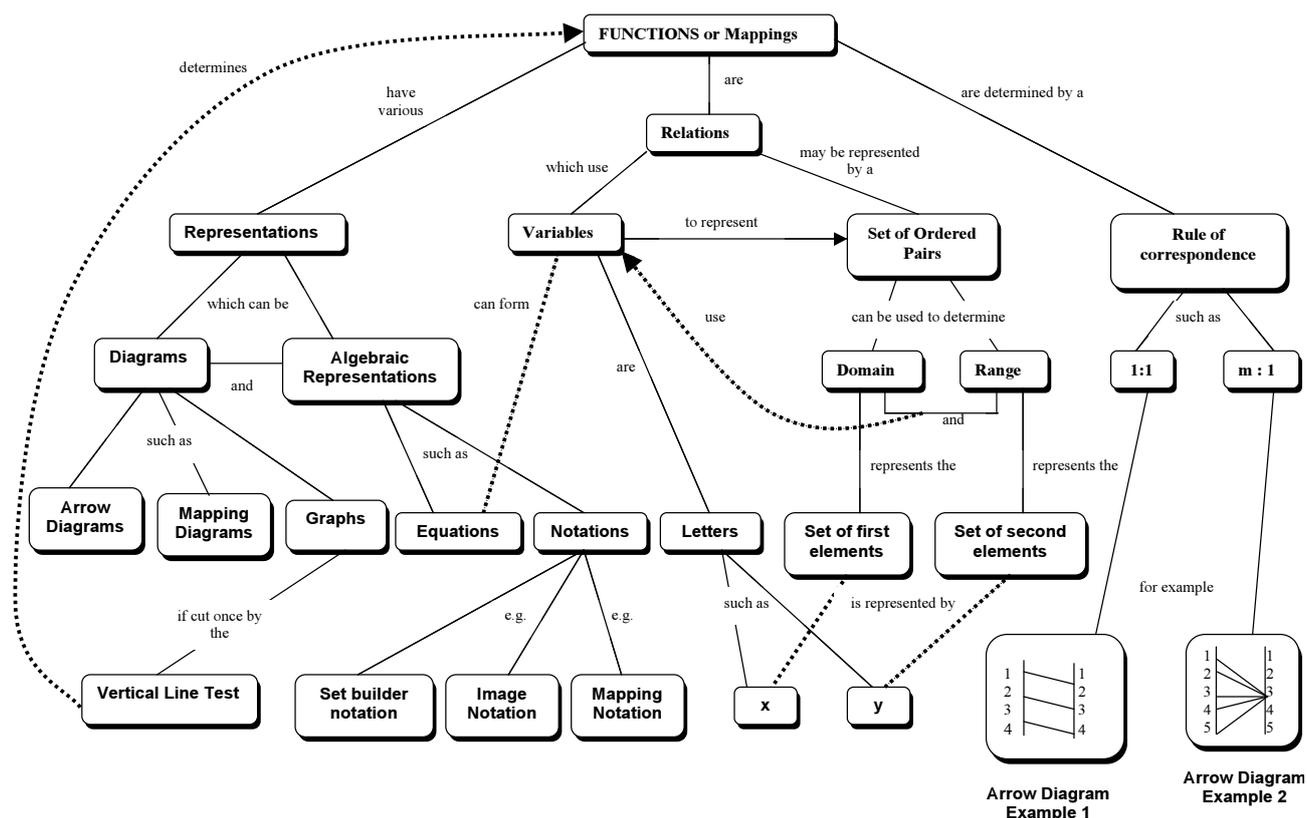


Figure 2. Dora's first concept map.

Table 1. Proposition list from the first concept map.

Level# → Level#	Propositions from Dora's concept map 1
1 → 2	1 Functions or mappings are relations.
1 → 3	2 Functions or mappings are determined by a rule of correspondence.
1 → 3	3 Functions or mappings have various representations.
3 → 4	4 Representations which can be diagrams and algebraic representations.
1 → 2 → 3	5 Functions or mappings are relations which use variables.
1 → 2 → 3 → 3	6 Functions or mappings are relations which use variables to represent set of ordered pairs.
1 → 3 → 4 → 5	7 Functions or mappings have various representations which can be diagrams such as arrow diagrams, mapping diagrams, graphs.
3 → 4 → 5 → 6 → 1	8 Representations which can be diagrams such as graphs if cut once by the vertical line test determines functions or mappings.
3 → 4 → 5	9 Representations which can be algebraic representations such as equations, notations.
3 → 4 → 5 → 6	10 Representations which can be algebraic representations such as notations; for example, set builder notation, image notation, mapping notation.
3 → 5	11 Variables can form equations.
3 → 5 → 6	12 Variables are letters such as x , y .
3 → 4 → 4	13 Set of ordered pairs can be used to determine domain and range.
4 → 5	14 Domain represents the set of first elements.
4 → 5	15 Range represents the set of second elements.
5 → 6	16 Set of first elements is represented by x .
5 → 6	17 Set of second elements is represented by y .
1 → 3 → 4	18 Functions or mappings are determined by a rule of correspondence such as 1:1, $m:1$.
1 → 3 → 4 → 6	19 Functions or mappings are determined by a rule of correspondence such as 1:1; for example, arrow diagram example 1.
1 → 3 → 4 → 6	20 Functions or mappings are determined by a rule of correspondence such as $m:1$; for example, arrow diagram example 2.
4 → 4 → 3	21 Domain and range use variables.
6 → 1	22 Vertical line test determines functions or mappings.

interaction can also take place between students themselves if working in pairs or collaboratively in small groups. Alternatively, the teacher can design more effective tasks that specifically redress the misconceptions. Thirdly, if propositions are all correct, the teacher can re-assign it as an enrichment task to be extended as new concepts are learnt over subsequent lessons. A variation would be for the teacher to delete links and linking words and ask students to construct individualised concept maps (or work in pairs or small groups). An exploratory option is to give students the opportunity to examine their own conceptual understandings in-depth at the completion of a topic, then construct individual concept maps using their own lists (or if preferred, one given by the teacher such as those in syllabus documents). Since student-constructed concept maps indicate their level of conceptual understanding and fluency with the topic's mathematical language, they can be presented in class to initiate mathematical dialogues, communications and discussions amongst the students as they learn collaboratively from each others' work and share ideas.

Whereas Figure 2 visually depicts Dora's perception of the integrated, hierarchical, network of interconnecting nodes, these extra dimensions are not easily discernible in the linear list of Table 1. I propose in this paper that the multi-dimensional aspects of the diagrammatic concept map 'offer a wider scope for multiplicity of interpretation' (Mason, 2001), and organization particularly as the task of concept mapping demands much cognitive processing of information and reflective critical thinking whilst arranging the same concepts in a meaningful hierarchy, linking and describing interconnections. In practice, constructing a hierarchical concept map explicitly pushes students to a higher level of thinking and reflection, which are desirable skills to cultivate and develop for effective problem solving and an essential part of working and thinking mathematically. In contrast, solving a problem by simply executing a procedure or applying a formulas such as finding the derivative using the power rule without fully comprehending the meanings of underlying concepts is to miss out on an aesthetic appreciation of calculus and indicative of a procedural, limited view of derivatives.

After presenting her first attempt in class (consisting of myself and her peers), Dora continued to revise her concept map for the second and third time by adding nodes, revising some labels and including illustrative examples. Her peers also took turns in presenting their concept maps for critique. Subsequent discussions and social interactions (student-student and teacher-student) focussed on critiquing whether or not displayed interconnections and linking words were mathematically sound and correct. As an extension, all of them were asked to expand their first maps to include more relevant nodes and illustrative examples. Dora's fourth attempt is redrawn in Figure 3 with a proposition list in Table 2. A comparison of Figures 2 and 3, and Tables 1 and 2 shows an increase in meaningful propositions (from 22 to 30) with the inclusion of more nodes (from 28 to 39) and links (from 34 to 51), an extra hierarchical level (from 6 to 7) and 5 more illustrative examples and 3 graphs. The final concept map had evidently expanded, becoming more complex with more integration and differentiation between concepts resulting in more meaningful propositions. Dora had also reversed the ranking of concepts 'Functions' and 'Relations' and included concept 'First elements appear only once.' Given her background as a teacher, her final concept map attempted to capture the typical 'functions' terminology at early secondary level.

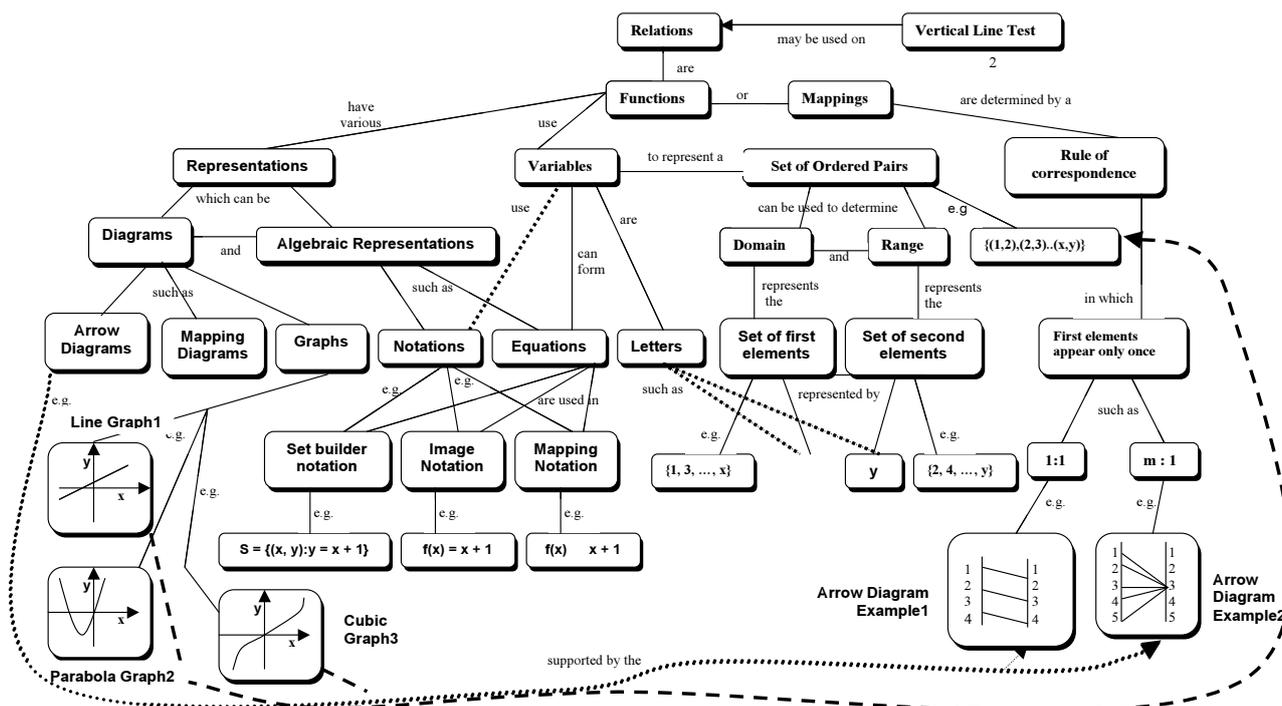


Figure 3. Dora's final concept map.

Table 2. Proposition list from the final concept map.

Level# → Level#	Propositions from Dora's concept map 1
1 → 2	1 Relations are functions.
1 → 1	2 Vertical line test may be used on relations.
2 → 2 → 3	3 Functions or mappings are determined by a rule of correspondence.
2 → 3 → 3	4 Functions use variables to represent a set of ordered pairs.
2 → 3	5 Functions have various representations.
3 → 4 → 4	6 Representations which can be diagrams and algebraic representations
2 → 3 → 4 → 5	7 Functions have various representations which can be diagrams such as arrow diagrams, mapping diagrams, graphs.
2 → 3 → 4 → 5 → 6	8 Functions have various representations which can be diagrams such as graphs; for example, linear graph1, parabola graph 2, cubic graph 3.
2 → 3 → 4 → 5 → 7	9 Functions have various representations which can be diagrams such as arrow diagrams; for example, arrow diagram example1 , arrow diagram example 2.
2 → 3 → 4 → 5	10 Functions have various representations which can be algebraic representations such as notations, equations.
5 → 6	11 Notations; for example, set builder notation, image notation, mapping notation.
5 → 6	12 Equations are used in set builder notation, image notation, mapping notation.
6 → 7	13 Set builder notation; for example, $S = \{(x, y): y = x + 1\}$.
6 → 7	14 Image notation, for example, $f(x) = x + 1$.
6 → 7	15 Mapping notation; for example, $f(x) \rightarrow x + 1$.
3 → 5 → 6	16 Variables are letters such as x, y .
3 → 5	17 Variables can form equations.
3 → 5	18 Variables use notations.
3 → 4	19 Set of ordered pairs can be used to determine domain and range.
3 → 4	20 Set of ordered pairs for example $\{(1,2), (2, 3), \dots (x, y)\}$.
4 → 5	21 Domain represents the set of first elements.
4 → 5	22 Range represents the set of second elements.
5 → 6	23 Set of first elements is represented by x .
5 → 6	24 Set of first elements for example $\{1, 3, \dots x\}$.
5 → 6	25 Set of second elements is represented by y .
5 → 6	26 Set of second elements for example $\{2, 4, \dots y\}$.
3 → 5 → 6	27 Rule of correspondence in which first elements appear only once such as 1:1, $m:1$.
5 → 6 → 7	28 First elements appear only once such as 1:1; for example, arrow diagram example 1.
5 → 6 → 7	29 First elements appear only once such as $m:1$; for example, arrow diagram example 2.
6 → 1	30 Linear graph 1, parabola graph 2, cubic graph 3 supported by the vertical line test.

Vee diagram examples

To introduce vee diagrams in problem solving to a Year 10 class, I used the following problem:

Find the equation(s) of the line(s) which pass through (3,-3) and forms with the coordinate axes a triangle of area 6 square units. Find equation(s) of the line(s) in general form.

Guided by the telling questions in Figure 1, I explained how the problem statement can be used to complete the sections: Object/Event, Focus Question, Records and Concepts by asking questions such as, 'What are the mathematical concepts used in stating the problem?', 'What are you asked to find?', 'What is the given information?' which is also consistent with Polya's first principle for problem solving namely 'understanding the problem'. The other three principles are: (2) devising a plan; (3) carrying out the plan; and (4) looking back (Polya, 1973). Polya's four principles provide an overview of the process of completing a vee diagram with more specific-section questions in Figure 1. Of

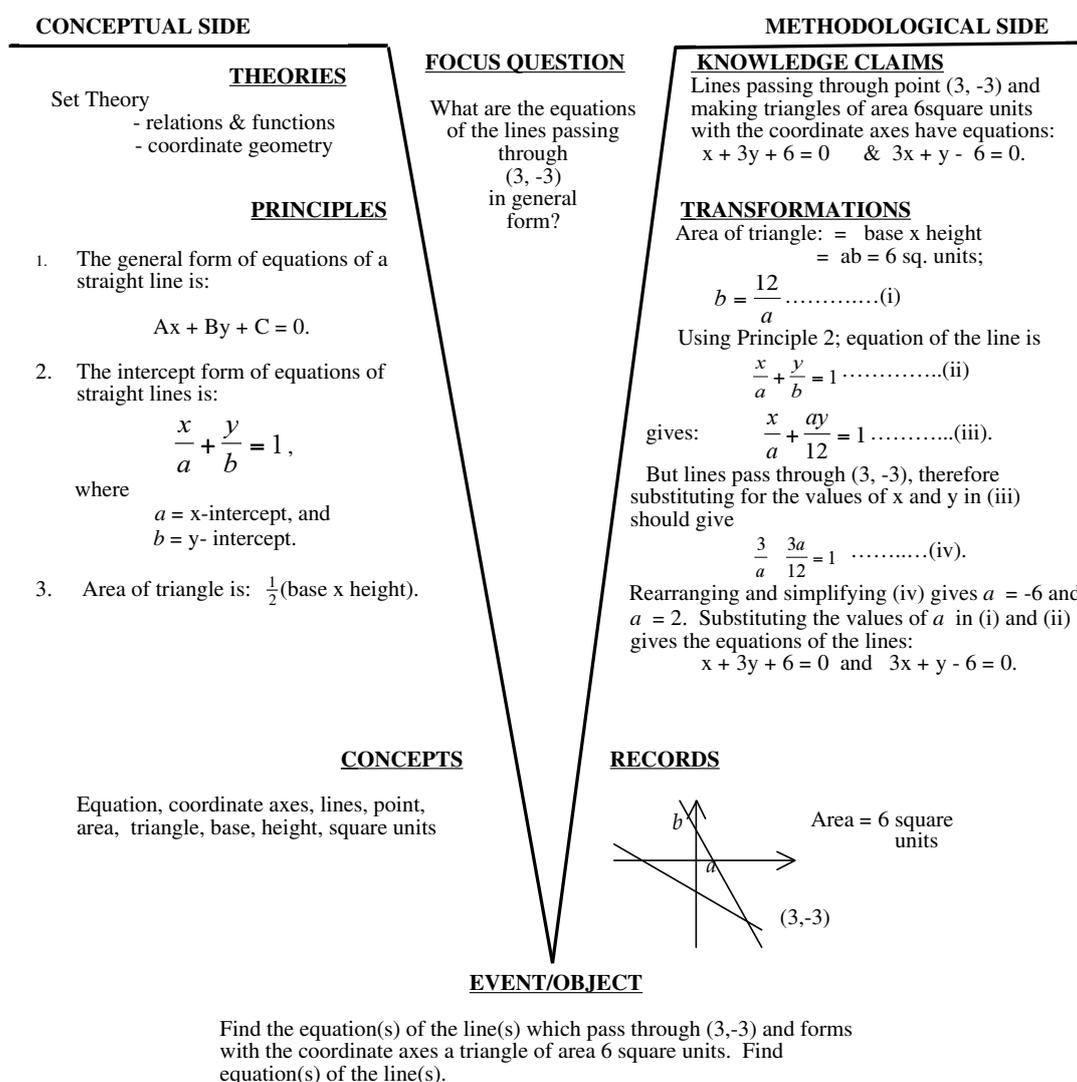


Figure 4. Vee map of problem (method 1).

fundamental importance is the enculturation of students to the ‘thinking part of the problem solving process (which) is typically suppressed’ in textbooks by ‘embedding conceptualization directly in the flow of the problem solving process’ (McAllister, 1994). Accordingly, the teacher should initiate a brainstorming session in which students are asked to make suggestions, conjectures and pose questions to ‘crack the code,’ interpret and analyse the problem statement for the intended meaning, relevant concepts and principles as they explore potential solutions whilst simultaneously slotting emerging information into the relevant sections of the vee diagram. By overtly drawing students’ attention to the different sections, connections between displayed conceptual and methodological information are reinforced and consolidated. Shown in Figures 4 and 5 are vee diagrams illustrating two methods of solution. Subsequent discussions can focus on clarifying, confirming and articulating connections between sections. Extension work may include explorations for more methods and relevant underlying principles with students using vee diagrams to record findings and in subsequent class presentations, effectively communicate their ideas. Alternatively, students may be asked to identify other relevant principles missing from the ‘Principles’ lists of Figures 4 and 5. Another option will be to ask students to pose their own problems and then construct vee diagrams to display relevant conceptual and methodological information.

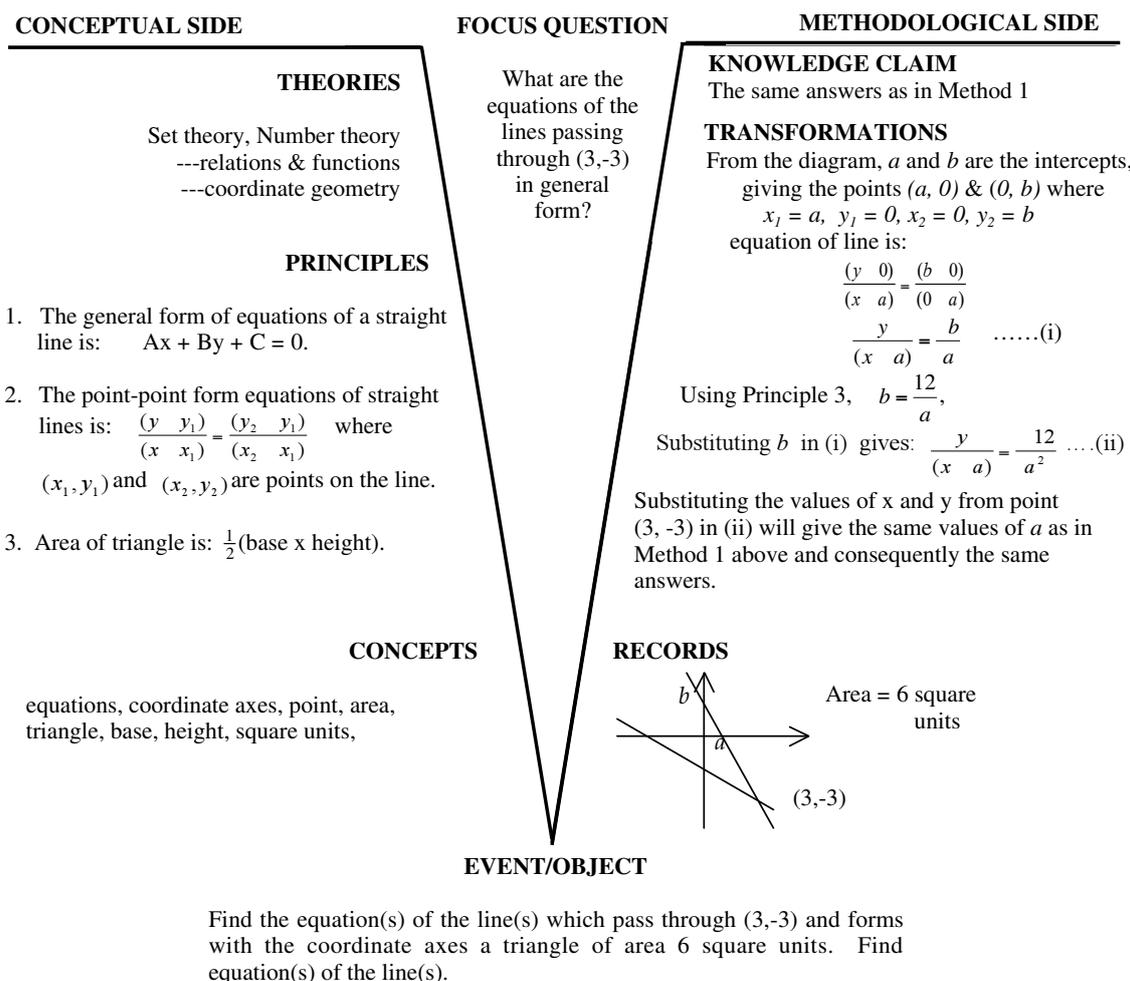


Figure 5. Vee map of problem (method 2)

Earlier on, I presented two examples of concept maps to illustrate the knowledge structure of some concepts on ‘functions.’ Another use of concept maps in association with problems is to display the conceptual framework on the LHS of vee diagrams to effectively communicate the connections between concepts, mathematical language, and procedures as shown in Figure 6. This problem concept map can be extended further to incorporate students’ findings from their own investigative and exploratory activities as suggested above. Resulting concept maps will reveal connections between concepts and procedures that cross multiple topics.

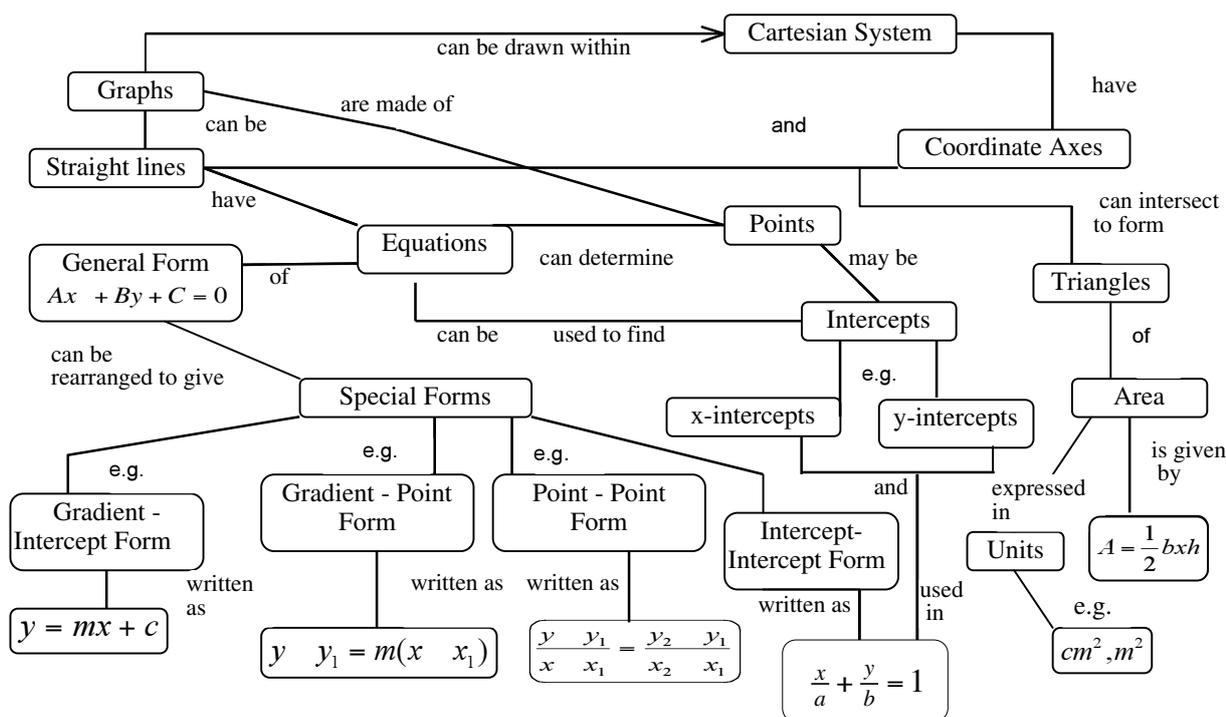


Figure 6: Concept Map of the Problem

Summary

With the current focus in mathematics education on the importance of developing students’ conceptual understanding, fluency with the language of mathematics, critical thinking, and working mathematically, teachers are constantly expected to design challenging and investigative tasks that can engage and motivate students in their learning of mathematics. An integral part of creating exemplary and conducive learning environments in mathematics classrooms is for teachers to be innovative and creative in the ways they teach and assess students. In this paper I have demonstrated how the Novak-type concept maps and Gowin’s vee diagrams can be used in mathematics classrooms as learning, teaching and assessment tools as they have been found to be quite effective in many international classrooms across many disciplinary areas as evident by the number of presentations (approximately 150 papers and posters) accepted for the First International Conference on Concept Mapping held on 14–17 September 2004 in Spain, see <http://www.cmc.ihmc.us>.

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Working mathematically in NSW classrooms: An opportunity to implement quality teaching and learning*

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In the new mathematics syllabuses in New South Wales [NSW], Working Mathematically is described as encompassing the five interrelated processes of Questioning, Applying Strategies, Communicating, Reasoning, and Reflecting. These processes support the development of mathematical concepts across all of the content strands. A document released by the Department of Education and Training, *Quality Teaching in NSW Public Schools*, describes elements of quality teaching that support quality learning. These documents provide similar advice to teachers in relation to selecting appropriate teaching and learning experiences that engage and challenge student thinking.

Introduction

Beginning in 2000, a review of existing mathematics syllabuses from Kindergarten to Year 10 was undertaken by the Board of Studies (BOS) in New South Wales. This review identified critical elements that required consideration for the development of the new syllabuses. One recommendation was to develop a continuum of learning that reflected the development of mathematical concepts across the compulsory years of schooling. Another recommendation was to embed problem solving and the processes of *working mathematically* into the content so that teachers could plan learning experiences that supported regular student engagement with these processes. In previous syllabuses, *problem solving* and Working Mathematically had been written as separate strands, with some educators suggesting that they became an added extra rather than as central to the purpose for learning mathematics (e.g., Pegg, 1997).

The writing team began work on the new syllabuses in 2001, with regular input from researchers, consultants and teachers. This process was also informed by a review of relevant literature (Owens & Perry, 2001), and a series of papers presented at a symposium (BOSNSW, 2001). A K–10 Scope and Continuum of Key Ideas was developed and incorporated into the two syllabuses that were released to schools at the end of 2002 (BOSNSW, 2002a; BOSNSW, 2002b). In each of these syllabuses, Working Mathematically is described on every page in each of the content strands of Number, Patterns and Algebra, Data, Measurement, and Space and Geometry.

In 2003, the Department of Education and Training (NSW DET) released a discussion paper about quality teaching (NSWDET, 2003a) and an annotated bibliography

* This paper has been accepted by peer review.

(NSWDET, 2003b). The proposed framework for quality teaching containing three dimensions and eighteen elements was informed by the research project based on Productive Pedagogies from Queensland (Queensland School Reform Longitudinal Study, 2001). Considerable funds were released to schools in 2003 and 2004 to support the implementation of quality teaching in NSW classrooms. Some schools developed projects that incorporated the introduction of the new mathematics syllabuses with the implementation of quality teaching. There are similarities, between the recommendations about working mathematically in the new syllabuses and the elements described in *Quality Teaching in NSW Public Schools*. This common advice reinforces the characteristics required to engage students in meaningful learning experiences that challenge student thinking. This paper describes some of the writing team deliberations for developing a working mathematically strand in the new syllabuses in NSW and considers the similarities in the advice between the new syllabuses and the quality teaching documents. Further advice for teachers is provided including promoting deep learning and the development of problem-solving processes.

Working mathematically in the NSW syllabuses

The writing team was confronted with the task of determining the relevant processes for the working mathematically strand. Closely aligned with this was the need to develop a rationale for mathematics in the curriculum. A section of the final rationale states:

Mathematics is a reasoning and creative activity employing abstraction and generalisation to identify, describe and apply patterns and relationships. It is a significant part of our cultural heritage of many diverse societies. The symbolic nature of mathematics provides a powerful, precise and concise means of communication. Mathematics incorporates the processes of questioning, reflecting, reasoning and proof. It is a powerful tool for solving familiar and unfamiliar problems both within and beyond mathematics (BOSNSW, 2002a, p. 7).

This statement suggests the need for students to be actively engaged in investigating mathematical ideas in order to develop deep learning and to be able to solve problems. So if developing problem-solving competence is one aim of the curriculum, students need to experience a variety of problem contexts that will allow them to develop a range of problem-solving strategies. The problem-solving process involves many processes including identifying information, describing what needs to be found, planning a way forward, applying strategies, refining and reviewing the approach, justifying the procedures and solutions, and communicating the solutions. From this, a set of verbs can be constructed that embraces such an active approach to learning (see Table 1). This list is not exhaustive and could be readily extended to include a vast array of experiences that are all important within the context of learning mathematics.

Table 1. Processes for active engagement in mathematics learning.

reasoning	creating	abstracting	generalising	identifying
describing	applying	questioning	reflecting	proving
planning	justifying	investigating	representing	observing
thinking	solving	engaging	appreciating	communicating

In writing the new syllabuses and developing the working mathematically strand, the writing team looked at the content and outcomes of other States and Territories in Australia as well as several curriculum documents from overseas. For example, in Western Australia, there is a working mathematically strand with four processes: *contextualise mathematics*, *mathematical strategies*, *reason mathematically*, and *apply and verify*. In Victoria, there is a *reasoning and strategies* strand with two processes: *mathematical reasoning* and *strategies for investigation*. In Queensland, there is no process strand. However, five outcomes in the introductory pages refer to processes and suggest that these should be achieved by the end of the compulsory years of schooling. In England, there is a *solving problems* strand with outcomes grouped into five sections for Grades 1–3 and six sections for Grades 4–6.

Documents that provide advice for curriculum development in Australia and the USA were also consulted. The *National Statement* (Australian Education Council (AEC), 1991) outlines the three process strands of *attitudes and appreciations*, *mathematical inquiry*, and *choosing and using mathematics*. The *National Profile* (AEC, 1994) includes a working mathematically strand with six processes: *investigating*, *conjecturing*, *using problem-solving strategies*, *applying and verifying*, *using mathematical language*, and *working in context*. In the United States of America, the *Principles and Standards* (National Council of Teachers of Mathematics (NCTM), 2000) describes five process strands for pre-kindergarten to Grade 12: *problem solving*, *reasoning and proof*, *communication*, *connections*, and *representations*.

There is considerable overlap between the processes used in these documents, which is not surprising given the extensive advice in the literature (e.g., Hiebert & Wearne, 2003). The final set of processes of *questioning*, *applying strategies*, *communicating*, *reasoning* and *reflecting* was determined on the basis of this advice as well as teachers' familiarity with some of these processes in the previous *Outcomes and Indicators* document (BOSNSW, 1998). While these processes can each be described separately, it is clear that they are interrelated and that many rich learning experiences will incorporate more than one process. If teachers provide such opportunities for students, not only will they be engaging them in working mathematically, but they will also be implementing some of the elements of quality teaching.

Quality teaching and working mathematically: Considering the similarities

The recently released document *Quality Teaching in NSW Public Schools* (NSWDET, 2003a) describes the dimensions and elements of quality teaching. Table 2 lists the three dimensions of the model with the six elements contained in each.

Table 2. Dimensions and elements from Quality Teaching in NSW Public Schools (NSWDET, 2003a).

Intellectual quality	Quality learning environment	Significance
Deep knowledge	Explicit quality criteria	Background knowledge
Deep understanding	Engagement	Cultural knowledge
Problematic knowledge	High expectations	Knowledge integration
Higher-order thinking	Social support	Inclusivity
Metalanguage	Students' self-regulation	Connectedness
Substantive communication	Student direction	Narrative

There is not enough space in this paper to describe each of these in detail. However, a careful examination of each of the eighteen elements suggests that the working mathematically processes incorporate many of these ideas if implemented as intended. To illustrate with one example, one of the *intellectual quality* elements, *substantive communication*, refers to the extent to which ‘students are regularly engaged in sustained conversations about the concepts and ideas they are encountering’ (NSWDET, 2003a, p. 11). The working mathematically processes of *questioning* and *communicating* each require student discussions in order for students to question mathematical ideas and interpretations, describe their understanding, challenge solution methods, and much more. Table 3 is an attempt to match each of the processes of working mathematically to the elements of quality teaching.

Table 3. Matching the processes of working mathematically with the elements of quality teaching

Working mathematically process	Description of the process (BOSNSW, 2002a, p. 19)	Elements of quality teaching (NSWDET, 2003)
Questioning	Students ask questions in relation to mathematical situations and their mathematical experiences	Problematic knowledge Substantive communication Engagement Student direction
Applying strategies	Students develop, select and use a range of strategies, including the selection and use of appropriate technology, to explore and solve problems	Deep understanding Higher-order thinking Social support
Communicating	Students develop and use appropriate language and representations to formulate and express mathematical ideas	Deep knowledge Metalanguage Substantive communication
Reasoning	Students develop and use processes for exploring relationships, checking solutions and giving reasons to support their conclusions	Deep understanding Higher-order thinking Explicit quality criteria High expectations
Reflecting	Students reflect on their experiences and critical understanding to make connections with, and generalisations about, existing knowledge and understanding	Deep understanding Higher-order thinking Background knowledge Knowledge integration Connectedness Narrative

This matching suggests that there is a particular emphasis on the dimension of *intellectual quality* in the working mathematically processes. However, elements from each of the other dimensions are also evident. White and Mitchelmore (2004) indicate that the particular elements of *background knowledge* and *connectedness* help to demonstrate to students the point of doing mathematics at school.

Since the syllabuses and the quality teaching documents provide advice that encourages teachers to plan learning experiences that engage students through problematic learning experiences, some schools have seized the opportunity to design projects that support the implementation of both. By reflecting on practice and working in collaborative teams, teachers have been able to design units of work that incorporate the working mathematically processes and other aspects of the new syllabuses, as well as elements of quality teaching (see descriptions of some of the projects at www.qtp.nsw.edu.au/crosspriority/ProjectResources.cfm?r=43).

Quality teaching and learning: Some advice for teachers

Implementing Working Mathematically in classrooms so that quality teaching and learning can occur requires considerable effort. Frequently this effort begins with reflection on your current practice and a desire to achieve better outcomes for students. It also requires meeting a series of challenges. These include providing more problem-solving opportunities for students even though they may actively resist this approach, finding the best tasks for the full range of student needs and interests, and providing the time for students to struggle with the tasks and underlying mathematical ideas.

In the NSW syllabuses, when students are engaging with working mathematically, they will ‘develop knowledge, skills and understanding through inquiry, application of problem-solving strategies including the selection and use of appropriate technology, communication, reasoning and reflection’ (BOSNSW, 2002b, p. 12). This suggests regular problem-solving experiences. Indeed, the processes attempt to describe what it is to do mathematics as a mathematician would.

When mathematicians become interested in a problem they:

- play with the problem to collect and organise data about it
- discuss and record notes and diagrams
- seek and see patterns or connections in the organised data
- make and test hypotheses based on the patterns or connections
- look in their strategy toolbox for problem solving strategies which could help
- look in their skill toolbox for mathematical skills which could help
- check their answer and think about what else they can learn from it
- publish their results.

(Williams, 2002, p. 304).

Engaging students in problem-solving experiences can be one of the challenges for teachers. While young children have a natural curiosity and more readily engage in such experiences (Mannigel, 1992), older students can be quite resistant to doing questions that require thinking about challenging ideas (e.g., Tobin & Imwold, 1993). These students frequently view mathematics as unrelated facts to be memorised, and procedures to be practised, with questions having only one correct answer. This usually reflects the experiences they have had in learning mathematics at school, particularly if the main activity of most lessons is to complete repetitive sets of drill and practice questions that require little thinking, reasoning or reflecting on knowledge and understanding. Unfortunately, some teachers hold similar views, or believe that implementing problem solving is too difficult because of a range of perceived constraints (Anderson, Sullivan & White, 2004). It is through the incorporation of working mathematically into syllabus documents, as well as professional development support that such views can be challenged.

Another challenge for teachers is to design tasks that include aspects of the syllabus content and at the same time, engage students in worthwhile mathematical activity, providing opportunities for students to question, apply strategies, communicate, reason and reflect. There are many sources of such tasks and so the issue can often be one of finding the time to look at a range of resources to find the right tasks for a particular group of students. To assist teachers, recommendations for working mathematically have been included on all of the content pages of the syllabuses. One example for Stage 2 *data* (BOSNSW, 2002a, p. 87) suggests that students learn to:

- pose suitable questions to be answered using a survey (questioning);
- interpret graphs found on the Internet or in the media (applying strategies, communicating);
- discuss the advantages and disadvantages of different representations of the same data (communicating, reflecting).

Another example from Stage 4 *patterns and algebra* (BOSNSW, 2002b, p. 96) suggests that students learn to

- compare and describe similarities and differences between sets of linear relationships (communicating, reasoning)
- recognise that not all number patterns form a linear relationship (applying strategies, reasoning).

One of the most difficult challenges for teachers is to provide time for students to struggle with problematic situations and not to step in too quickly (Hiebert & Wearne, 2003). This goes against what many of us believe is good teaching. We believe that we are there to support and assist students. Of course this is true, but if we do all of the thinking behind questions, when do the students learn to unpack questions and determine which strategies are the best to use?

The new syllabuses in NSW support quality teaching and learning in several ways since they acknowledge working mathematically as central to learning about, and doing mathematics, and they integrate working mathematically into the content. In addition, they recognise quality teaching and the practices of good teachers. Quality learning involves having a deep understanding of mathematical ideas and being able to use this flexibly and creatively. Quality teaching requires teachers to provide opportunities for students to be able to develop deep understanding, flexibility and creativity in using mathematical ideas. I suggest that we go a long way towards achieving this aim if we incorporate working mathematically into our teaching on a regular basis.

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The teacher's role in collaborative learning*

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After devising a suitable task and organising students into groups that will work harmoniously together, what then? How can teachers best promote effective interaction within groups? What can be done to avoid groups becoming frustrated and stuck, or bored and off-task? Drawing on observations made during recent research on collaborative learning, this paper discusses some of the strategies teachers can use while groups are at work: what to look for when observing from a distance; when and how to interact with a group; asking 'good' questions; and what to say if they ask if they are on the right track, or claim to have finished.

Introduction

It is last period in the afternoon, and Ms James' Year 10 class are working in groups on an investigative activity. During the final part of the lesson, students from different groups report on the conclusions they have reached, and the whole class becomes engaged in discussing a number of key points. One point generates disagreement, and the discussion becomes more heated, with many different students eager to have a say. The bell rings, and Ms James says, 'I think we might take that up tomorrow,' but some people want to continue the discussion then and there. While some students pack up their books and go home, a group gathers round the board. For more than five minutes they continue to talk, write, and draw graphs and diagrams as they attempt to explain and justify their ideas and convince their classmates. Focussed on the topic, they seem unaware of time passing.

During a recently-completed research project (Barnes, 2003) I was privileged to be present as an observer in the classrooms of experienced teachers who used collaborative methods in teaching mathematics. I saw numerous occasions when, as in the incident described above, students became deeply engaged in the mathematics they were doing—not just looking for answers, but trying to understand, and to explain and justify their thinking. As they worked, many showed signs of excitement, delight, or quiet satisfaction. What I saw fits with the descriptions of excellent teaching of mathematics given in the *Standards of Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2002). Students in the classes I observed were clearly being 'empowered to become independent learners ... motivated to improve their understanding of mathematics and develop enthusiasm for, enjoyment of, and interest in mathematics'. The approach adopted by the teachers in the study fostered communication skills, encouraged collaboration, and valued active engage-

* This paper has been accepted by peer review.

ment with mathematics (Standard 3.1). Thus one way of implementing the AAMT Standards would be to adopt an approach similar to that of the teachers I observed — using a combination of small group collaboration and whole-class discussion, where students are not practising previously-taught procedures, but developing new mathematical concepts or using recently-acquired concepts in new ways.

Planning for collaborative work

Introducing this form of learning may not be easy, however, as Nothdurft (2003) reported. Her students at first resisted changes that placed responsibility on them to make sense of mathematical ideas for themselves. They made it clear that they did not value collaboration with their peers, preferring instead to remain dependent on the teacher to give them rules and procedures that they could memorise and practise. These students' concerns echo those of a group I interviewed some years ago (Barnes, 1995) who asked, 'Why can't they just tell us the formula and let us get on with doing the maths?'. The key question then is: how can teachers best plan for and implement collaborative learning in their classrooms?

In the planning phase (AAMT, Standard 3.2) teachers must organise students into groups that are likely to collaborate well, select suitable activities, and train students to work together. *Groups* need to be well-balanced, containing students with differing personalities and strengths who will nevertheless be able to collaborate effectively. Ideally, each group should have one responsible student who will be sensitive to others, try to keep them on task, and ensure that everyone's voice is heard. A *suitable task* should be challenging but accessible. It will be most effective if there are several possible entry points, so that students with different thinking styles and levels of expertise can all make a start, but it should not be so easy that any individual could complete it more efficiently alone. Tasks need to be intrinsically interesting and rewarding, and also open-ended, allowing groups the possibility of extending or generalising the initial problem. *Training* involves negotiating with the class a code of behaviour for group work. Everyone should understand why the teacher is introducing collaborative learning, and agree to a set of social/mathematical norms. These should include treating others with respect, encouraging everyone to contribute, listening courteously and attentively to what they have to say, valuing constructive criticism, and justifying any assertions. And everyone should be helped to understand that expressing disagreement without giving supporting argument is unacceptable. Aspects of planning for collaborative work are discussed in more detail in Barnes (in press), along with practical strategies that were seen to be effective in classes I observed.

Implementing collaborative activities

Once groups have been formed, tasks selected, and efforts made to establish a culture of collaboration, what then? According to the AAMT *Standards*, 'Excellent teachers of mathematics... challenge students' thinking and engage them actively in learning... initiate purposeful mathematical dialogue with and among students... Their teaching promotes, expects and supports creative thinking and mathematical risk-taking... and involves strategic intervention and provision of appropriate assistance.' (Standard 3.3)

While the students are working in groups, how can the teacher best facilitate within-group collaboration? What form should strategic intervention and assistance take and when are they appropriate? It is tempting for teachers, while the class is working in

groups, to spend all their time circulating the classroom, interacting with each group in turn. However, from my observation, it seemed helpful if teachers paused first and observed the whole class, noting how well different groups were collaborating, and identifying where problems were likely to arise. They then took further time to listen to the discussion within a group before joining it and talking with the students. In this section I discuss first what to look for when observing a group at work, and then how best to engage them in discussion to promote purposeful dialogue, creative thinking and risk-taking, while avoiding adding to the frustration that many students experience when confronted with challenge.

Positioning during group work

Positioning theory (Davies & Harré, 1990; Harré & van Langenhove, 1999) provides a theoretical framework which can be used to study the interactions among students engaged in collaborative learning activities. Harré and his colleagues argue that during conversational interactions, people can be thought of as presenting themselves and others as actors in a drama, with different parts or 'positions' assigned to the various participants. Positions are not fixed, but fluid, and may change from one moment to another during an interaction. This distinguishes positions from fixed roles, such as 'recorder' or 'reporter' sometimes assigned to students. Participants in a discussion may actively seek a position, or may have it assigned to them by others. If a position is assigned, they may accept or contest it.

In my study, I identified a range of ways in which students positioned themselves, or were positioned by others, during small-group discussions. I made a list of these and described the associated behaviours. The list was derived by careful and repeated analysis of forty-five videotaped lessons from three contrasting schools. In the final stages of analysis, no new positions were identified, suggesting that the list is relatively complete. The main positions found are as follows:

<i>Manager:</i>	Initiates work, invites ideas, interprets instructions, gives orders or makes suggestions about who should do what, or how they should tackle the task.
<i>Facilitator:</i>	Acts to keep the group functioning smoothly, gives social support, ensures that nobody is ignored, tries to avoid or resolve conflict.
<i>Expert:</i>	Either makes authoritative mathematical statements, and decides what is correct, or is asked for help by others who accept what they say as authoritative.
<i>Spokesperson:</i>	Speaks to the teacher on behalf of the group, for example explaining what they have done, clarifying what is wanted, or asking questions.
<i>Critic:</i>	Seeks explanations, looks for alternative methods, disputes assertions made by other students. Points out flaws in reasoning or inaccuracies in calculations.
<i>Collaborator:</i>	Engages actively in the discussion, working closely with others. Often uses collaborative forms of talk such as speaking in chorus or completing another's sentences.
<i>Helper:</i>	Carries out routine tasks on behalf of another, usually a Manager. Acts as a subordinate, under the other person's direction.
<i>In need of help:</i>	Either claims not to understand, and explicitly or implicitly asks for help, or accepts an offer of help from another and pays attention to the explanation given.

<i>Entertainer:</i>	Initiates and sustains off-task activity — talk, gossip, banter, or play, causing a significant distraction from the group’s work.
<i>Networker:</i>	Monitors events in other parts of the room, or listens to the talk in other groups. Joins with other groups in mathematical or off-task activity.
<i>Outsider:</i>	Either tries to join in the discussion, but is interrupted or ignored; or says nothing for a long time, and gives no sign of seeking to participate.

What would we like to see?

In thinking about how teachers might use this list, it helps first to reflect on which positions are desirable and which undesirable. Only three — Entertainer, Networker, and Outsider — are clearly inimical to effective collaboration. When students take up a position as Entertainer, they distract themselves from the mathematics and disrupt the work of others in the group. Similarly, when students take up a position as Networker, they fail to participate fully in their own group, and distract members of other groups from their work. Finally, being positioned as Outsider means that a student is unable to participate fully in group discussions. This decreases their opportunity to learn, and others’ opportunity to learn from them.

No other position is unequivocally problematic, but some can be undesirable for students to occupy very frequently or for extended periods. For example, while it is desirable for any student to be able to take up the position of Expert at times, it is less satisfactory if one individual is accepted as the resident Expert in a group to the exclusion of others, because that decreases the others’ opportunities to contribute. It is also undesirable if the same student is always positioned as Spokesperson for a group, because this denies other students opportunities to articulate their thinking. Equally, while everyone should be encouraged to feel comfortable about occasionally being In Need of Help, if one student takes up this position too frequently, they may be becoming too reliant on other people, and failing to make an effort to think for themselves.

The most desirable positioning for effective collaborative work appears to be a pattern of maximum fluidity and flexibility, with no student occupying a position to the exclusion of others. Thus we might hope to see all members of a group sharing in managing and facilitating the group’s activities and all members regularly taking positions both as Collaborator and as Critic. The positions of Expert, Spokesperson, Helper and In Need of Help should be open to many different individuals, and efforts made to avoid anyone taking up, or being assigned, positions as Entertainer, Networker, or Outsider.

Watching and listening

The list of positions and their descriptions can be a useful guide, suggesting what to look and listen for when observing groups at work. For example: Are any students taking up positions as Entertainer or Networker? Is one individual dominating by assuming the position of Expert most of the time? Is any student continually positioning him/herself as In Need of Help? Is any student being ignored or interrupted by other group members? A positive answer to any of these questions indicates a need for early intervention.

Are group members using collaborative forms of talk, or giving other indications of shared thinking and close engagement with one another and the task? Are some people taking a stance as Critic, and encouraging others to justify their claims, or think of alternative ways to tackle a problem? If these questions can be answered in the affirmative, it may be a good idea to praise the group’s good collaboration, but leave them to continue working independently for a little longer.

Joining the group

When the teacher joins a group, it is important not to allow one individual to become the sole Spokesperson, especially if they have also frequently taken a position as Expert. If this seems to be happening, questions can be directed explicitly to other people. Others can be asked, in turn, to explain their understanding of what the group has done. I noticed that teachers in my study sometimes moved around the group, and physically took up a position beside students who had been less involved, in order to engage them in the discussion. They also made a point of crouching with their heads at student level, to avoid towering over them.

If one student is positioned too often as Expert, it is important to find ways of recognising the expertise of others. Teachers in my study used a variety of strategies for this. One looked carefully at the written working of a student who had been finding it hard to get others in her group to pay attention to her suggestions. She looked at, and praised, what the girl had written, and asked her to explain it to the rest of the group. Another teacher showed a new graphic calculator technique to a quieter and less confident student. This student was then responsible for teaching it to other members, and became the resident expert within her group on this technique. Problems set in practical contexts can sometimes let students bring to the mathematics classroom expertise from other areas of life, or other school subjects. An example arose in a task dealing with a spill from an oil tanker. The groups had to produce media reports that included the mathematical results (about the rate the oil was spreading), but could also contain other relevant material. Students in different groups had different interests and expertise, and the final reports varied greatly. Some mentioned the risk to the environment, and steps being taken to rescue seabirds and marine life. Others talked about the effects of the spill on oil prices and the stock market, while another group used their knowledge of chemistry to discuss how the slick might be broken up and dispersed. 'Real world' problems like these can create opportunities for people who would not usually achieve expert status in mathematics to receive recognition from their peers, and a consequent boost to their self-esteem.

What can a teacher say in response to comments like, 'We're stuck'? It does not seem very productive to say, 'Here's a hint,' and give explicit suggestions on solving the problem. This closes off alternative approaches that the group might have adopted, and the next time they encounter a difficulty, they are even more likely to rely on receiving a hint. More helpful is to respond with questions: 'What have you tried?', 'Have you identified what you need to do to answer the question?', 'Can you express the problem in a different way?', 'What possible methods might you use?'

Other questions that groups often ask are, 'Is this correct?' or, 'Are we on the right track?'. Again, the teachers I observed tended to turn these questions aside, and respond with further questions. Some of the following may be appropriate, depending on the task: 'Does your answer seem reasonable?', 'How could you check your solution?', 'Is there another way you could work it out?', 'Are you all satisfied that the reasoning is correct?', 'Could you draw a diagram or graph and use that to make an estimate?'

Sometimes a group decides that they have completed the assigned task. Often this is a cue for someone to position him/herself as Entertainer. If a teacher notices that the group is off-task, they make excuses, saying, 'We've finished.' The teachers in my study responded to this in two ways. First, they asked the group to make sure that every member could explain how they had obtained their solution, and could justify the methods used. Then they challenged the group to extend or generalise the task or the result in some way. It was thus rare to find a group sitting around with nothing to do, waiting for others to finish.

There is no need to wait until all groups have completed a task before having a reporting session. Indeed, whenever some groups are stuck, or uncertain what to do, the best strategy may be to ask all groups to give brief progress reports. This can result in cross-fertilisation of ideas, and reinvigorate the group discussions. In these circumstances, groups should be asked to report on just one thing: a useful fact they have discovered; a graph or diagram they have drawn; or even something they have identified that they need to know, but don't know yet. While groups are given a few minutes to select and brief a reporter, the teacher can make a quick check on each group's progress, and decide on the most useful order for reporting, so that those that have made least progress go first. Groups that follow can be asked to avoid repetition. Unless they disagree with the previous group, they should talk only about what has not already been reported. This keeps the reports interesting to everyone.

Conclusion

Collaborative learning can be a way of making mathematics vital for students. Adopting some of the steps suggested here may assist teachers to plan and implement collaborative activities in their classrooms. Being aware of the ways students position themselves, and are positioned by others, during discussions; observing groups before interacting with them and making effective use of questioning and whole-class discussions—all of these may help to promote more effective collaboration, so that students may be as excited by their mathematics and as deeply absorbed in thinking about it as those in the incident described at the beginning.

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The empty number line: Making children's thinking visible*

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This paper explores the use of the empty number line as an aid to recording children's thinking strategies for mental computation. An instructional sequence will be proposed for introducing and developing a more sophisticated use of the empty number line with two-digit addition and subtraction. The learners' perspective will be considered by nine year-old Emily, as she reflects on the benefits of using the empty number line and recalls some instructional experiences that were detrimental to her understanding of mental computation.

The implementation of a reshaped *Mathematics K–6 Syllabus* (Board of Studies, New South Wales, 2002) formalised the introduction of not only new mathematical content into primary mathematics, but also some new instructional 'tools'. One such tool is the empty number line (or blank number line). In so doing, NSW, along with an increasing number of other states and countries (e.g., England and New Zealand) has followed an international trend initiated by the Netherlands (Treffers, 1991).

While already proven to be a powerful tool for supporting the development of children's mental strategies in addition and subtraction (Beishuizen, 2001), effective use of the empty number line by teachers is often hampered by a lack of knowledge surrounding its strengths and weaknesses. Actual syllabi have limited scope to provide extensive background knowledge, yet such professional knowledge is essential for instruction utilising such tools to be most effective. According to the *Standards for Excellence in Teaching Mathematics in Australian Schools* (Australian Association of Teachers of Mathematics [AAMT], 2002), excellent teachers of mathematics, 'understand how mathematics is represented and communicated, and why mathematics is taught' (Standard 1.2). They have:

...a rich knowledge of how students learn mathematics... of current theories relevant to mathematics... appropriate representations, models and language. They are aware of a range of effective strategies and techniques for: teaching and learning mathematics... (Standard 1.3)

Additionally, the professional practice of excellent teachers is described by the Standards as being characterised by 'a variety of appropriate teaching strategies' (Standard 3.2) and the promotion of 'mathematical risk-taking in finding and explaining solutions' (Standard 3.3).

* This paper has been accepted by peer review.

The main aim of this paper is to develop teachers' professional knowledge surrounding the use of the empty number line through an explanation of its origins, rationale for its development and adoption into curricula around the world. By itself, such an increase in knowledge will not necessarily ensure excellence in teaching, but it will allow teachers to understand *why* they are teaching it and *how* to use it more effectively. To assist with classroom implementation, strategies for introducing and developing more sophisticated use of the empty number line, will be presented and 9 year old Emily will reflect on some of the instructional pitfalls she found detrimental to her understanding of the empty number line. While the mathematical content focus in this paper will be on the development of mental computation for addition and subtraction to one hundred, the empty number line can also be used to assist development of multiplication and division knowledge (see Bobis, Mulligan & Lowrie, 2004).

What is an empty number line?

The 'empty number line' is a visual representation for recording and sharing students' thinking strategies during mental computation (NSW Department of Education and Training [DET], 2002). Starting with an empty number line (a number line with no numbers or markers), students only mark the numbers they need for their calculation. It has been used most commonly as a tool for recording mental strategies for two-digit addition and subtraction computations. For example, Figures 1(a), 1(b) and 1(c) show three ways to record various solution strategies for $48 + 26$ on an empty number line.

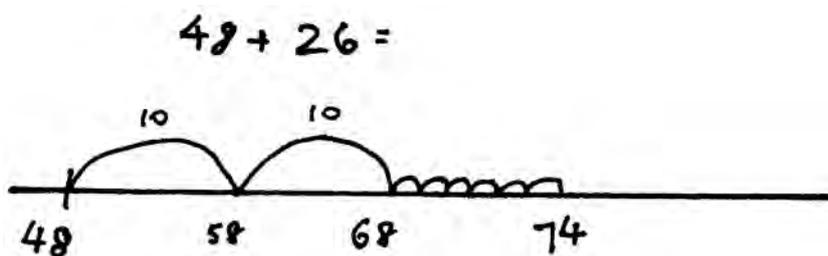


Figure 1(a). Using two jumps of 10 and six single jumps.

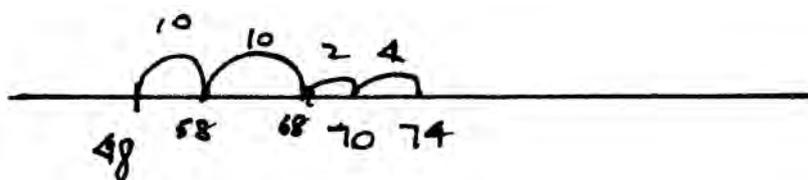


Figure 1(b). Using two jumps of 10, a jump of 2 and 4.

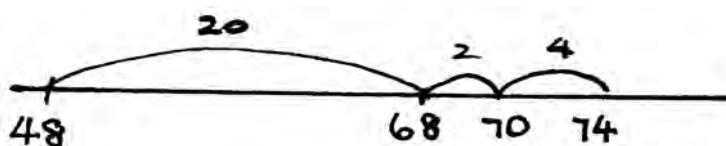


Figure 1(c). Using one jump of 20, a jump of 2 and 4.

Origins and rationale for the empty number line

The empty number line has its first recorded use in the Netherlands as far back as the 1970s (Gravemeijer, 1994). Early experiments with the empty number line were not successful, possibly due to it being introduced via a measurement situation and the analogy of a rigid ruler made students uncomfortable approximating the position of numbers on a line with no given calibrations. However, successful experiments were conducted by Treffers (1991). He found individual students were easily able to learn how to use an empty number line to record and make sense of a variety of solution strategies for two-digit addition and subtraction.

The empty number line was developed out of a need for a ‘new’ tool to help overcome problems experienced by children with two-digit arithmetic. Such problems included the common ‘procedure-only’ use of base 10 materials when modelling the computational procedures and when utilising the standard written algorithms — particularly for subtraction when regrouping was involved. For example, a common error made when solving $53 - 26$ using either base 10 materials or a standard algorithm, is for a child to calculate $6 - 3$. Children learn from an early age that they must take the smaller number from the larger one.

Gravemeijer (1994) presents three benefits of using the empty number line. First, he argues the need for a *linear* representation of number. Base 10 materials, such as Dienes Blocks, clearly reflect situations dealing with quantities, but those dealing with distance or measurement are better suited to a linear representation such as the empty number line. Second, he makes the point that the empty number line reflects more closely intuitive mental strategies used by young children. For instance, children naturally tend to focus first on counting strategies to solve number problems up to 100 — counting-on or counting-down. More proficient mental calculators use a combination of counting strategies (usually in chunks of 10) with partitioning strategies. Partitioning involves children ‘taking apart’ numbers in flexible ways to make them more convenient to calculate mentally. These strategies normally approximate the jumps on a number line. Figure 2 illustrates the jumps involved to solve $53 - 26$. Note that a combination of counting-back in chunks of 10 was made before the ‘6’ was partitioned into two lots of ‘3’ to bridge the decade more easily.

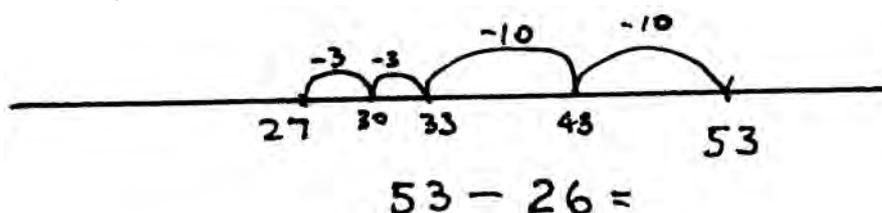


Figure 2. Recording of jumps to solve $53 - 26$.

A third reason for adopting the empty number line, is its potential to ‘foster the development of more sophisticated strategies’ (Gravemeijer, 1994, p. 461). Gravemeijer argues that as children record their thinking strategies, the line functions as a scaffold for learning because it shows what parts of the calculation have been completed and what parts remain. In this way, students’ thinking becomes visible to teachers and other students. Hence, we are able to ‘see’ what mental strategies are being used and where errors might be occurring. From this information, instructional decisions can be made to assist the development of more efficient strategies.

Another advantage of utilising the empty number line, not explicitly mentioned by

Gravemeijer but a natural progression of his thinking, is the way it can provide a stimulus for classroom discussion and sharing of mental strategies. Students can actually explain their strategies by showing others. This makes the empty number line a very powerful tool to enhance communication in the classroom.

When introduced effectively, the benefits of using the empty number line are obvious even to students. Emily, currently in Year 4, was introduced to the empty number line when in Year 3. She considers it to be ‘easier to learn and remember than the pencil and paper method’ and ‘if you make a mistake, it’s easier to find it’. She also recommends that teachers ‘get them (students) to use it at an early age so that they can answer harder questions in higher grades’. From Emily’s perspective, the empty number line is ‘easier’ to use because she can understand how it works and because it keeps a record of each step in her thinking, allowing her to track errors and think of what to do next.

A sequence for instruction

Introducing the empty number line for the first time, assumes that children are already familiar with a linear representation of number (a number line with numbers). Buys (2001) recommends that the empty number line be introduced via a string of structured beads that alternate in colour every 10 beads. Figure 3 illustrates how a string of beads modelling counting in tens (off the decade) can be used to introduce the same jumps on an empty number line.

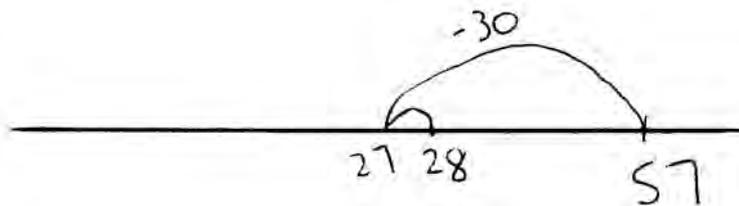


Figure 3. Modelling counting in tens using a bead string and an empty number line.

Before using the empty number line to record more complex mental strategies involved in 2-digit addition and subtraction, there are some prerequisite counting skills and knowledge that should be introduced to children. Two essential strategies that children must understand and use effectively before a more sophisticated use of the empty number line is possible include:

- counting in tens (on and off the decade); and
- jumping across tens (or bridging tens).

Within the *Mathematics K–6 Syllabus* (BOSNSW, 2002), both these counting strategies are included in the Stage 1 Number strand under outcomes NS1.1 and NS1.2 where the empty number line is suggested as a possible tool for explaining and recording their use.

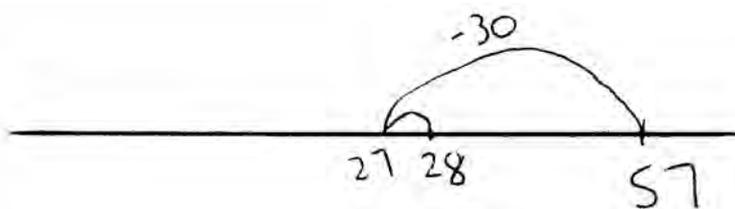
Counting in tens on and off the decade (as already modelled in Figure 3) allows a student to start with any number and count forwards or backwards in multiples of 10. When first introduced to this skill, children can use manipulatives, such as the bead string or bundles of ten popsticks, to model the counting process and record their counting on either a hundreds chart or as jumps on an empty number line. Figure 4 demonstrates how the hundreds chart can be used to record the jumps involved in $56 + 30$.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 4. Jumps of tens recorded on a hundreds chart.

Bridging tens requires that children are able to flexibly partition numbers. For example, to solve $8 + 5$, the first number remains as a whole and the 5 is partitioned and added in parts. It makes it easier, if a part of the 5 is added to the 8 to 'make 10' before the final part is added. Hence, $8 + 2 = 10$; $10 + 3 = 13$. This same strategy can then be applied when bridging 10s in higher decades (e.g., $38 + 5 = 43$; $38 + 2 = 40$; $40 + 3 = 43$).

Based on the strategies for counting in tens and bridging tens, students can be introduced to the *jump strategy* (or sequential strategy) for two-digit addition and subtraction. The fundamental characteristic of this strategy is that one number is treated as a whole, and a second number is added or subtracted in manageable chunks of tens and ones. For example, Figure 1(a), (b) and (c) demonstrates how $48 + 26$ can be solved in increasingly more sophisticated ways. For calculations such as $57 - 29$, it is often more efficient to apply a compensation strategy such as subtracting 30 and adding 1 (see Figure 5).


 Figure 5. Applying a compensation strategy to solve $57 - 29$.

Note that for instructional purposes, only the numbers needed are recorded on the empty number line and that the number of jumps decreases with increased sophistication of strategy use. Emily recalls some confusing experiences with the empty number line in Year 3 when a teacher provided pre-drawn number lines starting at zero and with all the

10s marked (see Figure 6). Confusion resulted, with Emily thinking the teacher wanted her to start at the first number marked on the line: zero. Once again, in Year 4, a textbook exercise caused confusion and resulted in the execution of a less sophisticated strategy (counting in ones) because too many unnecessary numbers were provided (see Figure 7).

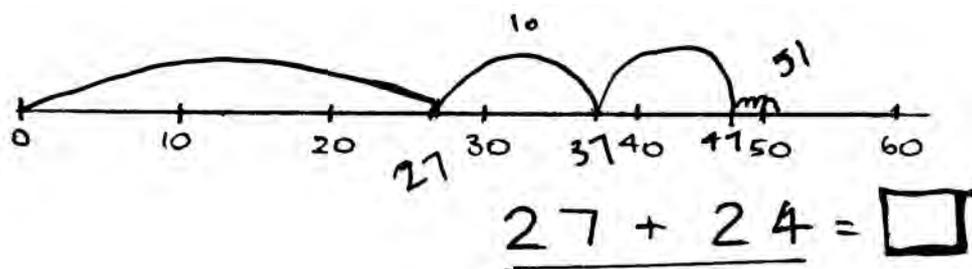


Figure 6. Pre-drawn number line containing unnecessary numbers may cause confusion.

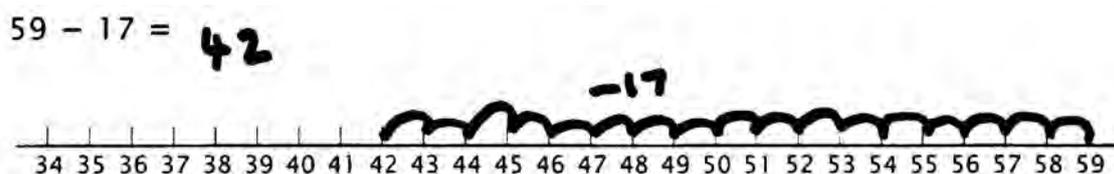


Figure 7. Too many numbers may result in the use of a less efficient strategy

Conclusion

While the empty number line is introduced to support the development of mental strategies, it eventually should be replaced by ‘number language’. For example, the various solutions for $48 + 26$ represented in Figure 1(a), (b) and (c), should eventually be replaced by a numerical-only recording such as:

$$\begin{aligned}
 &48 + 26 \\
 &48 + 20 = 68 \\
 &68 + 2 = 70 \\
 &70 + 4 = 74
 \end{aligned}$$

With each level of use, it is important to emphasise verbalisation of the various strategies modelled on the empty number line. Such communication will assist the sharing of abbreviated strategies and nurture the application of mental strategies without the support of an empty number line. Teachers need to be aware that children will be ready to abbreviate their strategies at different ages and grades. Generally, the time it would take for an ‘average’ child to become proficient with the more sophisticated strategies outlined in this paper, would take about two years with ‘good’ instruction. As adults, who already understand the mathematics and are merely using the empty number line to model our thinking, we can not expect children (who may not understand the mathematics) to see the representation the same way we do. Hence, care must be taken not to introduce more sophisticated strategies before children have a good understanding of prerequisite knowledge. At the same time, teachers have to cater for students’ whose thinking needs to be challenged by gradually introducing higher-order solution strate-

gies. Finally, children will vary their strategy use according to the numbers involved. The rigid application of just one tool or one procedure will severely limit children's ability to apply mental strategies in flexible and fluent ways. A variety of strategies and representational tools are needed.

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Working mathematically to revitalise assessment*

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The dominance of outcomes-based approaches to education has put more pressure on teachers than ever before to ensure that students meet expected or mandated outcomes. Teachers must be able to identify their students' current mathematical knowledge and understanding, and know how to intervene in order to ensure their students' development towards the outcomes. Typically, however, mathematics outcomes defined by curriculum statements provide few indications of students' partial understanding, and this is especially true of working mathematically outcomes. One approach to this difficulty is to design assessment tasks against an identified developmental continuum, and to assess students' progress using a scoring rubric to link the underlying continuum with the task. Such an approach has the potential to integrate assessment, teaching and learning in such a way that teachers can make dependable and defensible judgments about their students and plan appropriate further intervention.

Introduction

Mathematics curriculum documents from all states and territories in Australia currently describe outcomes that students are expected to meet. Some of these outcomes are framed in terms of mathematical content such as those within strands such as Number, Space and Measurement. Others are part of a set of process outcomes, often called Working Mathematically, in which students are expected to be assessed on their competence to do mathematics and communicate mathematically. Outcomes may be established through research, experience, and tradition and are provided to teachers in the expectation that they will ensure that their students reach the appropriate level or stage at the anticipated point in schooling. Progress is determined by assigning students to particular levels or stages based, often, on teachers' classroom decisions. In such an approach, teachers are implicitly expected to be able to make judgments about their students' current mathematical knowledge, understanding, and competence and to plan programs that will ensure that students can progress to the next outcome. Outcomes, however, are usually identified for two-year stages in learning, and few details are provided to teachers about the interim stepping stones that define students' progress. Alternatively, teachers are provided with a plethora of indicators, which often leads to a 'checklist' approach in which teachers 'tick off' behaviours, and assume that an outcome has been met when sufficient boxes have been checked.

* This paper has been accepted by peer review.

Linking teaching and assessment

In recent years, there have been many calls to link classroom assessment more explicitly with teaching (e.g., Shepard, 2000). Formative assessment that provides feedback to teachers and students has been shown to improve students' outcomes when it provides meaningful feedback and changes teaching and learning (Black & Wiliam, 1998). Such assessment needs to be close to the classroom, and informed by teachers' own judgments about their students. Teachers, however, make judgments about their students based on their beliefs and interpretation of classroom experiences (Morgan & Watson, 2002), and may inadvertently miss significant working mathematically behaviour, especially when this behaviour is related to students' partial understanding. Assessment processes, therefore, need to assist teachers' judgments and provide meaningful information about worthwhile mathematical behaviour without swamping teachers with excessive detail. The issue is how this might be accomplished in practice. Can we design assessment that is rigorous and defensible, and that also gives teachers sufficient information to allow them to provide appropriately targeted intervention? One approach is to consider the assessment processes used in vocational education, which focus on competency.

Competency-based assessment

Competence is underpinned not only by skill, but also by knowledge and understanding, and involves both the ability to perform in a given context and the capacity to transfer knowledge and skills to new tasks and situations (Mayer, 1992). It leads to a focus on outcomes in such a way that information is obtained about current performance, but also predicts how that performance can translate into a 'real world' situation. Competence assessment generally requires three components: a task within which there are clearly described expected standards of performance; clear connections to the curriculum; and opportunities to consider wider application of the particular skill or knowledge (Griffin, 1997). The assessment process is thus embedded in a learning sequence that can lead to higher levels of competence.

When considering working mathematically outcomes, however, the progression is not always clearly defined. There is only a sketchy 'road map' for teachers. One way round this difficulty is to use a generic developmental continuum of competency that can be realised in different mathematical situations. An empirically identified continuum that has been shown to work in a variety of mathematical situations (Callingham & Griffin, 2000) is shown in Table 1.

To use this continuum, a scoring rubric, a guide or rule for making judgments, must be developed that links different levels of the continuum to a task or sub-tasks within a teaching sequence. In this way, the assessment process is an integral component of teaching, informing teachers about their students' mathematical behaviour as they teach. In addition, the behaviours typical at each level of the continuum suggest ways in which teachers can modify their immediate teaching, and where students need to go next, providing the feedback suggested by Black and Wiliam (1998).

Table 1. Generalised continuum of competence.

Description	Student behaviour
J <i>Making conjectures</i>	Suggests extensions of the original problem, or changes the task parameters to create new situations.
H <i>Generalisation or relationship use</i>	Expresses the generalisation of the problem solution in symbolic form, and applies it to new situations.
G <i>Generalisation or relationship recognition</i>	Recognises a generalisation of the solution strategy and expresses this generalisation in words.
F <i>Rule or process use</i>	Applies a personal rule to extensions of the initial task.
E <i>Rule or process recognition</i>	Recognises a rule underpinning the structure of the task.
D <i>Pattern or structure use</i>	Applies repeating elements in the task structure to solve problems.
C <i>Pattern or structure recognition</i>	Recognises repeating elements in the task structure.
B <i>Element use</i>	Recognises classes of task elements.
A <i>Element identification</i>	Recognises similarities in different elements.
0 <i>No apparent understanding</i>	No recognition of the elements of the underlying task structure.

Assessment through teaching

How might such an embedded approach to assessment work in practice? To illustrate the process, let us take a task that is available from an existing commercial source and develop assessment that could inform teachers about their students' development at key stages during the lesson.

The lesson is *Garden Beds* (Maths 300, n.d.), which addresses aspects of pattern and algebra, and area and perimeter. A story shell about a gardener who wishes to surround his shrubs with a border of tiles provides interest and relevance for students, who see similar situations in garden makeover programs on television. The lesson starts with an arbitrary number of 'plants' around which tiles are placed. Students are asked to work out how many tiles are needed. This process is repeated several times until all students recognise how the tiles need to be placed to surround the plants. At this point students are set a challenge:

Imagine you are working in the shop that sells tiles. Customers come in to buy tiles. When they come into the shop they often tell us how many plants they have. Some customers might have lots of plants - some might even have as many as 100 plants to protect. How can you find out how many tiles each customer will need?

Students are provided with appropriate concrete materials, asked to keep a record of their findings, and to explain how they arrived at their solution. Each of these activities addresses some aspect of working mathematically, and provides an assessment opportunity for teachers, without having to set up a special assessment event.

Students can address the *Garden Beds* problem in many different ways. They may count

tiles, find patterns, develop rules linking the number of plants with the number of tiles, and express these rules as generalisations. Each of these approaches is valid, and indicates different development against the underlying continuum. By anticipating these possible outcomes, teachers can develop a rule or guide for scoring, a scoring rubric, which describes these mathematical behaviours and maps them onto the underlying continuum. One such rubric is shown in Table 2, together with suggested intervention strategies derived from typical student behaviours.

Table 2. Scoring rubric for first task in Garden Beds.

Student response	Continuum Level	Interpretation	Intervention
Detailed explanation of answer and arrangement includes symbols or equations that relate the tile arrangement to the symbolic expression for the arrangement shown (e.g., $T = 2P + 2 + 4$).	H Generalisation or relationship use	The student can express the generalisation of the problem solution in symbolic form, and apply it to new situations, justifying this by reference to the generalisation. The student has mastered the particular problem type and is ready to learn how to change the problem type to explore new ideas.	Intervention should focus on encouraging students to change the problem type by asking ‘What would happen if...’ and changing the conditions of the problem to create a new situation.
Explanation of answer and arrangement relies on relationship appropriate for the table arrangement (e.g., the number of tiles is double the number of plants plus two for the ends and 4 for the corners).	F Rule or process use	The student’s own rule is applied to extensions of the initial task. The student can extend the solution obtained to a limited range of other tasks having a similar structure to the initial task, and is ready to learn how to form a generalisation that could be transferred to other settings.	Intervention should focus on providing students with opportunities to develop a general rule or process that can be applied to other settings with similar contexts.
Explanation of answer uses patterns to find appropriate numbers (e.g., it goes up by 2 each time).	D Pattern or structure use	The student recognises the underlying principles in the structure of the task, and can apply these in a familiar setting, such as a straightforward extension of the initial task. The student is ready to learn how to identify a rule that links the repeating elements together.	Intervention should focus on encouraging students to formulate rules or describe the processes used in concrete situations, and to record and express these rules in their own ways.
Students model garden beds using concrete materials and count tiles to find out appropriate numbers.	B Element use	The student recognises classes of task elements and is ready to learn how to combine these into a pattern or structure. Problem attempts are presented as single drawings or phrases that relate to one element only of the task.	Intervention should focus on providing opportunities for students to extend and record patterns, and identify structures and talk about these to peers and teachers.

Note that comparative language, such as ‘achieves some aspects of the task’ or ‘partially solves the problem’, is not used in the rubric; nor are quantitative aspects included, such as ‘gives two different rules’. Rather, there is a focus on describing significant mathematical behaviour that may be demonstrated by students as they work on the task. This language is important because it reduces the emphasis on quantity, inherent in expectations of multiple rules for example. The rubric also recognises that all the described behaviours are important steps along the way to full understanding, and removes a sense that a solution that does not provide a generalisation is wrong. Such solutions instead are seen as demonstrating students’ partial understanding of the underlying mathematics.

The next task in the *Garden Beds* lesson could be to focus on how the information that students generated was recorded. Was it unsystematic or organised? Did the recording support identification of the underlying relationship? Developing a recording process that clearly identifies that the number of tiles is dependent on the number of plants is another stepping stone towards identifying and using algebraic relationships. Students could be challenged to present their information in different ways. Again, by anticipating the quality of different responses, teachers can develop a scoring guide to help them make judgments about the performances of their students. One rubric for assessing this aspect is shown in Table 3.

Note again that this rubric avoids comparisons or subjective language, but instead focuses on the underlying mathematical behaviour. The target in this rubric is recording processes, rather than the communication of the findings, but is expressed in similar ways to the rubric shown in Table 2, and again links directly to the underlying continuum. The highest levels of the rubric also reward important mathematical behaviour, including symbolic expression, rather than irrelevant factors such as presentation or length of explanation. A concise mathematical record showing control of the underlying mathematical ideas is recognised as a high level of response. At the same time, however, approaches that indicate more limited understanding, such as those relying on drawing diagrams only, are also recognised as partial understanding. In this way a wide range of performance is expressly catered for and acknowledged.

Not every level of the continuum is addressed in either of these rubrics. In undertaking the Garden Beds task teachers could identify other aspects of their students’ work that mapped onto other levels of the continuum. The assessment can be modified to take account of these judgments by developing further rubrics that relate to the other levels identified. In this way the process becomes a flexible tool for describing on-going learning, rather than simply a means of summarising students’ performances.

Further extensions of the original task could be assessed in the same way. These might include different kinds of linear relationships, or non-linear relationships, extensions into different social contexts, or other learning areas such as science.

Conclusion

Embedding assessment in classroom practice is essential if aspects of working mathematically are to be addressed. Using a competency based approach underpinned by a generalised continuum of competence can provide teachers, and students, with a basis for making judgments about performances. By using descriptors of typical behaviours at different levels of the continuum of competence, teachers obtain feedback about their students’ mathematical behaviour that allows for appropriate, targeted intervention, and gives clear direction for future development. Teachers can defend their judgments by referring to the underlying behaviours, and showing students what their performance

Table 3. Scoring rubric for organisation of information in Garden Beds task.

Student response	Level	Interpretation	Intervention
Systematic and correct methods that include the relationship expressed in a different way (e.g., formula, graph).	H Generalisation or relationship use	At this level the student can express the generalisation of the problem solution in different symbolic forms, and apply it to new situations, justifying this by reference to the generalisation. The student has mastered the particular problem type and is ready to learn how to change the problem type to explore new ideas.	Intervention should focus on encouraging students to change the problem type by asking ‘What would happen if...’ and changing the conditions of the problem to create a new situation.
Systematic correct methods that include the underlying rule (e.g., table of values that includes a rule statement).	E Rule or process recognition	At this level the student recognises a rule underpinning the structure of the task and is ready to learn how to apply that rule consistently and extend the use of the rule. The rule is likely to be expressed in words or diagrams, using non-technical language that summarises the student’s own approach to the problem.	Intervention should focus on providing extensions to the original task and encouraging students to apply their rules consistently to these extensions.
Systematic correct methods that rely on patterns (e.g., table of values expressed numerically only, with no rule stated).	C Pattern or structure recognition	At this level the student recognises the repeating elements in the structure of the problem, and is ready to learn how to recognise the underlying principles.	Intervention should focus on providing opportunities for students to discuss the patterns and structures that they identify, and to record these patterns in systematic ways.
Systematic correct methods that do not clearly indicate the underlying relationship (e.g., list; diagram).	A Element identification	At this level the student recognises similarities among different elements and is ready to regroup these according to personal ideas.	Students should be encouraged to explain and record their own element classifications, and to describe those classifications that might help to solve the problem posed.
Recording incorrect or not systematic	0 No apparent understanding	At this level there is not enough information to describe the student’s work. If the task was attempted, it is likely that the student did not recognise the elements of the underlying structure of the task.	If the task was attempted, intervention should focus on helping the student to record systematically the essential elements that underpin the task.

indicates and what they need to do next. The generalised approach can be linked to different curriculum frameworks, including those addressing essential learnings, and to different mathematical contexts.

The approach to assessment described here is coherent with the AAMT *Standards* (AAMT, 2002) in several domains. Use of a range of assessment processes are explicitly recognised as an important aspect of teachers' *professional practice*. By focussing on Working Mathematically, teachers can develop and extend their *professional knowledge* of their students through recognising and acknowledging the mathematics their students know and use. Teachers' knowledge about their students' mathematical learning may also be enriched, particularly through using a learning sequence that addresses Working Mathematically in different mathematical contexts.

Teaching and learning mathematics through exploration that encourages initiative, creativity and confidence in students will only become widespread if it is supported by similar assessment processes. It is time to work mathematically to revitalise assessment.

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Working mathematically: The role of graphics calculators*

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The newly revised NSW Mathematics 7–10 Syllabus emphasises the importance of students becoming more actively engaged in their learning. The Working Mathematically strand of the syllabus encompasses five interrelated processes (questioning, applying strategies, communicating, reasoning, reflecting) designed as a framework for classroom activities. This paper discusses some factors associated with the use of graphics calculators as tools for working mathematically. Issues discussed are teacher and student roles, collaborative practices, cognitive conflict, and teachers' knowledge and beliefs.

Introduction

The National Statement on Mathematics for Australian Schools (Australian Education Council, 1991) identified three areas of mathematical thinking or 'ways of knowing' (p.26): *Attitudes and Appreciations*, *Mathematical Inquiry* and *Choosing and Using Mathematics*. These three strands are separated from the content strands of the document and broadly describe how mathematical knowledge is developed, applied and communicated. They include processes such as observing and generalising patterns, problem solving and mathematical modelling, conveying mathematical ideas, and explaining and justifying conclusions.

Although the language and terminology may vary from place to place, the mathematics curricula for all Australian states and territories now include outcomes that are consistent with the philosophy of the *National Statement* (Callingham & Falle, 2003). In New South Wales, the recently revised *Mathematics Years 7–10 Syllabus* (Board of Studies, 2003) incorporates a Working Mathematically strand that encompasses the processes of questioning, applying strategies, communicating, reasoning and reflecting. The working mathematically outcomes are seen as integral to learning mathematics, both in applying existing knowledge and in developing new concepts and skills.

The use of technology in the classroom is another important curriculum issue. One of the goals identified in the *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) is for the use of technology 'to be regarded as a normal part of doing mathematics' (p. 23). The statement also notes that computers and calculators provide students with opportunities to investigate mathematical ideas and achieve a richer understanding. Particular reference to graphics calculator technology is

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made in *Graphics Calculators and School Mathematics: A Communiqué to the Education Community* (Australian Association of Mathematics Teachers, 2000). The document describes how informed use of graphics calculators can become a catalyst for pedagogical change leading to learning characterised by investigation, collaboration and an understanding of mathematics as richly connected concepts.

The aim of this paper is to consider some of the factors associated with the use of graphics calculators in supporting the ideas of Working Mathematically.

Teacher and student roles

The introduction of graphics calculators does not automatically guarantee a more questioning and reflective classroom environment. However, there is a good deal of evidence to support the view that the use of this technology can lead to significant changes in classroom dynamics (e.g., Farrell, 1996; Goos, Galbraith, Renshaw & Geiger, 2000; Harskamp, Suhre & Van Struen, 2000; Simonsen & Dick, 1997).

The graphics calculator is a powerful tool for exploring mathematical concepts. Students can enter a symbolic expression and immediately display the corresponding table of values and graph. Split screens allow for even more explicit links among the different representations of functions to be made. Visual representations are particularly helpful in enabling students to access complex and realistic algebraic models that may lie well beyond their symbolic manipulation capabilities. Use of the graphics calculator allows students to progress from simple algorithms to more sophisticated tasks in which realistic data is translated into graphical models, analysed and interpreted.

Graphics calculators can generate graphs and tables of values quickly and reliably, providing immediate feedback to students. It often requires only a small number of keystrokes to effect quite dramatic changes in graphical and numerical data representations, facilitating the exploration of open-ended questions. Numerous graphs can be generated in a short time frame affording students greater opportunities to recognise patterns and establish relationships among the different forms of a concept.

The power of the technology can therefore promote a shift in focus from a more teacher-centred approach to one of inquiry and self-discovery by students. Farrell (1996) analysed videotapes of thirty-six mathematics lessons in which graphics calculators or computers were sometimes used. She found that students' activities changed when they were using the technology. Students investigated more and employed problem-solving strategies such as planning, implementing strategies, and checking or verifying solutions. In other words, the graphics calculator became a vehicle for working mathematically.

The traditional instructional paradigm of teacher exposition, worked examples and student practice can be transformed by the use of graphics calculators. Farrell (1996) also reported that traditional, expository teaching styles were often replaced by investigation activities undertaken by students acting independently of their teacher. Discovery learning characterised by organisation and classification of data, investigation and generalisation, and student reflection leading to the formulation of new mathematical concepts became the norm. The working mathematically processes of questioning, applying strategies, reasoning and reflecting are clearly evident in such an approach.

Simonsen and Dick (1997) studied teachers' own perceptions of their teaching practices as they used graphics calculators in the classroom. The teachers reported that when graphics calculators were used, their lessons were more likely to be student-centred and they asked many more 'What if...?' questions of their students. The calculator provided a shared reference that promoted a more cooperative approach to learning in which the

teacher facilitated student inquiry rather than directed it. Minor trends to emerge from the study included an increased student discussion of mathematical ideas and increased involvement and enthusiasm on the part of students.

The teacher's role can change significantly to accommodate the technology. The teacher need no longer be the central focus of classroom interaction and can become a technical assistant, collaborator, facilitator and catalyst (Heid, Sheets & Matras, 1990). The teacher is then responsible for choosing appropriate calculator activities and promoting their mathematical content. The teacher becomes an opportunist, constantly seeking to draw attention to connections among the various representations of mathematical concepts provided by the calculator, and shaping the relationship between the calculator display and mathematical knowledge (Guin & Trouche, 1999).

Collaborative practices

Collaborative group work is a critical component of Working Mathematically because it encourages communication and student reflection as differing points of view are expressed, considered and evaluated. The graphics calculator has an important role to play in group activities as a kind of conversation piece for sharing mathematical ideas and making their thought processes publicly available in the classroom. The technology facilitates social interaction in the classroom because it acts as a common point of reference for students they discuss their ideas and results.

Farrell (1997) found that students worked together more often when they used technology than when it was not used. This working together was sometimes planned (at the direction of the teacher, for example) but was more often spontaneous (such as when students leaned over to examine the display of a peer's calculator and discussed how it was obtained). Students used the calculator screen to share their discoveries, justify their reasoning to one another and negotiate meaning for the mathematical concepts they were examining. The calculator screen became the means by which the ideas were shared and explained among students and with the teacher.

Graphics calculators were designed primarily as personal tools and Doerr and Zangor (2000) caution that the tendency for some students to use their calculators as more private devices may inhibit group work. Interactions among group members may break down because it is relatively easy for individual students to explore their own conjectures without the need to consult their peers. If students continually transform their calculator display in light of newly made discoveries without communicating the results to others then it can become increasingly difficult for the group to continue functioning effectively.

However, the graphics calculator overhead projector panel can act as a shared device to counterbalance the situation just described and foster communication (Forster & Taylor, 2003; Goos et al., 2000). Guin and Trouche (1999) describe the role of a 'sherpa student' (p. 209) who uses the overhead panel to project his or her calculator display so that everyone can observe it. The teacher can then ask the student to interpret the panel display and explain how the calculator was used to obtain it. Other students are also able to question this student and make their own comments or suggestions (Forster, Taylor & Davis, 2002). Guin and Trouche (1999) note that the use of the overhead panel favoured classroom debates and reinforced the social aspects of knowledge construction thus invigorating the reflection phase of the lesson.

Groups can also use the overhead panel to present their findings and explain their reasoning to the teacher and their peers. Partial solutions can be shared and completed with whole-class input while shortcomings in the presenters' work can be identified and cor-

rected (Goos et al., 2000). Members of the class can suggest alternative solutions, both in terms of mathematical and calculator approaches, to enrich the classroom discussion. In addition, the teacher is able to observe students' thought processes and assess their level of understanding. Reflective discussions also afford the teacher an opportunity to reinforce correct thinking and emphasise key mathematical concepts.

Cognitive conflict

The technical limitations of graphics calculators can sometimes produce unexpected results that provide a unique opportunity to develop students' analytical skills. Contradictions often arise between results obtained by hand and those displayed on the calculator due to the rectangular shape of the viewing window and the relatively small number of pixels that comprise the screen (see Mitchelmore & Cavanagh, 2000). Rather than avoid situations that confront some of these characteristics of the technology, teachers could include examples where the calculator results conflict with what students expect and then encourage the students to resolve the inconsistencies (Guin & Trouche, 1999).

Examples of unexpected results include partial views (where the initial viewing window does not display all of the important features of a graph) and rounded trace coordinates (that do not permit irrational values to be shown exactly). There are also unequally scaled axes (where, for instance, perpendicular lines do not appear at right-angles) and issues related to the resolution of the screen (where, for example, parabolas may appear flat near the vertex or asymptotes on graphs of rational functions may be distorted). Negotiating the differences between mathematical concepts and their corresponding calculator displays can lead to lively class discussion that is consistent with the principles of working mathematically.

The identification, interpretation and resolution of inconsistencies between the pen and paper environment and the graphics calculator screen allows students to sharpen their mathematical thinking skills and improve their understanding of the technology. In the *identification phase*, students first need to explain carefully the mathematical result they expect to see and then examine the calculator display, noting the precise nature of any discrepancies. It is important here for students to justify their initial hypotheses, even in light of apparently contradictory evidence from the calculator.

In the *interpretation phase*, teachers can help students to understand the unexpected calculator results by considering some of the important internal processes of the technology and its limitations. These might include some discussion of how the calculator produces a graph and the difficulties inherent in representing the graph of a continuous function using discrete pixels. In considering the shortcomings of the machine, students are more likely to develop a healthy scepticism of the calculator's output and recognise the dangers in making conclusions based on an ill-considered acceptance of the calculator's initial display.

In the *resolution phase*, students need time to explain the calculator's output and consider how the situation might be resolved. Armed with a more sophisticated understanding of how the calculator operates, students will be better placed to suggest alternative approaches that could produce a more appropriate display. For example, this may mean zooming out to show a more complete graph or simply ignoring the jagged formation of a curve due to the screen resolution. Another possible strategy is for students to decide that the best way forward is to put the calculator aside and continue working by hand. The ability to discern whether or not to use the calculator is an important skill that can only be developed if students encounter situations where both the limitations and potential of the calculator are evident.

Caution is required, however, and tasks need to be carefully designed and introduced gradually lest the conflict engendered by these examples imposes too great a cognitive load on students and becomes a barrier to future learning. There is a strong case for avoiding difficulties in the early stages and then structuring exercises to draw explicit attention to the limitations of the technology in an incremental way. However, cognitive conflict can act as a powerful stimulus for learning when teachers encourage students to make predictions about what they expect to see on the calculator screen, contrast their forecasts with what is actually displayed, and resolve the inconsistencies. By questioning the calculator's output and reflecting on the constraints of the technology, students' not only learn to interpret the calculator's output in a more discerning way but their mathematical reasoning is enhanced as well.

Teachers' knowledge and beliefs

A number of studies report that a teacher's philosophy of mathematics and mathematics education largely determine the way in which they use technology in the classroom (e.g., Simmt, 1997; Tharp, Fitzsimmons & Ayers, 1997). The results of these studies suggest a strong correlation between a teacher's views of what it means to do mathematics and his or her views on the use of calculators. For example, Tharp, Fitzsimmons and Ayers (1997) found that teachers who hold a rule-based view of mathematics are less likely to regard calculators as enhancing instruction and may even see the technology as a hindrance to effective teaching.

Doerr and Zangor (2000) found that teachers' confidence in their own knowledge and skills was another important factor in shifting the locus of control in the classroom and allowing students freedom to use their calculators in exploring concepts and working mathematically. Teachers who could operate the calculator successfully and had experienced success in using the technology were more likely to incorporate graphics calculators in their teaching. They were also more comfortable in allowing students to explore concepts on their own and discover new ways of operating the machine.

Simmt (1997) observed six high school teachers using graphics calculators to teach a unit on quadratic functions. Using videotapes of lessons (with and without graphics calculator usage) and subsequent interviews with the teachers, she investigated the teachers' rationale for using graphics calculators and the relationship between the teachers' philosophy of mathematics and their use of the technology. She found that the teachers' most common reasons for using graphics calculators were that they saved time and helped to motivate students.

Simmt (1997) also noted that the graphics calculator became an extension of the methodology normally used by the teachers. No new methods or approaches were evident and the teachers reported that they did essentially the same kinds of activities whether using the calculators or not. The teachers' choices about the kinds of activities they used, the style of questions they asked and their interactions with students all reflected their personal views of mathematics education. Their strong preference for algebraic solutions meant that the teachers favoured a more traditional pedagogy and did not encourage graphical solution strategies.

Simmt (1997) concluded that merely providing the tool had little impact on instruction because teachers were not challenged to consider how they might adapt their classroom practices to take full advantage of the technology. She suggested that assisting teachers to reconsider their beliefs about teaching and learning mathematics is also necessary when introducing the graphics calculator as a teaching tool.

Professional development that only focuses on instruction in the operation of the calculator will be of little benefit in encouraging teachers either to take up graphics calculators or change their classroom practices (Waits & Demana, 2000). Familiarity with the technology is a valid starting point, but attention also needs to be given to the inevitable pedagogical challenges that the technology brings (Goos et al., 2000). In other words, professional development must not only show teachers how graphics calculators can assist the processes of working mathematically in the classroom, it must also inform teachers about the kinds of instructional practices that are best suited to using the technology in this way.

Attempting to implement both a new technology and a new pedagogy simultaneously is a challenging and long-term process (Huffman, Goldberg & Michlin, 2003). Lasting change takes time and teachers will require more on-going opportunities to reflect on their instructional practice with graphics calculators if they are to incorporate the technology in ways that are consistent with the principles of working mathematically (Tharp, Fitzsimmons & Ayers, 1997). A supportive environment that encourages teachers to share their own insights and classroom experiences and gives them some say in the design of curriculum materials will better equip teachers to take full advantage of the opportunities presented by the introduction of graphics calculators.

Teachers' knowledge of the limitations of the graphics calculator is also important (Cavanagh & Mitchelmore, 2003). One of the great advantages of the technology is that it allows students to explore concepts freely in whatever ways they choose without the direction of the teacher. Many teachers are uncomfortable with such an approach because they fear that in such an unpredictable environment students may stumble across unexpected calculator results that the teachers themselves do not fully understand. Not wanting to run the risk of appearing foolish in front of their students, some teachers may try to wrest back control of the calculator activities and the students' freedom to investigate and work mathematically is thus diminished.

Improved understanding of how the calculator operates (how it produces a graph, for instance) can increase teachers' confidence, leading to a more flexible approach in the classroom (Doerr & Zangor, 2000). Teachers who feel more comfortable in dealing with unexpected calculator displays are more likely to encourage students to question their calculator-based results. Promoting a critical attitude to the calculator's output does not guarantee that students will develop a similar outlook, but it does have the potential to unlock the calculator's power as a tool for improving students' understanding of mathematical concepts.

Conclusion

The graphics calculator is a powerful tool for improving classroom instruction and promoting spontaneous, self-directed learning. The introduction of the technology can foster new roles for teachers and students and encourage collaborative learning experiences that enable students to build a richer understanding of many mathematical concepts. In using the graphics calculator, students can also gain an enhanced appreciation of the tool itself and its limitations. However, care must be taken and it is clear that teachers will need ongoing support through carefully designed professional development programs if the benefits of the technology are to be maximised. These programs should consider pedagogy that complements the use of the technology and allow teachers time to reflect on their own practice. Teachers will then discover that graphics calculators can become a powerful means of supporting the principles of working mathematically in the classroom.

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Assessing highly accomplished teachers of mathematics*

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This paper provides information on the development by The Australian Association of Teachers of Mathematics in collaboration with Monash University and state mathematical associations of a set of standards for teachers of mathematics and an associated assessment process. Following a discussion on the motivation for developing such standards, examples of the experiences and insights of other organisations in Australia and elsewhere in developing their own standards are presented. The complexity of describing excellent teaching is acknowledged, as are the concerns that some writers express about the process and possible pitfalls. Early evidence is provided that teachers can use such standards to make consistent judgments on documentation developed by teachers, using the standards criteria.

Introduction

It is argued that standards for the teaching profession will increase public esteem of the profession. Public, credible standards — and the assessment of individuals against these — are fundamental to the high esteem in which other professions are held by the public. The way in which other professions practise self-regulation enhances their professionalism, and adapting this model has the potential to produce the same positive outcomes for mathematics teachers (Bishop, Clarke & Bennett, 2000).

Some significant claims have been made about the power of standards and their assessment; for example, in the USA:

The National Board for Professional Teaching Standards is leading the way in making teaching a profession dedicated to student learning and to upholding high standards for professional performance. We have raised the standards for teachers, strengthened their educational preparation through standards, and created performance-based assessments that demonstrate accomplished application of the standards. (NBPTS, 2004)

The Australian Association of Mathematics Teachers (AAMT) has been discussing the development of professional teaching standards for many years and collaborated with Monash University in a federally-funded research study from 1999 to 2001. Morony

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(1999) defined ‘professional standards’ as they apply to this study and provided reasons for developing them:

In the current context in education in this country, professional standards are a statement of what teachers need to know and be able to do in order to do their work at an acceptable level. They go beyond mere statement of required initial qualifications, although these may be important. They can be standards for entry into the profession. Or they can be about accomplished or expert performance of the work of the profession, linked to monetary rewards. The reasons for having professional standards can range from building community confidence in teachers’ work and thereby improving the status of teachers and teaching to supervision of individual staff members by principals. Ensuring that students have appropriately skilled teachers is essential — public statements of standards can help in this. (p. 42)

There were two aspects that motivated the AAMT involvement in the process of developing standards for teachers of mathematics and an associated assessment process. The first was that it would develop the profession through:

- improving the status of teachers;
- increasing knowledge of the work teachers do;
- articulating what it means to be an excellent teacher of mathematics;
- building a professional language to support reflection and discussion; and
- providing a ‘benchmark’ for which to aspire.

The second argument relates to the individual perspective. Ingvarson (1998) claimed that in identifying standards of excellence, teachers would be able to design their own professional development to attain them. Morony (1999) regarded this kind of arrangement of professional development to be an essential characteristic of true ‘professional practice’, quite distinct from approaches focusing on ‘implementation of new policies’. Professional standards can help develop individual professionals through:

- providing a ‘road map’ to identify and plan personal/team professional development needs;
- amplifying the importance of teachers’ professional knowledge;
- providing a framework to evaluate practice; and
- endorsing and valuing the work of accomplished teachers of mathematics.

Development of standards for excellent teachers of mathematics: The experiences of others

Several countries have developed standards for professional excellence or accomplishment. The United Kingdom model comprises six standards. This model assesses excellence in

1. student achievement;
2. subject knowledge;
3. planning;
4. teaching, managing pupils and maintaining discipline;
5. assessment; and
6. advising and supporting other teachers.

Teachers are assessed through a portfolio, classroom observation, an interview, consultation with the teacher’s principal, and references from colleagues (Department for Education and Employment, UK).

Canada also has a system for recognising advanced teachers. In 1999, the Ontario College of Teachers produced a model for assessing excellence in teaching that comprises five standards. These are: commitment to students and student learning; professional knowledge; teaching practice; leadership and community, and ongoing professional development. The standards were produced in consultation with the teaching profession and the public (The Professional Affairs Department, 1999).

In the USA, the National Board for Professional Teaching Standards (NBPTS) has offered a wide variety of certificates for excellent teachers since the early 1990s. All of the NBPTS standards emphasise that accomplished teachers are aware of what they are doing as they teach and why they are doing it. They are conscious of where they want student learning to go and how they want to help students get there. Furthermore, they monitor progress towards these goals continuously and adjust their strategies and plans in light of this constant and complex feedback. Accomplished teachers set high and appropriate goals for student learning, connect worthwhile learning experiences to those goals, and articulate the connections between the goals and the experiences. They can analyse classroom interactions, student work products, and their own actions and plans in order to reflect on their practice and continually renew and reconstruct their goals and strategies. Excellent teachers are assessed through a portfolio and a one-day attendance at an assessment centre (NBPTS, 2000).

In Western Australia, advanced teachers, called Level 3 Teachers, are assessed on five competencies. First, they are expected to utilise innovative and/or exemplary teaching strategies that promote high levels of student participation and involvement. Second, they should employ consistent exemplary practice in developing and implementing student assessment and reporting processes. Third, excellent teachers should engage in self-development activities to critically reflect on their own teaching practice and on teacher leadership. Fourth, they should enhance other teachers' professional knowledge and skills through professional development sessions, mentoring, and supporting colleagues. Fifth, excellent teachers are expected to provide high-level leadership in the school community. Teachers are assessed through a portfolio and attendance at a one-day assessment interview (Martin, 2001).

The development of the AAMT Standards

AAMT and Monash University began to establish a set of standards for excellence in teaching mathematics in Australia in 1999. AAMT took a leading role in the development of the mathematics standards. Its affiliated organisations identified a group of approximately fifteen experienced mathematics teachers in New South Wales, South Australia, Tasmania and Victoria. These teachers were selected by the teachers' associations in each state through expressions of interest or identification by the affiliated associations. They formed a nucleus of expertise that was regularly consulted over the three years of the project. The teachers formed focus groups that met at least twice-yearly over the three years to develop the standards and to design and trial the assessment strategies. The researchers from Monash University recorded the teachers' opinions and judgements and produced the standards based on this information. The standards are thus based in real teaching practice, produced and affirmed by the profession.

An important benefit of professional standards for excellence in teaching is that they may also provide an alternative career path for excellent teachers. At the current time, Australian mathematics teachers who wish to advance in their careers must generally move through the administrative stream, abandoning classroom teaching. An alternative

pathway, established through accreditation in accomplished teaching, would provide an opportunity that is not available to many teachers. Some mathematics teachers would welcome the opportunity to progress to more senior positions while remaining in teaching, and a professional standards framework would facilitate such an objective.

Assessing complex performances

As previously stated, the description of the accomplished teacher of mathematics represents only part of the use of standards. The use of standards for the evaluation of teachers is a key aspect. However, teaching is a complex performance and there are many challenges in assessing accomplishment (Delandshire & Petrosky, 1998).

During the final phases of the AAMT project, teachers explored the assessment based on the standards and number of tensions were discussed including the following:

- *comparability versus creativity* — Can I exercise creativity in presenting evidence of my accomplishments to a group of assessors who need to show consistency in their judgments?
- *meaningful versus manageable* — How can I give evidence of my accomplishment in ways that do justice to my achievements and yet avoid excessive amounts of time in assembling and presenting evidence? How much evidence do my assessors really need? What do I need to tell them to ensure that they fully understand the nature of my achievements?
- *accomplishment versus ongoing critical inquiry* — To what extent can I use my portfolio as a means of engaging in genuinely critical inquiry into my own teaching, and not simply to demonstrate my accomplishments?
- *personal goals versus school policy* — Is it possible to demonstrate personally meaningful accomplishments beyond the professional development goals required by my school or those required by government policy?

Since the publication of the standards, there have been a number of initiatives including the establishment by AAMT of the National Professional Standards Committee – Mathematics which is chaired by Professor John Mack and consists mostly of practising teachers. The model which they are working on is based on clear principles, namely that the assessment process will be:

- rigorous and valid;
- adaptable to and applicable in all teaching contexts;
- fair to all candidates no matter what their teaching situation;
- equally accessible to teachers across the country;
- controlled by the candidate insofar as this is possible; and
- oriented towards contributing to professional growth of the candidate.

In 2003, the Commonwealth Department of Education and Training (DEST) funded AAMT to trial and evaluate the model. The *Teaching Standards Assessment Evaluation Project* (TSAEP) model required the candidates to:

- respond to unseen questions that simulate teaching decisions through an assessment centre;
- submit a portfolio of their work and achievements as a teacher consisting of a professional journey (reflective essay), a case study of one of two students' learning, an example of current teaching and learning practices, validation (report of a classroom observation or video of their teaching) and documentation (awards, references, testimonials etc.); and
- take part in an interview.

The model involves a moderation or ‘team consensus approach’ to assessment where individual assessors accumulate evidence to make holistic judgments based on the standards, and then meet to reach consensus.

Critiquing the process

There have been a number of critiques of the NBPTS process (Delandshire & Petrovsky, 1998). Researchers have argued that teachers involved in developing professional standards (i.e., those ‘insiders’ who have been members of the small groups charged with the responsibility of formulating standards on behalf of the profession) relate to the standards in a fundamentally different way from teachers who receive them (i.e., those ‘outsiders’ whose performance must henceforth be judged by standards that have been formulated by others; cf. Petrosky, 1998). Another body of research argues that such documents, when they are lifted out of the context of the discussions that produced them, inevitably read like a set of bureaucratic requirements which teachers perceive as being fundamentally alien to their authentic professional commitments (Clandinin & Connelly, 2000). Other researchers have noted the way professional standards have been interpreted as culturally-loaded statements formulated by middle class educators in middle class schools, with almost no relevance to teachers working with minority groups (Petrosky, 1998).

In response to some of these concerns, Monash University is currently involved in a follow-up project funded through an Australian Research Council (ARC) Linkage grant that aims to examine critically key assumptions relating to professional standards and the evaluation of teachers’ professional practice by portfolio assessment in Mathematics and English. The preparation of portfolios as a means of demonstrating accomplishment of professional standards is becoming an increasingly common practice, as is demonstrated by the NBPTS, but such processes still require scrutiny if they are to be successfully implemented.

Australia currently provides a unique opportunity to gain insights into the value of subject-specific professional standards and portfolio preparation for teachers, as professional standards have been developed independently by Mathematics, English and Science teacher associations, in comparison with the United States where the NBPTS took carriage of the process. Although documentation exists about the professional growth experienced by US teachers when preparing portfolios for certification, this tends to take the form of stories told by teachers who have achieved certification. For teachers who receive NBPTS certification, the standards have validity precisely because these teachers have been deemed to have accomplished them (Buss, 2000; Ingvarson, 1998). They have accepted the formal protocols for certification developed by the NBPTS, involving certain assumptions about standards and their measurement. Otherwise they could not access the professional rewards that the NBPTS makes available to them.

An especially innovative aspect of the current Monash project relates to the way that it explores the views of ‘outsider’ and ‘insider’ teachers with respect to the *Standards* and assessment of portfolio items. The ‘outsider’ teachers will construct accounts of their professional practice in the form of portfolio items to be read by ‘insider’ assessors. The research will record how the ‘outsider’ teachers and ‘insider’ assessors evaluate any particular portfolio items as representative of their nominated domains of accomplishment. The research will thereby provide valuable data on professional standards, the ways in which teachers might demonstrate their achievement of those standards, or if indeed the nominated domains have validity as areas of teacher accomplishment, and the processes

in which assessors engage when making judgements about evidence of accomplishment and its measurement.

Some early results

While it is early stages in relation to the analysis and publication of results, one of the important findings of both projects is that teachers can make consistent judgements on the evidence provided using the standards as the criteria. Of course more rich data will emerge from each of these projects particularly in relation to the perspectives of different participants, but at this stage at least it is clear that the process provides opportunities for both candidates and teacher assessors to reflect on their own teaching.

Concluding comments

In this paper, I have outlined some of the directions that are being taken in the Australian context in relation to the development of a standards-based process for the assessment of highly accomplished teachers of mathematics. There are many complexities in this process. As well as the complexities of evaluating teaching, there are the multiple agendas of those involved. Resolving or just managing the professional, the political and the systemic contexts is difficult. There are many unanswered questions and on-going challenges. Having said that, it is a journey that has been and continues to be 'worth taking'. Having the opportunity to engage with teachers in extended discussion in relation to excellence has been a privilege. Importantly, in the way standards development and implementation has been conceptualised by AAMT, the profession – the teachers of mathematics – are in the forefront. The voice of the teacher has been the loudest.

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Written algorithms in the primary years: Undoing the ‘good work’?

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The teaching of conventional written algorithms in primary schools dominates the curriculum with concerning effects on both student understanding and self-confidence. In this paper, I summarise research findings and the opinions of key writers, with particular emphasis on the potential dangers of introducing conventional algorithms too early, and share research data from a follow-up study to the Victorian Early Numeracy Research Project. I make the argument that there is far more important work to be done in these years in developing concepts and strategies for mental computation, and offer some practical suggestions.

Advantages and disadvantages of teaching algorithms in the early years of primary school

Given that the teaching and regular practice of written algorithms is widespread in primary classrooms, there must be a number of reasons for their prominent role in everyday mathematics activity. Plunkett (1979), Thompson (1997), Usiskin (1998) and other writers offered several reasons for this. These included:

- algorithms have been traditional primary mathematics content around the world for many years;
- algorithms are powerful in solving classes of problems, particularly where the computation involves many numbers, where memory may be overloaded;
- algorithms are contracted, summarising several lines of equations involving distributivity and associativity;
- algorithms are automatic, being able to be taught to, and carried out by, someone without having to analyse the underlying basis of the algorithm;
- algorithms are fast, with a direct route to the answer;
- algorithms provide a written record of computation, enabling teachers and students to locate any errors in the algorithm;
- algorithms can be instructive;
- for the teacher, algorithms are easy to manage and assess.

These, at first glance, appear to be a powerful list of reasons why the teaching of conventional written algorithms has been traditional and should continue in the primary years. However, a number of writers have identified potential dangers to teaching conventional written algorithms to primary children (Kamii & Dominick, 1998; McIntosh, 1998;

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Northcote & McIntosh, 1999; Thompson, 1997; Usiskin, 1998). These can be summarised as follows.

- They do not correspond to the ways in which people tend to think about numbers; for example, in the context of use of most conventional algorithms, the '4' in the number 547 is treated as '4' and not '40'.
- They encourage children to give up their own thinking, leading to a loss of 'ownership of ideas'. The original purpose of algorithms in previous centuries was for clerks to be able to carry out a large number of calculations in a short period of time. Thinking was not the focus, but rather quick and reliable answers. Technology changes the relative importance of algorithms — some become more important, some less important. Most clerks today, given a large number of calculations, would use either a calculator or pre-prepared spreadsheet to carry out these calculations.
- The traditionally-taught algorithms may no longer be the most efficient and easily learned. There is evidence (e.g., Groves & Stacey, 1998; Shuard, 1990) that children in classrooms where the regular use of calculators for conceptual development is encouraged may develop an algorithm for subtraction that blends place value understanding and negative numbers (e.g., for $354 - 278$, a child might use an algorithm that involves the following steps: $300 - 200 = 100$; 50 take 70 is -20; 4 take 8 is -4; so the answer is 100 take 20 take 4, giving 76).
- They tend to lead to blind acceptance of results and over-zealous applications. Given the focus on procedures that require little thinking, children often use an algorithm when it is not at all necessary. Hope's (1986) example of finding $\$100 - \99.95 using the conventional algorithm is a classic case of this.

There is also the issue of relevance. Adults use formal written computation for only a small proportion of their calculations. Northcote and McIntosh (1999) found, in a survey conducted with two hundred adults over a twenty-four-hour period, that only 11.1% of all calculations involved a written component. It has become increasingly unusual for standard written algorithms to be used anywhere except in the mathematics classroom. Most calculations required only an estimate. They also found that for 60% of all calculations, only an estimate of the correct answer was needed. The ways in which conventional algorithms are traditionally taught discourage the application of number sense by estimating first or assessing the reasonableness of the answer afterwards.

An example from research of the possible detrimental effect of teaching written algorithms in the early years on children's mental strategies and number sense

Narode, Board and Davenport (1993) conducted a year-long study of nineteen first, second and third-grade students, involving videotaped interviews. All students were asked to solve two-digit addition and subtraction computations embedded in simple story problems and in familiar contexts, such as stones or marbles. The students were asked to solve each problem, first using base 10 blocks and then mentally or with paper and pencil as they chose. The students were also asked whether they knew of any alternative ways to solve the problem.

Interestingly, almost all children interviewed before instruction in addition and subtraction algorithms used invented strategies which used traditional, front-end approaches (not the usual right-to-left order).

The researchers discussed the case of Jamie (a second grade girl), who was interviewed on several occasions during the school year. It is important to note that Jamie had not met

conventional algorithms prior to second grade, but was introduced to them during second grade.

Early in the school year, Jamie successfully added 19 and 26 mentally: 'I know I have 30 because I have a group of ten and two more tens. Then if I take 1 from the 6 and give it to the 9, I'll have another group of 10. That leaves five left, so the answer is 45.' After five months of school and work with conventional algorithms, Jamie attempted to add 34 and 99 by beginning to group the 9 tens and 3 tens, then stopped and said, 'Oh, I have to add the ones first.' She then grouped the units, and traded for a ten to solve the problem.

In the last month of the school year, asked about the possibility of solving the problem by adding the tens first, Jamie emphatically stated, 'No, you never add the tens first.' She was invited to suggest another way that the problem could be solved. Jamie suggested that another way to solve the problem might be to know the answer from memory. Finally, she was confronted with her own invented strategy, from earlier in the year, as a strategy 'someone used' to add $49 + 19$ ('I think of $50 + 19$ and then subtract one to get 68'). When asked if she thought this method might work, she replied 'If you know that way, it's okay, but it's much, much better to just add the ones first.' (p. 259)

This one example is a powerful demonstration of the way in which a child can move from trusting their understanding of numbers and flexible strategies to following a single procedure without much hesitation. Narode, Board and Davenport (1993) summarised their findings:

We believe that by encouraging students to use only one method (algorithmic) to solve problems, they lose some of their capacity for flexible and creative thought. They become less willing to attempt problems in alternative ways, and they become afraid to take risks. Furthermore, there is a high probability that the students will lose conceptual knowledge in the process of gaining procedural knowledge (p. 260).

The *Early Numeracy Research Project* (ENRP) was established in 1999 by the (then) Victorian Department of Education, with a Prep to Grade 2 mathematics focus.

The ENRP became a collaborative venture between Australian Catholic University, Monash University, the Victorian Department of Employment, Education and Training, the Catholic Education Office (Melbourne), and the Association of Independent Schools Victoria. The project was funded to early 2002 in thirty-five project ('trial') schools and thirty-five control ('reference') schools.

There were two main features of the ENRP that made it different in important ways from previous projects: the ENRP growth points and task-based interview. The framework of growth points provides a means for *understanding* young students' mathematical thinking in general, and the interview provides a tool for *assessing* this thinking for particular individuals and groups. The main project concluded in early 2002 (for further information, see Clarke, 2001), but the focus here is on a small follow-up project and the data which emerged from it.

It was agreed that in November 2002, a sample of Grade 3 students would be interviewed by the research team at Australian Catholic University, to be followed by another interview period in November 2003, with Grade 4 students. These students would be chosen from those who had participated in six previous interviews, at the beginning and end of the Prep, Grade 1, and Grade 2 years. Six hundred and thirty Grade 3 children

were interviewed in 2002, and five hundred and seventy-two of the same children (now at the end of Grade 4) were interviewed in 2003.

The mathematical domain of *addition and subtraction strategies* provided data that was of some concern, in relation to the traditional emphasis on teaching written algorithms in Grades 3 and 4.

The six growth points for the domain of addition and subtraction strategies are:

1. Count-all (two collections)
Counts all to find the total of two collections.
2. Count-on
Counts on from one number to find the total of two collections.
3. Count-back/count-down-to/count-up-from
Given a subtraction situation, chooses appropriately from strategies including count-back, count-down-to and count-up-from.
4. Basic strategies (doubles, commutativity, adding 10, tens facts, other known facts)
Given an addition or subtraction problem, strategies such as doubles, commutativity, adding 10, tens facts, and other known facts are evident.
5. Derived strategies (near doubles, adding 9, build to next ten, fact families, intuitive strategies)
Given an addition or subtraction problem, strategies such as near doubles, adding 9, build to next ten, fact families and intuitive strategies are evident.
6. Extending and applying addition and subtraction using basic, derived and intuitive strategies
Given a range of tasks (including multi-digit numbers), can solve them mentally, using the appropriate strategies and a clear understanding of key concepts.

The data for the relevant growth points for Grade 4 students at the end of the school year are given in Table 1.

Table 1. Percentage of Grade 4 students achieving each ENRP growth point for domain of addition & subtraction strategies (November 2003).

Growth points	Reference schools % <i>n</i> = 174	Trial schools % <i>n</i> = 398
0		
1	0.6	1.0
2	12.6	8.0
3	7.5	4.8
4	20.1	25.4
5	46.6	41.2
6	12.6	19.6

If the increasingly-common argument that students should not be taught conventional written algorithms until they are able to add and subtract two-digit numbers in their head (which underpins the latest UK mathematics curriculum is accepted, then this means that around 40% of trial school and reference school students were not ready for this content by the end of Grade 4. These are the students who have not yet grasped both basic and derived strategies, with many counting by ones for all such problems. This is not necessarily a criticism of the teaching or the students, but may say something about readiness for these ideas.

It is also worth noting the statement in the most recent New South Wales Mathematics syllabus (Board of Studies New South Wales, 2002) that ‘formal written algorithms are introduced after students have gained a firm understanding of basic concepts including place value, and have developed mental strategies for computing with two-digit and three-digit numbers’ (p. 9). I strongly agree with this position.

Of course, students can be increasingly encouraged to record the various steps in their calculations, in ways that make sense to them. The danger is not so much with the written form, but the imposition of the teacher’s method for recording, which, as was shown earlier, can have unfortunate consequences. In this way, students are developing and gradually refining their own invented algorithms, in conversations with their peers and the teacher.

Are any student-invented algorithms okay?

It is a natural process for children to record their thinking on paper, as the numbers become too large for everything to be retained in their head. As students start to develop their own algorithms, a question arises: are any student-invented algorithms acceptable? How should children’s invented algorithms be treated in the classroom?

Early in school, given that the algorithm leads to a correct answer, the answer is probably, ‘Yes: they are okay’; but over time, we want to encourage children to consider whether the procedures are:

- efficient enough to be used regularly without considerable loss of time;
- mathematically valid;
- generalisable (can the algorithm be applied to the full range of problems of the type being solved?) (Campbell, Rowan & Suarez, 1998).

Occasionally, teachers claim that, ‘only the brighter children can create their own algorithms’. Those involved in projects that encourage children to create their own algorithms dispute this, but even if it were true, the encouragement for children to do so will likely yield a range of algorithms. These can be shared publicly and discussed, and children who are unable to create a written method of their own will at least have a range of options from which to choose for their own use.

When should conventional algorithms be presented to students?

I believe that there is no place for formally introducing conventional algorithms to children in the first five years of school. If they arise during classroom problem solving (and they almost certainly will, given the input of parents and siblings into the process), they can be considered and discussed.

By giving arithmetic a problem solving focus, and by providing a whole range of problems for children to solve (preferably in story contexts of interest to children), we redefine the role of students, in the words of Lampert (1989), from the task of ‘remembering what to do and in what order to do it, to a problem of figuring out why arithmetic rules make sense in the first place’ (p. 34).

The cognitively guided instruction (CGI) problem types (e.g., Fennema, Carpenter, Franke, Levi, Jacobs & Empson, 1996) provide one basis for creating such story problems. My suggestion would be to use a variety of such problem types, with increasingly large numbers, challenging children to solve them, by any method that makes sense to them. Through sharing their methods, children can make a start on the process of evaluating various methods for their mathematical validity, their efficiency, and their generalisability.

ty, though not in these terms! In time, when they meet conventional algorithms (in upper primary, if at all), they will be in a strong position to compare the various possibilities on a fair basis, without feeling pressured to discard all that they have learned.

Concluding thought (from 1830)

The learner should never be told directly how to perform any operation in arithmetic... Nothing gives scholars so much confidence in their own powers and stimulates them so much to use their own efforts as to allow them to pursue their own methods and to encourage them in them (Colburn, 1912, p. 463).

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Supporting the Standards for Excellence in Teaching Mathematics in Australian Schools through the Specialist Schools Network in England and iNET, International Networking for Educational Transformation

Graham Corbyn

Mathematics and Computing Specialist Schools Trust, UK

The Specialist Schools Trust consists of a network of over 2300 secondary schools in England, the majority of which have mathematics as a specialist subject and support the drive to promote innovation in mathematics education not only within individual schools but across local and wider communities as well.

The international developments in mathematics recently have been highlighted by a number of exciting initiatives in New York, Bulgaria, South Africa and Copenhagen. As a result of iNet and the Transformation through Global Networking Conference in Melbourne during July 2004, many opportunities to embark on new ventures leading towards innovation in mathematics with Australian schools became apparent and through this paper I hope to be able to create a new partnership with Australian teachers to share ideas across the globe which will support mathematics education internationally.

This paper describes briefly the strategies adopted in specialist schools to promote mathematics with an overview on teaching and learning; effective use of ICT; the relevance of mathematics and its role in the work place as well as in local and wider communities. I finish with the issue in England regarding the role of statistics in mathematics and propose this area as a focal point to begin our discussions in light of our new partnership with the Australian Bureau of Statistics (ABS). It is intended that much of this paper will relate closely to the notions within the *Standards for Excellence in Teaching Mathematics in Australian Schools* as published by the AAMT in January 2002 and through the partnerships that have been established through iNET in Australia, the ABS and the AAMT, I hope this paper will promote discussions that will lead towards a collaborative strategy taking global transformation in mathematics education one stage further.

Specialist schools and iNET

The Specialist Schools Trust is the lead body for the Government's specialist schools programme. Our mission is to build a world-class network of innovative, high performing secondary schools in partnership with business and the wider community.

The Trust is at the head of a network of over 2000 affiliated secondary schools, including the majority of the 1955 specialist schools which we believe is now the largest affiliated network of schools in the world. Our task is, to use the power of that network to make those schools increasingly effective and successful. Our strength lies not only in our activ-

ity at the centre but in the power of the network itself and its ability to connect school with school to drive forward a common ethos and focus on a program of continuous improvement and leadership. Our two key words are ‘excellence and diversity’. We would like each of our schools to flourish in its particular community.

To become a specialist school, schools need to go through a demanding application process. Schools can apply for any of ten specialisms or if they wish a combination of any two specialisms. If successful, schools are given extra funding from the government to develop their chosen specialism with the aim of raising whole school achievement. As well as an annual grant, there is also a one-off capital grant generally used to re-furbish and/or build new classrooms. The application involves presenting a four year action plan specifically stating targets which are to be met through a number of objectives linked to the specialist subjects and ‘themes’ including a focus on teaching and learning and developing the community dimension. These targets are reviewed at the end of each four-year phase when a similar application is written so that schools can be considered for re-designation for a further four years. The vision and ethos presented by the school has to be of the highest order and work towards individual schools or even federations of schools becoming a recognised centre for excellence in any given region, providing a wide range of opportunities geared for all in their school and in their communities.

From a mathematics point of view, the specialist school movement has proved to be significant in terms of specialist schools being at the ‘cutting edge’ in mathematics. Table (1) shows the number of schools and their specialism over the last few years.

Table 1. The number of schools with specific specialisms.

Specialism	September 2002	September 2003	September 2004
<i>Arts</i>	173	230	305
<i>Business and Enterprise</i>	18	81	146
<i>Engineering</i>	4	14	35
<i>Humanities</i>	–	–	18
<i>Languages</i>	157	189	203
<i>Mathematics & Computing</i>	12	77	152
<i>Music</i>	–	–	5
<i>Science</i>	24	121	225
<i>Sports</i>	161	229	283
<i>Technology</i>	443	503	545
<i>Combined</i>	–	10	38
Total	992	1454	1955

Mathematics is obviously a key focus subject in all mathematics and computing schools and the fact that since its introduction in 2002, this particular specialism has seen the largest percentage increase of all the specialisms, reinforces the view that mathematics is seen as a significant subject area. Arts, languages, sports and technology colleges started before 2002, which is why numbers are higher in these specialisms.

Furthermore, it is essential to understand that there are typically two or more focus subjects in each specialism. With this in mind, mathematics is a focus subject in business and enterprise, engineering, mathematics and computing, music, science, technology and most combined specialism schools. Therefore from September 2004, over 1000 schools will have set challenging targets to develop the teaching and learning of mathematics in their school and across their communities. This is almost one third of all

secondary schools in England and the first subject to have reached this milestone. Its inclusion in such a large number of schools and specialisms supports the notion that mathematics underpins many areas and is such an integral part of any individual's education.

Mathematics and computing specialist schools (MCSSs) are expected to raise standards and develop further interest in mathematics across the whole ability range, which will lead to whole school improvement. This includes a clear focus on developing innovative approaches in the teaching and learning of mathematics as well as sharing good practice across the curriculum particularly through considered and effective use of ICT. In a general sense, all schools should have similar aims, and good practice is evident in mathematics departments of other specialisms, but the MCSSs are at the forefront of development and will promote new ideas and strategies for all those schools in the specialist schools network by presenting and sharing ideas through a range of conferences, network meetings and seminars arranged by the Specialist Schools Trust. The notion of all specialist schools having a key role within mathematics has been endorsed fully by the Specialist Schools Trust and the success of the first mathematics conference in April 2004 and the presence of the Secretary of State for Education recognised this fact.

Schools, without doubt, respond to achieving specialist status with pride and are thrilled to be involved and have opportunities to network nationally across a wide range of themes. Of course, it is impossible as Subject Leader for mathematics and computing, to be able to offer support to every single school or individual mathematics teacher and working with key strategic partners including the mathematics associations, societies and government is a main factor in the success of the mathematics and computing specialism. Maintaining the links with these partners is through our introduction of the *Mathematics Expert Panel, Head Teacher Steering Group for Mathematics and Computing Schools* as well as our *Lead Practitioner Network for Mathematics* as briefly described later in this paper. To be able to expand this network internationally would be fantastic. For example, the *Young Leaders in Mathematics in Australia* working with lead practitioners in England could prove to be a very powerful collaborative experience!

To complete this overview, the Specialist Schools Trust is committed to working collaboratively globally. Typically this is carried out through iNET whose mission is 'to create powerful and innovative networks of schools that have achieved or have committed themselves to achieving systematic, significant and sustained change that ensures outstanding outcomes for all students in all settings'.

iNet (Australia), as it develops (as part of this global network), is committed to strengthening the capacity of school leaders to work across sectoral boundaries and further details on iNET Australia can be found at:
www.sst-inet.net/countries/australia/default.aspx

Standards for Excellence in Teaching Mathematics in Australian Schools (as adopted by the AAMT Council Meeting in January 2002)

The document as referred to above, covers three domains:

- professional knowledge;
- professional attributes;
- professional practice.

With reference to the statement, it is clear that much of what is expected of teachers in Australia is similar to what is expected in England. To possess a strong knowledge base and to be able to draw upon a range of teaching and learning strategies is paramount.

Planning is also essential so that teachers can ensure that each individual is making progress and developing an understanding of mathematics. For this reason understanding how mathematics is learned is vital.

The teaching and learning of mathematics in specialist schools is a top priority and indeed it is what happens in the classroom that will raise standards in mathematics. A range of strategies has been adopted in England in recent times to ensure that effective teaching and learning is taking place. Examples include the adoption of three part lesson plans based around mental starters, a core objective and a plenary. Each lesson encourages an interactive approach through the use of a variety of resources and ensures that students have ownership of what they have learned and are involved.

There has been a big focus on question and answering so that teachers can effectively assess student understanding and continue the lesson in response to individual or class needs. There has been a move away from use of textbooks in mathematics simply to encourage students to collaborate and share their ideas through group discussions. In essence, this has been extended into cognitive acceleration programs such as *Cognitive Acceleration in Mathematics Education (CAME)*, published by Kings College, London which encourages students to think about specific problems and begin to build bridges between similar problems to improve their understanding. Of course, intervention strategies play a part in this but it is a skill for the teacher to manage such groups so that students generate the findings for themselves and then have the confidence to share with others.

Professional development of teachers has to be continuous rather than one-off courses and teachers given time to review and evaluate. It is felt in England, that mathematics is special and teachers of mathematics require more specific training courses that will provide them with extended opportunities to try new strategies and provide them with the chance to try new resources in the classroom, especially in the use of the latest technologies. It is also clear that teachers need time to practise new ideas and also be given opportunities to share with others what worked well or what did not work well. The British Government is committed to setting up a national infrastructure for professional development of teachers of mathematics in the form of a National Centre for Excellence for the Teaching of Mathematics (NCETM), which will oversee a number of Regional Mathematics Centres (RMCs).

The Specialist Schools Trust's mathematics networks are heavily involved in professional development of teachers of mathematics and indeed currently run training courses for mathematics teachers run by mathematics teachers in schools that have been awarded regional training status. In essence, teachers have designed a number of 'toolkits' consisting of mathematics resources that are shared and discussed during specific training days. Once a school has received the 'toolkit' training then it is able to use the 'toolkit' to support the teaching and learning of mathematics across their communities as necessary.

The community dimension (as referred to in 2.3 of the *Standards*) is a major part of any specialist school plan. Typically about one third of the specialist school funding has to be linked to the community plan. The responsibility of specialist schools is to work with both the local community and wider communities, including developing international links if appropriate. The community plan is equally as important as the school plan in any specialist school. Therefore, MCSSs are fully committed in becoming active partners in a learning society with their local schools and communities. The community dimension is divided into working with a number of local partner primary schools and at least one partner secondary school as well as actively engaging and working with universities, business and industry to promote the use of mathematics and computing outside of school. The wider community dimension includes international developments and examples have included mathematics teachers attending conferences in Copenhagen as well as

video-conferencing to schools in South Africa.

The community dimension also includes working with adult communities. Examples include schools working with single mothers, Muslim women, the prison service, unemployed, small businesses and so on. In many of these cases, mathematics is delivered to support specific mathematical needs, so for example, some communities require support in understanding personal finance and calculating annual percentage rates whereas other communities require low level numeracy skills.

The new technologies that are now available have without doubt revolutionised the teaching and learning of mathematics. To be responsive to these new technologies forms part of domain 3, professional practice, within the *Standards* statement. Specialist schools are fortunate that they are able to use some of their funding to buy new technologies although they have to justify why and how it is to be used and have to build in a training programme for teachers to ensure that the new technologies are successfully implemented. Over the last couple of years, there has been a move towards effective use of information and communication technology (ICT) to support the teaching and learning of mathematics. It is usual to see a number of interactive whiteboards in mathematics departments in specialist schools but there is a question of whether it is being used effectively. It should not just be a projection screen or an expensive board to write on, but a tool which can be used to promote mathematics as an exciting and fun subject, yet encourage students to interact with the board to generate discussions and gain a more profound understanding of mathematics. The advent of PC tablets has given students the opportunity to interact at their own desks as opposed to walking to the board at the front of the room. This has been successful due to the improvement of wireless connectivity and has also set the scene for small group work and further collaborative strategies at student level.

Graphics calculators are another example. These are no longer just number crunching machines that can draw graphs. Graphics calculators can now be linked to peripheral devices such as rangers, which will plot movement onto a distance-time graph, viewed on an interactive whiteboard. This gives students the opportunity to ‘feel’ what is happening when they move in certain directions at certain speeds and therefore provides a real understanding for what is happening. Some applications such as reflection symmetry, coordinate programmes, can be downloaded directly onto calculators from the Internet, providing more opportunities for small group work and discussion.

In July 2004, Leanne Dale (Lead Practitioner in Mathematics) and I had the privilege of visiting the Australian Bureau of Statistics (ABS) in Melbourne and identified possible ways forward in raising the profile of statistics in Australia. It seems that there is plenty of scope for collaboration in developing Internet based resources and expanding the censusatschool project, which has proved successful in Australian schools. The reaction of two Australian students to Leanne’s use of a PC tablet when she showed them how we teach statistics suggested that we have much to share regarding effective use of technology and pedagogy. However, although we seem to do some things well, we do have issues, and the following section of this paper therefore relates specifically to the role of statistics in England and highlights the issues we face. I would be particularly interested in teachers’ reactions in Australia as to what is being proposed.

The role of statistics within the mathematics curriculum and its impact on other curriculum areas

Statistics and indeed any topic related to handling data has always been part of the national curriculum for mathematics. In February 2004, a report looking at issues within mathematics education, *Making Mathematics Count*, recommended that statistics should be embedded across the curriculum and hinted at the removal of statistics from the mathematics curriculum altogether. This paper highlights some of the issues that schools in England are facing as a result of this recommendation.

One key issue is the role of statistics and the interpretation of recommendation 4.4 from Professor Adrian Smith's Inquiry:

The Inquiry recommends that there should be an immediate review by the QCA and its regulatory partners of the future role and positioning of Statistics and Data Handling within the overall 14–19 curriculum. This should be informed by: (i) a recognition of the need to restore more time to the mathematics curriculum for the reinforcement of core skills, such as fluency in algebra and reasoning about geometrical properties and (ii) a recognition of the key importance of Statistics and Data Handling as a topic in its own right and the desirability of its integration with other subject areas.

The *National Curriculum for Mathematics* is divided into four *attainment targets*, one of which is referred to as *data handling*. The notion of having these separate attainment targets, in my opinion, has encouraged mathematics education to see statistics and handling data topics as an isolated area and therefore not fully integrated into other areas of mathematics. The introduction and recent popularity of GCSE Statistics, a qualification based on statistical methods mostly delivered through the mathematics curriculum supports the move to provide statistics as a subject in its own right.

There are a number of issues regarding the GCSE Statistics course based around:

- the different examination structures between GCSE Mathematics and GCSE Statistics;
- the different assessment criteria used in coursework for GCSE Statistics in comparison with GCSE Mathematics.

Briefly, GCSE Statistics enables all students to have the opportunity of reaching the standard grade C mark whereas this is not true for GCSE Mathematics. It can be argued that GCSE Statistics is popular simply due to the structure of the assessment and course and little thought into what the subject can offer. The idea that GCSE Statistics supports the GCSE Mathematics handling data coursework if they are run parallel to each other is seen as another benefit. However, in terms of extending mathematics through statistics, this is a different issue and one that is frowned upon by many!

Many students see handling data topics as one of the most enjoyable aspects of mathematics. This is simply because they have opportunities to experiment, research and test particular hypotheses. The practical side of mathematics is one which many feel needs to be developed so that the enjoyment factor can be enhanced. This is why the suggestion to remove statistics from the curriculum has been met with surprise. Although time to deliver the curriculum is an issue, the removal of this aspect could prove to be serious in terms of raising the profile and enjoyment of mathematics at a time when it needs to be seen as an exciting subject relative to everyday life. Although enjoyment is a key issue, it has also been suggested that more algebra and geometrical reasoning be added to the curriculum to help provide students with the skills needed to move on into universities to

study mathematics related courses. In some minds, the removal of statistics and replacing it with algebra is just too much to bear!

However, it is widely recognised that statistics, in particular, does have a broader role and elements of statistical analysis are required in many other subject areas. Although, the advantages of delivering statistics to relevant curriculum areas can be seen, the issue in the UK is who will be expected to deliver the statistics elements relevant to each curriculum area? The shortage of mathematics teachers in the UK is common knowledge, so is it realistic to expect more mathematics teachers to be involved in supporting the teaching and learning of statistics across the curriculum?

One thing is clear: statistics does have a role within the mathematics curriculum and across the curriculum. The issue is how it can be integrated effectively and how each aspect can be delivered and assessed. A new infrastructure will need to be put in place to accommodate the changes and only when the proposed National Centre for Excellence in the Teaching of Mathematics (NCETM) (Recommendation 6.12) is in place, will we be given some idea of the future direction of mathematics education in England.

Teaching trend and regression with computer technologies*

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This paper describes the use of technology for teaching and learning trend and regression. We discuss findings in the research literature and report the introduction of the regression principle using two Java applets, in one of Craig Davis' Year 12 classes. As well, we describe the use of graphics calculators for work on regression in the class.

Introduction

Computer technologies, including hand-held graphics and CAS calculators and the internet, are used increasingly in classrooms, and the benefits for teaching and learning algebra and calculus are well documented (e.g., Davis & Forster, 2003). This paper addresses the benefits for teaching and learning statistics, which are less well recognised. In particular, we discuss instructional approaches to trend and regression. Findings in the research literature and from a recent research inquiry in one of Craig Davis' (first author) classes are discussed.

We focus on graphical approaches and consider four aspects of them: the real-life contexts of the data, the geometric structures in graphed data, the symbols for statistics constructs on the graph, and whether graphs can be manipulated directly on the screen. The four aspects of graphical work are discussed under separate headings in the review, but in practice they are inter-related and influence learning in combination.

The social context of technology-use also strongly influences students' progress. Important social considerations are whether the pedagogy is student inquiry or demonstration by the teacher; and whether open or closed questions are used to guide students' thinking. The social aspects of technology-use are beyond the scope of this paper but detailed accounts are provided in Ben-Zvi and Arcavi (2001), McClain and Cobb (2001), and Davis and Forster (2003).

Literature on teaching and learning trend and regression

Real-life contexts

Ainley (2000) reports that, prior to any instruction on trend, eight-year-old students

* This paper has been accepted by peer review.

recognised trend relationships on a scatter plot of (age, height) data that they produced on spreadsheets. Explanations for why they perceived trend easily were: the heights of points above the horizontal axis indicated the height measurements, so literal interpretation of the graph was valid; and students were familiar with age and height changing continuously, so inferring the existence of points between the points shown on the scatter graph (for a continuous trend relationship) made sense.

Ben-Zvi and Arcavi (2001) report that a class of mixed ability thirteen-year-old students were slow to perceive the trend in Olympic times for the men's one hundred metre sprint. The students were provided with the fastest times for the sprint in each Olympics in the twentieth century. They graphed the data on spreadsheets but a lot of guidance was needed before the perceived trend was relevant. The problems were that the data were discrete, so a continuous trend was counterintuitive, and sprint times for successive Olympics tended to decrease but did not always decrease (i.e., they showed local irregularity).

Doerr (1999) found that Grade 10 and 12 students rejected trend was relevant for analysing data that were clearly random. They generated the data by turning out M&Ms in a cup, counting the number of Ms showing, and adding this number of M&Ms plus the original ones to the cup, and repeating the exercise. The students were asked to plot the results of the trials on their graphics calculators and predict the results of future trials. Students did not accept that the regression models that they fitted were valid for prediction because they knew the results of each trial were random within a range. So, the students were reluctant to accept as valid the models which they utilised routinely when randomness in data was not so obvious.

Chu (1996) reports a study where undergraduate students developed regression models for pricing diamond rings. Prices of forty-eight rings, and weight in carats of the diamonds in them, were provided. Typically, students fitted a linear model to the data and were uncritical of its limitations. They did not explain the negative intercept, which suggested that the price for a ring without a diamond was negative. Discussion led to establishing that an exponential model was a better fit.

In summary, the contexts of data determine whether data:

- (a) are discrete or continuous;
- (b) consistently increase or decrease, or show local irregularity;
- (c) are clearly random or are not obviously random; and
- (d) display linear or other mathematical patterns.

These characteristics of data, which depend on context, influence the ways in which students understand trend and perceive its relevance. Further, height data support early understanding of trend because of the close physical resemblance between the context and the graph. In all the above studies, the contexts of data were familiar to students, and this familiarity underpinned students' analysis. A theme in the research literature is that using data from familiar contexts is essential for meaningful learning.

Geometric structures in graphed data

Figure 1a below illustrates the geometric structure of the data used by students in Ainley's (2000) study. Each value of the independent variable (age) was paired with a single value of the dependent variable (height) and covariation between adjacent points on the graph was consistently positive. The simple structure meant positive trend was easy to discern. Figure 1b illustrates local irregularity in data, similar to that for data used in Ben-Zvi and Arcavi's (2001) study. The rising path of the points in some parts of the graph hindered students from recognising the decreasing, linear trend. Focussing on specific parts of the

graph prevented them from perceiving the global property. However, removal of an outlier and rescaling the vertical axis resulted in students moving forward in their analysis. Both actions made the graph smoother (see Figure 1c), and the scale-change focussed students' attention on the graph as a whole because the points shifted as a group.

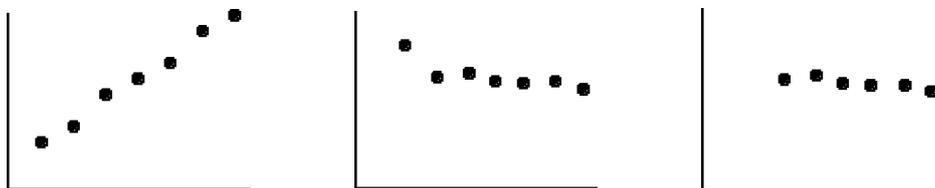


Figure 1. Data with (a) consistently positive covariation, (b) local irregularity and (c) after removal of an outlier and a scale change.

In a study by Cobb, McClain and Gravemeijer (2003), Grade 8 students were asked to analyse data for carbon dioxide concentration in air measured several times each year over twenty-two years, at a given location. Points on the graph were scattered as shown in Figure 2a. Most students identified an increasing, linear trend and could explain that it meant carbon dioxide levels had increased over time. They paid little attention to data above and below their conjectured trend lines. However, in another task with stacked data, which drew the eye in a vertical direction (see Figure 2b), students described trend and variation from the trend, and recognised the practical significance of both.

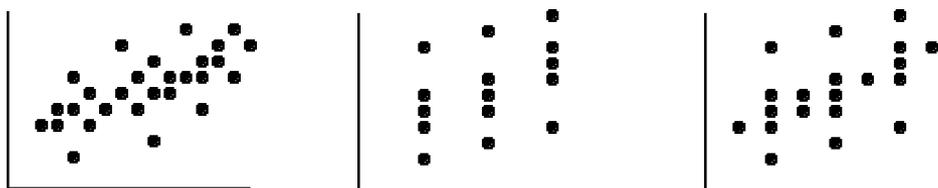


Figure 2. Data showing a positive trend and distribution about the trend.

In Chu's (1996) study, the diamond ring data were stacked as in Figure 2c. Generally, students analysed the data in terms of trend and variation from the trend.

In summary, data with a simple geometric structure — where points are close to being collinear — suits early work on trend. Stacked data can support recognition of trend and variation from the trend. Both ways of perceiving data are fundamental to statistical reasoning (Cobb et al., 2003).

Symbolisation of constructs

In Ainley's (2000) study, students superimposed lines on their spreadsheet graphs using the drawing tool, and/or utilised the 'line graph' capability, so that points on the graph were connected with segments. The lines and segments pointed to a linear trend, students referred to them when describing trend, and they used the segments for prediction. Hence, the symbols influenced the students' understanding of trend. In Ben-Zvi and Arcavi's (2001) study, students objected to connecting points on the graph of Olympic data because they knew predictions between available points were inappropriate (their reasoning was that the Olympics occur only every four years). In any case, utilising segments between points could encourage flawed understanding because they relate to particular points, whereas trend applies to data as a whole.

Specialised software was used in the Cobb et al. (2003) study and several tools were available for manipulating the graphs. For instance, students could partition data into vertical slices, and bars appeared at the extreme and median values for data in each slice (as in Figure 3). The intention was that students would perceive the data were distributed about the median values. However, students used the superimposed structures in visualising trend lines and did not focus on distribution. Using the same tool on stacked data resulted in the intended outcome: students recognised distribution as well as trend.

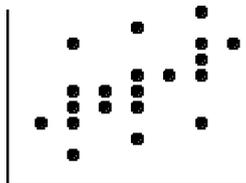


Figure 3. Data within vertical slices partitioned into equal sized groups.

The examples highlight that superimposed segments, lines through data, and lines that partition data can assist students perceive trend relationships. Using the symbols involves manipulating the graphs on the interface of the technologies, and it is widely accepted that the manipulation, as well as observation of the graphical display, contributes to learning (e.g., Ainley, 2000; Cobb et al., 2003).

Teaching linear regression

The comments below are based on systematic research in Craig Davis' 2003 Year 12 Applicable Mathematics class. Lessons were video-recorded; audio-recordings were made of students' one-to-one conversations; work samples and students' assessment scripts were photocopied. The claims in this paper are based on analysis of the video and audio data.

We note that the Year 12 Applicable Mathematics course in Western Australia specifies that students should understand the least squares principle qualitatively, and should study linear and exponential models, and residual analysis (Curriculum Council, 2003). Non-symbolic graphics calculators are required for the course.

Linear regression with Java applets

A three-phased approach, using Java applets and a hand-drawn graph on the whiteboard, was used to introduce the regression principle. Students were familiar with drawing scatter plots on graph paper and fitting lines by eye, but had not previously been taught the mathematics of regression.

The first phase involved whole-class discussion in relation to the Scatterplot applet produced by MATTI Associates (2003). The applet was accessed live on the Web using a laptop and the display was projected onto the whiteboard with a data projector. The display resembled Figure 4a, and summary statistics including the co-ordinates of the mean point (\bar{x}, \bar{y}) were shown at the side. Points could be added to the graph by clicking on it, and the line and statistics were automatically updated.

After adding several points, Craig asked, 'How do you calculate the line?'. The students responded that the line should be drawn through the mean point and its gradient calculated in such a way that the distances were minimised; however, they were unable to

explain how they would calculate the gradient, or which distances were involved.

In the second phase, Craig drew a sketch on the whiteboard showing that vertical distances were used (see Figure 4b) and he asked again how to calculate the line. Students suggested using the absolute values of the distances to the line and the squares of the distances, and to sum these.

The third phase involved demonstrating the squared distances using the ‘least squares’ applet from the National Council of Teachers of Mathematics’ website (NCTM, 2000). The display was similar to Figure 4c, and the numerical expression for the sum of the squares was given under the graph. The effect of adding points to the graph and dragging the line was demonstrated. When asked how to obtain the best line, students said to ‘lessen’ and ‘minimise’ the sum and gave suggestions on how to achieve this (increase the gradient of the line, etc.).

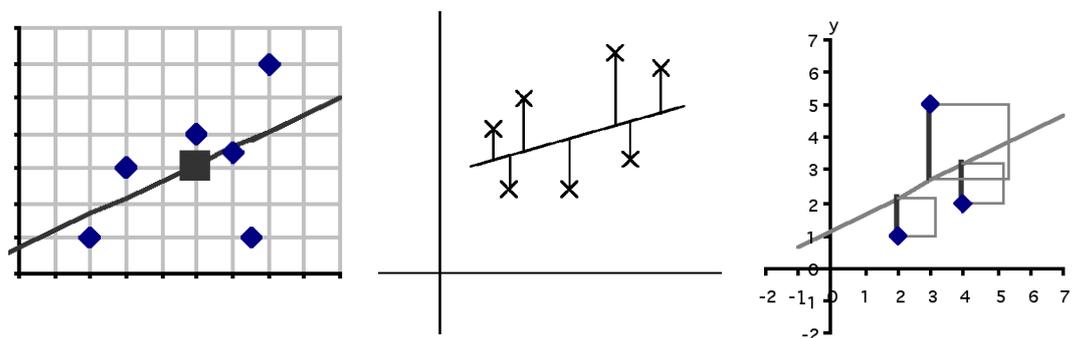


Figure 4. Graphs used for illustrating the least squares regression principle.

In summary, the three-phased approach attracted widespread interest and gave students the opportunity to infer the least squares principle themselves. We suggest the first phase was important because it started students thinking about how to calculate the line, and they were confronted with not being able to come up with a method. The second phase was important for defining the convention that vertical distances are used: students cannot be expected to spontaneously know conventions. The third phase served to illustrate the least squares principle, and gave students the opportunity to consider what minimising the squares involves. Using the NCTM applet only would have been quicker, but would not have allowed students as many opportunities to think through the regression calculation, so we suggest would not have been as conducive to understanding.

Aspects of the approach that supported visual learning were:

- the residuals on the hand-drawn graph (Figure 4b) and squares on the third graph (Figure 4c) pointed to the regression principle;
- the applet graphs could be manipulated; and
- changes in summary statistics and the sum of the squares could be seen.

A limitation of the approach was that students did not interact with the applets directly: learning through manipulating the graphs themselves and through their own inquiry was not open to them. The single computer and single Internet port in the room favoured the pedagogy of demonstration. Another limitation was that data were treated merely as points in space and were decontextualised. The applets encouraged a decontextualised approach because data entry was via the graphs, and scales were not visible (Figure 4a) or were preset on the second (Figure 4c).

Linear regression with graphics calculators

Graphics calculators were used for:

- (a) graphing data in order to view any pattern in them;
- (b) fitting appropriate regression models and predicting off the models; and
- (c) calculating and graphing residuals.

Procedures on the calculators were introduced with data that the class created, then, students used the technology to work application questions. The calculators were not used for introducing the regression or residual concepts, and seem to us to offer little that is useful for introductory work: they do not have inbuilt capabilities for showing the vertical lines for residuals or squares on the residuals, and the graphs cannot be manipulated directly.

All students progressed to being able to produce adequate graphs on their calculators, and to fit curves and predict from them. However, developing students' competence involved considerable class time. One difficulty was the chain of actions to produce a scatter graph on the Hewlett Packard calculators students were using was long, which meant that the production of graphs was initially error prone. Added difficulties were that students could not always decode the error messages, and axes were not generally visible on the graph. The lack of axes made interpretation difficult, particularly judging if points were outliers. Difficulty with determining outliers applied to scatter plots and residual plots.

Operating on column variables to calculate residuals was also introduced. The method is efficient for calculating residuals for multiple data. However, at least initially, some students had difficulty in following the commands for the calculation. Hence, the sophisticated use of the calculator, which was readily taken up by some, may have confused others. In summary, while graphics calculators facilitated graphing and calculation, multiple difficulties were encountered which impacted on students' progress in the statistics topic.

Conclusion

In review, we have focussed on graphical approaches for teaching trend, and described four aspects which impact on students' understanding:

- the real-life contexts of data;
- the geometric structures in graphed data;
- symbols used to indicate trend and distribution about a trend line; and
- ability to manipulate the graph.

Ideally, technology-based instruction takes into account all four aspects.

The MATTI and NCTM Java applets allowed direct manipulation of the graphs, and the NCTM applet displayed the squares that symbolised the least squares' regression. The applets did not favour the use of 'real-life' data. The graphics calculators suited the use of 'real-life data' but direct manipulation of the graph was not possible, and symbols for regression were not available. These properties of the technologies determined that the applets and not the graphics calculators were used for introducing regression; and the single Internet port and computer in the classroom determined that the introduction was by demonstration.

Hence, technologies enable and constrain instruction and critical awareness of their affordances for learning is warranted.

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Take 'em out of the equation: Student understandings of 'cancelling'

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Questioning during interviews with students about their responses to a set of algebra test items revealed a range of understandings of the concept of cancelling. Although most students could apply a 'cancelling' procedure, their explanations revealed a very limited understanding of when to cancel, often coupled with a false use of the concept to include negation of one term by another. This paper discusses some examples that illustrate how students understand the procedure of cancelling and the meanings they ascribe to it.

During interviews conducted as part of a study of students' early algebraic understandings, some students, unprompted, used the term 'cancelling' in two mathematically different contexts. Further questioning revealed a range of understandings, and misunderstandings, of the concept of cancelling and its applicability. Although most of the students could apply a 'cancelling' procedure, either wholly or in part, their explanations revealed a very limited understanding of when to cancel, often coupled with a false use of the concept to include negation of one term by another. This paper discusses some examples that illustrate how students understand the procedure of cancelling and the meanings they ascribe to it. The implications of these student understandings are discussed and some teaching strategies to assist students to understand the concepts behind mathematical terminology are suggested.

Background

Working mathematically includes learning to use the appropriate language of mathematical discussion, that is, to develop a particular 'register' (Pimm, 1987) in which students communicate mathematical ideas. Bills (2002) found that students who have greater mathematical success tend to emulate the speech patterns and use the vocabulary of their teachers as they gain greater understanding. However, as this study shows, the use of the correct vocabulary does not necessarily indicate a deep understanding of the mathematical concept signified by the words. Asking students to explain further what they meant by the terminology used, revealed the existence of underlying concepts that may well hinder their mathematical progress.

Students' algebraic conceptualisations have provided a rich field for research

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(Bednarz, Kieran & Lee, 1996; Kieran, 1992; Kuchemann, 1981). Much of this research has focussed on how students solve equations and the development of their understanding about algebra. To a lesser extent, the research has also considered the difficulties students encounter when learning to deal with algebraic expressions and equations (Kieran, 1992; Matz, 1982). Classroom experience would suggest that students find the concept of algebraic manipulation without a closed end-point a far more difficult idea to grasp than that of solving equations (Hall, 2003).

Hall (2003) examined students' reasoning for the final simplification of an algebraic fraction involving two quadratic expressions. The major concept with which they had difficulty was 'cancelling'. The research described in this paper suggests that this concept presents further difficulties in understanding by students and that these difficulties may be partly attributed to the ways in which the idea of cancelling and related concepts are conveyed by the language used in the classroom.

Understanding of one's natural language structures facilitates algebraic understanding (MacGregor & Stacey, 1994) as students learn to appreciate the syntactical and semantic structure of expressions and equations. Natural language in the classroom acts as a mediator between mental processes and the symbolic systems of mathematics and as a mediator between experience and mathematical concepts (Boero, Douek & Ferrari, 2002). It is through an increasing command and understanding of their everyday language that students are able to comprehend and articulate mathematical ideas and develop a mathematical register. Teachers provide a model of mathematical behaviour, including the use of appropriate vocabulary and linguistic structure that accurately conveys mathematical concepts.

Even the best-planned and clearly-explained lessons do not guarantee that students have learned what the teacher intends. The ideas of teachers are filtered by the students' own experiences, beliefs and behaviours. Therefore teachers need to have an understanding of ways in which their students might conceptualise mathematics in order that they may adjust their teaching so that students may better learn (Even & Tirosh, 2002). One way in which this understanding can develop is to examine the errors that students make by asking students to explain their thinking and by probing the understandings behind the language used. The following excerpts demonstrate how this technique revealed student conceptions of the process of cancelling.

Methodology

The audiotaped interviews, from which these excerpts are taken, were part of a larger study which focussed on the language used by secondary students in Years 8, 9 and 10 describing their responses to a range of algebra items. The items were adapted from selected syllabus examples in the NSW *Mathematics Years 7–10 Syllabus* (BOSNSW, 2001). Some items used by Kuchemann (1981) were also included. Thirty-three students from three different secondary schools in a NSW country town sat a pen-and paper test composed of forty of these items, and administered by their class teachers. From the results of the test, students who were variously successful, and who were willing to be interviewed, were selected. The interview protocol consisted of a set of standard questions based on test items, and these questions were supplemented by the interviewer as needed, depending on the responses of the students. Sixteen of the students interviewed spontaneously used the term 'cancelling' to describe the process they used in four questions which required students to transform expressions. The questions, identified by the numbers assigned them on the test were:

- Question 7: $(a - b) + b$
 (During the interview, students were also asked to respond to other transformations of that expression, namely:
 $a - b + b$; $b + a - b$; $-b + a + b$; $b - b + a$; $-b + b + a$.)
- Question 8: $\frac{4ab}{4b}$
- Question 12: $\frac{2}{a} \times \frac{3}{b}$
- Question 16: $\frac{2}{a^2} \times \frac{5a}{4}$

The responses from the interviews typify the following conceptions of ‘cancelling’:

- cancelling is a physical process;
- terms have to be the same letter or number in order to be cancelled;
- a negative number cancels a positive number of the same value;
- terms cancelled are seen as unnecessary;
- zero and one are confounded.

Each of these is discussed in the following sections. Students have been identified by a number, S1, S2 and so on. The interviewer is identified as I.

Discussion of examples of student explanations of ‘cancelling’

Cancelling is a physical process

The student operates on the appearance of the expression to make it look simpler.

- I: What does cancelling mean?
- S1: Taking it out, to make it simpler.
- I: What do you mean ‘taking it out’?
- S1: Oh... um, I don’t know why, I’ve just been taught like. If you have, if you have 100 divided by 10, you can like, 100 over 10. You can cross out one of those zeros on the top. It’s simplifying it.
- I: Why can you just cross things out?
- S1: I don’t know. It’s a maths thing. I’ve just been taught whatever you do to the top you can do to the bottom.

This student [S1] conceptualises the process as one that involves the mere physical removal of mathematical symbols. There is no connection to any mathematical ideas, but the following of a rule that is partly remembered and applied whenever an algebraic fraction is encountered. This appeal to authority occurred frequently, and is illustrated further in the responses of students S2, S6 and S11.

The following extract, from the responses of student S3, on the other hand, does associate the physical removal of terms with division, but only metaphorically. The use of the expression ‘like division’ indicates that the student does not fully understand that cancelling is, in fact, the mathematical process of division.

- I: You mentioned something about cancelling in Question 8, $[\frac{4ab}{4b}]$. What do you mean?

S3: It means there's one b there and one b there, so you can scratch them both out. It's like dividing them, and there's two fours so you can scratch both them out because they're the same, and there's no a . So you are just left with a .

Terms have to be the same letter or number in order to be cancelled

Student S3 recognised that in order to be 'cancelled' the symbols have to be 'the same'. If the symbols are different, then no cancelling can occur. This comment was articulated many times as the following three examples illustrate.

S4: You can get rid of something because you don't need it, to make the expression, the answer easier... if there are two similar things you can cancel them both out, but if they are different you can't cancel them but if they are the same you can.

Student S2 explained why, in Question 8, $[\frac{4ab}{4b}]$, b and 4 could be removed.

S2: Well, by cancelling them out because they cancel each other out.

I: What does it mean though?

S2: Because you have something that's on the top and something that's on the bottom that are the same. You can cut them out of the equation.

I: Why are you cutting them out? That doesn't sound very mathematical.

S2: I don't know. I have no idea. I kind of just remember what I'm told and use that.

Student S5 could deal with Question 8, but the presence of no identical terms in Question 16 presented a problem.

S5: Six times, oh 6 over ab $[\frac{2}{a} \times \frac{3}{b}]$. I just times the top and put it over the bottom.

The same with number 16, $[\frac{2}{a^2} \times \frac{5a}{4}]$. I do 10 divided by $4a$ squared... yep.

I: Could I do any cancelling in 16?

S5: No, there aren't any like terms, because like a squared. Oh, you could, you could put 10 over $4a$, can you? I think, I dunno.

Student S5 saw, with prompting, that the terms with a could be cancelled, but stopped short of factorising the 4 and 10, or recognising that there was a common factor which could be divided into both 4 and 10. Student S6 on the other hand saw a and a^2 as different terms, unable to be 'crossed off'. This may not be without a certain logic, as students have been taught that a and a^2 are different terms in the context of addition and subtraction.

S6: And 16? Wouldn't have a clue. You have to divide the 2 and a squared somehow and $5a$ in 4.

I: How is Question 16 different from Question 12?

S6: Maybe you could do 2 times $5a$ is $10a$ and a squared times 4 is $4a$ squared, so $10a$ over $4a$ squared.

The concept of cancelling as 'crossing off' identical symbols results in students extending their 'rule' to the elimination of positive and negative terms of the same value. For example, Question 7, $[(a - b) + b]$ on the test paper and the set of related expressions pre-

sented during the interviews, as described above, produced the following types of responses.

A negative number cancels a positive number

The students S8 and S5 had decided that all of the expressions presented as transformations of Question 7 $[(a - b) + b]$ could be simplified to a :

S8: Because you've got a plus b and so when you a minus b , it's minus b plus is just, there's no b , so like that cancels out.

S5: Except for that one $[-b + b - a]$, because the b s are cancelling each other out.

I: What do you mean cancelling each other out?

S5: Because of this one, it goes a minus b and then plus b , so, if you take it away and then put it back again, it, it doesn't do much.

I: Doesn't do much?

S5: No.

This last remark of student S5 that a particular term, if able to be 'cancelled', was unnecessary, was also made by students S4 (see above), S11 (see below), S10 and S9.

Terms cancelled are seen as unnecessary

This would indicate that the students understand expressions as instructions to do something rather than as statements of relationships between numbers. They have what Kieran (1992) calls an operational understanding of expressions.

S10: Well, the ones that have a minus b plus b , the b s like cancel. Like there's no point in minusing if you are just going to plus it again. And, yeah, like the same b plus a and then minus b again. There's no point in putting it there in the first place, you might as well just cancel it out.

S9: For the second one, the answer would just be a , because plus b and minus b sort of cancel each other out.

I: Cancel each other out?

S9: If you have minus b plus one $[b]$ makes zero.

I: Is zero the same as cancelling?

S9: Zero is, it doesn't really count as anything.

I: Zero doesn't count as anything?

S9: Well, you write it... you don't. You don't write it. It doesn't count for anything.

For student S9, there is a perception that zero is 'nothing' rather than 'no number' and so doesn't matter. That is, zero has no mathematical significance. Associated with this dismissal of zero are the statements made by Students S3 (above) and S11 and S12 (below).

Zero and one are confounded

S11: I'd cancel. For those, say 8 [Question 8], you'd cancel the b s and cancel the 4s because they immediately cancel each other out being in the division: b divided by b equals zero and 4 divided by 4 equals zero. Well, technically it equals one, but you don't need to write that.

I: What is the difference between one and zero?

S11: Oh, I don't know. It's in my head. I know it. I know the technical terms and stuff but...

I: If b goes into b once, if 4 goes into 4 once, how do I get zero out of that?

S11: I don't know. It's just the way I've been taught, the way that I know it. It's stored in the back of my head that you don't need that any more. Those cancel each other out and you end up with a .

Further confusion of division with subtraction, felt by student S12, may be because of partly remembered index laws:

S12: These ones? The b s cancel each other out?

I: What do you mean by cancel out?

S12: [Be]cause it's division and b minus b is just nothing, and so it's $4a$ over 4 which... I think you can cancel out the fours to a .

The examples chosen represent responses that indicate important misconceptions or muddled conceptions by students. They need not be statistically representative. Other students interviewed demonstrated similar understandings, but did not express their ideas so explicitly. It was the use of the technical term 'cancelling' that drew the writer's attention to these instances.

Conclusion

Some of the issues raised in these examples are, in no particular order: a student's appeal to authority as justification for the application of a poorly understood procedure; a persistent and pervasive operational concept of the manipulation of algebraic expressions; the tendency of students to abbreviate rules to phrases such as 'do the same to the top and bottom' where the subtleties, clarity and accuracy of meaning are lost; the implications of the use of natural language expressions such as 'get rid of' which implies a physical process to remove something which is unnecessary; and the lack of connections made between mathematical skills, in this case the lack of connected understanding of factorising, division within terms and the significant distinctions to be made between zero and one.

Teachers become complicit in the continuation of some of these misapprehensions by using natural language, interpreting the students' own, often casual lay language and even going along with it. This is not to advocate a pedantic use of strictly formal mathematical language, but rather that teachers develop a greater consciousness of the structure of their own language and the terminology they use or accept from students. Perhaps the use of the troublesome words such as 'cancelling' should be avoided. The word might be replaced with the conceptually more useful phrase, 'divide the same factor into the top and bottom of the expression'. This could avoid the students having to deal with 'like and unlike' terms such as a and a^2 . It would also consolidate their understand-

ing of factorisation and its power in enabling algebraic transformations.

Discussions in the classroom can help reveal student conceptions, but only if teachers listen carefully and avoid the temptation to dismiss (as merely colloquial or vernacular laziness) statements such as ‘[be]cause it’s division and b minus b is just nothing’ or ‘you can take it out of the equation’. Behind these words lie constructs that may well inhibit students’ mathematical progress. If teachers have an understanding of how language works, they may pay greater attention to the way students phrase explanations and what this may reveal about their understanding. They may also take care to model an appropriate mathematical register that ‘provide[s] students with appropriate linguistic expression to fit their thinking process’ (Boero, Douek & Ferrari, 2002, p. 263) to encourage the development of clear, logical, and accurate mathematical ideas.

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SAFEly does it with CAS: Where have we been, where are we going?

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The 'SAFE team' of Sue Garner, Allason McNamara and Frank Moya has used different types of computer algebra systems (CAS), integrated into the senior mathematics classroom in the new subject: Mathematical Methods CAS Units 1–4, as part of the Victorian Curriculum and Assessment Authority (VCAA) CAS Pilot Study. Each of the classes involved has undergone a clearly observable change in the way teaching and learning takes place. This paper focusses on the history, up to the current time, of the use of CAS in the Victorian mathematics classroom, and also looks to the future. The current experience of the author, as teacher, will be shared, with examples of the types of tasks and problems that are suited to a mathematics course where students have unrestricted access to CAS. Approaches to solving these tasks, using hand-held and computer-based CAS, are discussed. The use of computer algebra systems in the teaching, learning and assessment of senior mathematics, as part of the VCAA CAS Pilot Study, has resulted in a strong link between the classroom use of CAS and the high stakes assessment at the end of the VCE. It has been observed that the students using CAS are more confident in exploring previously unseen functions and move more easily between the representations of graphic, numeric and symbolic solutions.

Introduction

The author of this paper maintains that the integration of computer algebra systems (CAS) into senior mathematics, as part of the Victorian Curriculum and Assessment Authority (VCAA) CAS Pilot Study, has resulted in clearly observable changes to the way that teaching, learning and assessment takes place in the final two years of secondary school mathematics in Victoria. A teacher's response is, 'I don't any more teach the big introductory lesson with the notes and heading on top. That's gone. All changed in a year,' (Garner & Leigh-Lancaster, 2003, p. 376).

As CAS can perform algebraic manipulations, such as solving and differentiating a function, one of the most common questions asked by the educational community is whether students will lose 'by-hand' algebraic skills. This is discussed in Flynn, Berenson and Stacey (2002).

Extensive research has already been undertaken during the 1980s about the expected loss or gaining of skills with the free use of calculators in the classroom (AAMT, 1987). The current research extends this question to the use of scientific, graphic and now CAS

technology in the classroom (see Asp & McCrae, 2000). The introduction of the scientific and subsequently the graphic calculator in Victoria (www.vcaa.vic.edu.au/vce/studies/MATHS/caspilot.htm), also resulted in changes to teaching and learning styles, but it is our belief that the symbolic capability of CAS 'allows students and teachers alike to explore algebraic functions in ways that were previously unattainable at secondary school level, and allows teachers to introduce mathematics in different ways' (Garner, 2004a, p. 28).

The concept of the congruence (see Leigh-Lancaster, 2000) between teaching, learning and assessment illustrates how CAS has been accepted into the classroom with CAS not only being used in internal assessment tasks, but also being assumed in 'high stakes' assessment examinations at the end of the VCE in Victoria.

The VCAA CAS Pilot Study

Mathematical Methods CAS Units 1–4 were accredited in February 2001 as a new VCE subject, with pilot implementation beginning in three initial schools covering co-educational and single sex, city and regional, government, Catholic and independent. Seventy-eight students sat the first Units 3 and 4 (Year 12) examinations in this new subject, at the end of 2002. There has been an increasing student enrolment from 2003 to 2005. At 2006 the new VCE Study Design will take over, with an expectation of about a 10–15% uptake across Victoria of the new subject Mathematical Methods CAS. The impact of this will be felt over all three Year 12 mathematics subjects, as those students using CAS in Maths Methods will also be allowed to use them in the technology assumed sections of the exams in the other two subjects; the easier Further Mathematics and more difficult Specialist Mathematics.

The initial stage of the pilot proceeded under the auspices of the CAS-CAT project 2000–2002, with the Department of Science and Mathematics Education (DSME) at the University of Melbourne (see www.edfac.unimelb.edu.au/DSME/CAS-CAT) and VCAA, with the support of three calculator companies: Casio, Hewlett-Packard and Texas Instruments. The expanded VCAA Mathematical Methods (CAS) Units Pilot Study 2001–2005 (www.vcaa.vic.edu.au/vce/studies/MATHS/caspilot.htm) integrates the use of CAS in the curriculum, assessment and pedagogy of a senior secondary mathematics VCE subject. This paper shares some reflections on the change in the approach to teaching and learning that has been experienced in the classrooms of Ballarat Grammar, Methodist Ladies' College and Frankston High School in Victoria as a result of the early journey of using CAS in the classroom. There is now a steady flow of publications about the impact of CAS on the student, the teacher, and on assessment (see Evans, Norton & Leigh-Lancaster, 2003; 2004; Flynn & Asp, 2002; Garner, 2002, 2003, 2004b; Garner & Leigh-Lancaster, 2003; Garner, McNamara & Moya, 2003; Pierce & Stacey, 2002; Pierce, Herbert & Giri, 2004).

Observation of the expanded VCAA Pilot has led to a decision made at the VCAA Board meeting (October 2003) stating that:

...the current VCE Mathematics study structure and relationship between VCE mathematics studies continue for the next accreditation period, with the inclusion of Mathematical Methods (CAS) as a parallel and alternative study to Mathematical Methods, available for all schools from 2006. Thus, from 2006, Mathematical Methods (CAS) Units 1–4 can be implemented by all schools at a time suitable to them. (VCAA, 2004, p.1)

The new Study Design for the VCE to commence in 2006 includes the option of using CAS in all three Units 3 and 4 subjects: Further Mathematics, Mathematical Methods and Specialist Mathematics. A consultation process is currently ongoing about the introduction of a technology free one-hour examination for the latter two subjects (VCAA, 2003).

Where have we been?

It appears that any change in education is twinned with its critics and supporters. There will be teachers at either end of the spectrum: those who absorb change easily, and those who wait until research proves the efficacy of the change, or when curriculum developers make the final decision. Being part of the pilot study has provided an intense journey of learning and changing, both mathematically and technologically. Scientific and graphic calculators have been introduced into the mathematics classroom within the one generation of teachers. CAS now provides either a challenging incentive to change classroom practice yet again, or another reason to stick strictly to a transmission teaching style, passing the knowledge in one direction only from teacher to student. In the authors' experience this 'traditional' style of teaching does not gel with a power balance that is increasingly changing in favour of the student. The viewscreen, not the teacher, becomes the central focus of the class. The students are quick to learn the techniques of using CAS. It could be tempting to teach the calculator techniques as just another knowledge base to be learned. Galbraith (cited in Geiger, Galbraith, Goos & Renshaw, 2002) writes that we, as educators, should be careful of the tendency to merely replace maths with technology, thereby adopting the tyranny of the screen to replace the tyranny of the textbook. Teaching students the 'whiz bang' approach of all the processes that a calculator contains puts the procedural load on the student ahead, and at the expense of, the conceptual load (see Geiger, Galbraith, Goos & Renshaw, 2002).

Issues

The positive aspects for the student of using technology can include concentrating on conceptual aspects, using different representations simultaneously, and allowing realistic modelling. Negative consequences can include a more demanding curriculum, a diminished role for teachers and the loss of by-hand skills (Oldknow & Flower, 1996, pp. 43–46). It can be said that CAS is just another technology tool that follows the same journey of scientific and then graphic calculators, however we consider that this change has more far-reaching consequences:

Considering the scientific calculator using numerical representation, the graphics calculator extending to graphical representation, and the CAS calculator including symbolic representation as well, it could be said that this transition to CAS calculators will likely be more influential. This is because the transition is not just another part of a continuum, but that CAS incorporates all three representations and completes the continuum. (Garner, McNamara & Moya, 2003, p. 255)

Context of three schools

Students in Year 12 at Ballarat Grammar had access to the Casio FX 2.0+ calculator for over two years and sat their final examinations in November 2002 in the new subject Mathematical Methods CAS. Frankston High School and Methodist Ladies' College are

part of the expanded pilot with their students sitting their final Year 12 examinations using CAS respectively in 2003 and 2004. Frankston High students use the TI92/Voyager and the PC-based *Derive*, while MLC students are using the PC-based *Mathematica*. It has been found, in the three schools, that there is a wide variety in methods of learning that occurs in the CAS classroom, and the teachers have also experienced a changed culture in the senior mathematics classroom when students are using the CAS in all aspects of learning and assessment (see Garner, 2003a). Students, as part of the VCAA Pilot Study, learn the use of the Casio FX 2.0+, TI-89, TI-92, *Mathematica* or *Derive*-based CAS, but CAS is ultimately the vehicle for the exploring of the mathematical content. Students are content with learning the calculator skills and syntax as the learning progresses.

A significant example of this is defining a function. All CAS technology has the facility to define a function before a question is attempted, thereby allowing ease of use as the technology differentiates or solves using these functions. This avoids the typing in of a function on several occasions throughout a longer multi-stage problem. It also allows the defining of further functions that have been created in the solution. It has also been found that successful students are fully integrating this new technology into their mathematics learning and that they become masters at selecting when it is wise to use CAS, when it is better to use CAS and 'by-hand' skills combined, and when it is quicker to do a problem by hand alone.

Students

At Ballarat Grammar the 2001–2002 cohort of students in the pilot study were asked questions about their use of CAS. A question and some answers recorded in focus groups in February 2003 are presented here.

- Interviewer: Did you usually use the CAS menu or by hand algebra as your first attempt at an algebraic question?
- Students: Depends really.
- Interviewer: Can we go around and each say?
- Student 1: CAS.
- Student 2: It depended if it was a really complicated... just use CAS.
- Student 3: Definitely the same.
- Student 4: Yeah, it depended on the size of it and if you were going to write it out or just use CAS.
- Student 5: If it was just practice or homework type of thing, to try and keep my by hand skills up by doing it by hand and if I couldn't, do it by the calculator and then by hand; sometimes I just got too lazy.
- Student 6: Well, I'd have to say by CAS; my maths skills are pretty down hill I must say... it's just a confidence thing I thought... yeah.
- Student 7: It would just depend, when you first looked at it if you thought it was worth using CAS or not. Sometimes, I would get half way through a question and I was doing it on CAS and I realised it would be easier to do it by hand anyway.

Student 7's answer is a common response. It is tempting to use the CAS for all questions, but as students become wiser users they become clearer as to where the advantage is in having the open choice of a CAS in their hands at all times.

Colleagues

At the Mathematical Association of Victoria (MAV) Annual Conference in December 2003, a question was asked: why was the initial introduction of CAS described as ‘onerous’? The teacher responded:

Because [name] had the big job. She had to plan all the work and assessments, do all the marking and do all the networking and training by herself. She had to take the school along with her confidence: the school, administration, teachers, students, and parents. And she was delighted with the results. So that has given her the confidence for this year. Which is just as well because I have hounded her all year to ask questions, send students upstairs to interrupt her yet again to ask, ‘How does that particular syntax go?’ (Teacher A, spoken at MAV conference 5 December 2003).

That same teacher went on to say in answer to another question from the floor:

I was terrified because I was new to Methods as well as to CAS and I am well aware that the students know that [name] is the expert. I said to them, ‘Let’s go on a journey together.’ I now know that there were bits I should have emphasised more, but I am looking forward to next year. Yes, I have taught in a really different way and things keep cropping up (Teacher A, spoken at MAV conference 5 December 2003)

As in any innovation in teaching, the level of collegiality is vital in taking a school on a journey of change. In introducing CAS into the classroom, the confidence of teachers in the mathematics faculty, and the parents of the students, is most important, especially for the early parts of the journey. And most importantly the students need to be keen to travel the journey. The high stakes assessment that faces them at the end of VCE is often part of the path to a career that has been planned for some years. This means that students need to feel confident in their chosen subjects, including Maths Methods CAS.

Student voice

Listening to students as they work with CAS is informative and often amusing. These snapshots of student conversation were transcribed in Maths Methods classes during 2004.

- ‘How good is it? I’ve converted [name]. He loves it. [To another student] How do you work it out? In my head. Do you do Specialist? Yes. Well, you’re not allowed to join the party.’
- ‘You can do it [trig equations] by hand which is probably quicker and easier.’
- ‘Can you set the domain for the k value [parameter] in CAS? No, you have to think!’
- ‘I like [name]’s way of using the pi on six scale. That’s nice, I like that. But you have to have an idea of where you are going first.’
- [Talking about dependence on CAS] ‘It starts like that and you end up like grabbing it to get a fix! Can [name] use your CAS? He’s getting withdrawals.’

Where are we going?

Frank Moya, a teacher using *Derive*, is quoted in *The Age* as saying, ‘Traditionally mathematics courses have focused on how to perform particular mathematics procedures and algorithms. With this technology there can be a greater focus not just in how to perform an operation but when you should use that operation and why it works,’ (*The Age*, 2004a, p. 9).

Technology free exam

It is expected that assessments in Victoria from 2006 will have a technology-free component to balance the technology assumed exam. Decisions made at the VCAA Board meeting (October 2003) state that:

The VCAA approved, in principle, the recommendation from the Mathematics Expert Studies Committee that each of Mathematical Methods, Mathematical Methods (CAS) and Specialist Mathematics have a technology free examination and a technology assumed access examination (approved graphics calculator or CAS as applicable)... subject to consultation on models for these examinations with key stakeholders. (VCAA, 2003b, p. 7)

During a by-hand skills test during 2003, students were heard to say:

- ‘Can we just have our CAS please for two minutes?’
- ‘This is really hard: go on Garnsy, just give us two minutes, time us.’
- ‘Watch us fly with our CAS; see how much we can do.’
- ‘You do use by-hand in CAS. You use your hands to type it in!’

Analysis questions

The SAFE team is writing analysis questions for use as supplementary text material for the subject Mathematical Methods CAS Units 1–4. The questions *perfect party balloon* and *grey spot* in this section reflect what the team of authors had to consider when writing material for a course where CAS has changed how teachers teach traditional topics. *Grey spot* also shows how CAS can be used in new material introduced into the Maths Methods CAS course (Garner, McNamara & Moya, 2004b).

Making mathematics connect

Notes taken at the Heads of Faculty meeting held at Ballarat Grammar in June 2004, stated that in using CAS the following advantages are observed:

1. CAS suggests the mathematics (using exact and approximate solutions)
2. Observing patterns in CAS leads to the mathematics
3. The big picture approach is supported by CAS
4. CAS supports a more unified approach to the curriculum
5. Learning algebraic and CAS methods together leads to more complete and flexible understanding of a concept and algorithm. (Morphett, 2004)

The *perfect party balloon* question illustrates the above points, especially the use of exact and approximate solutions to observe patterns, and the use of the big picture to introduce the concept of differentiation. Figures 1 to 9 illustrate the step-by-step solution using the Casio FX2.0+.

Perfect party balloon

A birthday party is planned and balloons are bought. These balloons have been especially ordered to be exactly spherical in shape when blown up. The organisers at the party note that the formula for the surface area (A) of a sphere is $A = 4\pi r^2$.

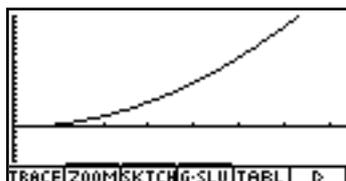
1. What is the exact value of the surface area of the balloon that has a radius of 1 cm?

$$A = 4\pi \text{ cm}^2$$

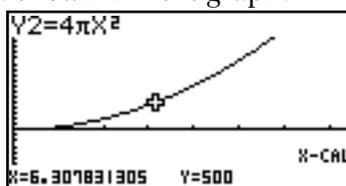
2. What is the exact surface area of the balloon that has a radius of 10 cm?

$$A = 400\pi \text{ cm}^2$$

3. Sketch a graph for $A(r) = 4\pi r^2$, $r \in (0,15)$.



Point (6.31,500) required in 4. shown in next graph.



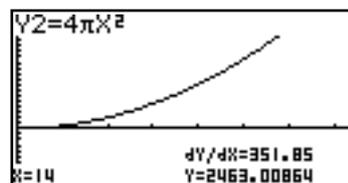
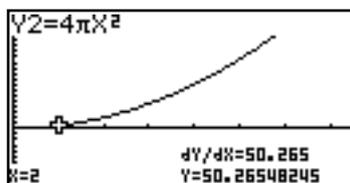
4. From the graph of $A(r)$, calculate the values of r , correct to two decimal places, when $A = 500, 1000, 1500, 2000, 2500 \text{ cm}^2$.

r	6.31	8.92	10.93	12.62	14.10
A	500	1000	1500	2000	2500

The children at the party become bored with their party games and become intent on finding the perfect party balloon. They fill in a table similar to the one below, tracing the rate of change of the surface area as the balloon increases in radius.

5. Fill in their table below by finding the gradient of the curve $A(r)$ at $r = 2, 4, 6, 8, 10, 12, 14$, giving your answers correct to two decimal places.

Screenshots showing the first and last gradient at $r = 2$ and $r = 14$ as required in the table below.



r	2	4	6	8	10	12	14
rate of change of surface area wrt radius	50.27	100.53	150.79	201.06	251.32	301.59	351.85

The children notice that these values increase approximately linearly, a pattern that intrigues them. To further investigate this relationship the children are shown that their calculators can calculate this rate of change value, calling it $\frac{dA}{dr}$, the derivative of A with respect to r .

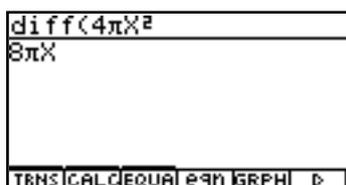
6. Using this $\frac{dA}{dr}$ notation, fill in the table below, using exact answers.

r	2	4	6	8	10	12	14
$\frac{dA}{dr}$ = rate of change of surface area wrt radius	16π	32π	48π	64π	80π	96π	112π

7. Find the general form for $\frac{dA}{dr}$, the derivative of A with respect to r .

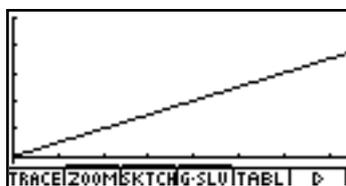
Show that the exact answers in the table above match the formula found.

$$\frac{dA}{dr} = 8\pi r$$



r	2	4	6	8	10	12	14
$\frac{dA}{dr}$ = rate of change of surface area wrt radius	16π	32π	48π	64π	80π	96π	112π
$\frac{dA}{dr} = 8\pi r$	$8\pi \times 2$ $= 16\pi$	$8\pi \times 4$ $= 32\pi$	$8\pi \times 6$ $= 48\pi$	$8\pi \times 8$ $= 64\pi$	$8\pi \times 10$ $= 80\pi$	$8\pi \times 12$ $= 96\pi$	$8\pi \times 14$ $= 112\pi$

8. Sketch the graph of $\frac{dA}{dr}$, for $r \in (0,15)$.

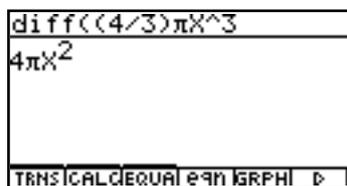


A further relationship is investigated, when the children are told that the formula for the volume of the balloon is

$$V = \frac{4}{3}\pi r^3$$

9. Find the general form for $\frac{dV}{dr}$, the derivative of V with respect to r .

$$\frac{dV}{dr} = 4\pi r^2$$

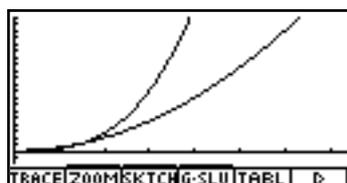


10. Write down the exact answers in the table below for $\frac{dV}{dr}$, for $r \in (0,15)$.

r	2	4	6	8	10	12	14
$\frac{dV}{dr}$ = rate of change of surface area wrt radius	16π	64π	144π	256π	400π	576π	784π

The children notice that these values increase in a pattern that they investigate.

11. Sketch the graph of $\frac{dV}{dr}$, and $V(r)$ on the same axes, for $r \in (0,15)$.



12. Help the children by commenting on the relationship between these two graphs.

Making mathematics easier

The example *grey spot* shows how to set up a piecewise linear function using one equation with the aid of a CAS, some matrices and absolute value functions. The solutions in this section are produced with *Mathematica*, a computer based CAS.

Grey spot

Granny Smith is travelling on a pensioners' bus, from Bendigo to Melbourne. It is supposed to travel at 100 km/h from Bendigo to Kyneton for 48 minutes and then stop at Kyneton for 1 hour. It then is supposed to travel at 90 km/h from Kyneton to Melbourne. This section of the journey usually takes an hour.

1. Write a rule for the distance, $g(t)$ km, Granny Smith is from Bendigo at time t hours if the journey goes as expected. Use a hybrid function.

$$g(t) = \begin{cases} 100t, & 0 < t \leq 0.8 \\ 80, & 0.8 < t \leq 1.8 \\ 90t - 82, & 1.8 < t \leq 2.8 \end{cases}$$

2. If the bus leaves Bendigo at 1.00 pm, what time will it be passing Woodend, a town which is 90 km from Bendigo? Give your answer to the nearest minute.

The bus will be passing Woodend at 2:55 pm.

$$g[t_] := 90t - 82$$

$$\text{Solve}[g[t] == 90, t]$$

$$\left\{ \left\{ t \rightarrow \frac{86}{45} \right\} \right\}$$

N[%]

$$\left\{ \left\{ t \rightarrow 1.91111 \right\} \right\}$$

$$0.91111 \cdot 60$$

$$54.6666$$

The above rule is the traditional approach used to solve problems of this nature. Try setting up the following, as g can also be written as an absolute value function, g_2 with the rule:

$$g_2(t) = a|t - v_1| + b|t - v_2| + c|t - v_3| + d, \text{ where } a, b, c, d, v_1, v_2 \text{ and } v_3 \text{ are real constants.}$$

3. Write down the corner values $v_1 \dots v_3$.

$$v_1 = 0, v_2 = 0.8 \text{ and } v_3 = 1.8.$$

4. Complete the missing rates of change values in the following matrix.

$$\begin{bmatrix} 1 & -1 & -1 & a \\ 1 & 1 & -1 & b \\ 1 & 1 & 1 & c \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 90 \end{bmatrix}$$

5. Solve this system of simultaneous equations.

$$a = 95, b = -50 \text{ and } c = 45$$

$$\text{Solve} \left[\begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 100 \\ 0 \\ 90 \end{pmatrix} \right] \{a, b, c\}$$

$$\{ \{ a \rightarrow 95, b \rightarrow -50, c \rightarrow 45 \} \}$$

6. Find the value of d correct to one decimal place, if $g_2(0) = 0$.

$$d = -41$$

$$g_2[t_] := 95\text{Abs}[t - 0] - 50\text{Abs}[t - 0.8] + 45\text{Abs}[t - 1.8] + d$$

$$\text{Solve}[g_2[0] == 0, d]$$

$$\{ \{ d \rightarrow -41. \} \}$$

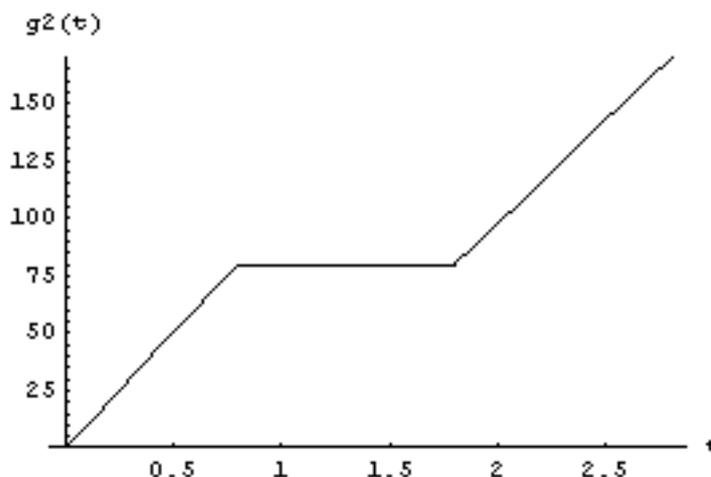
7. Hence write down the rule for g_2 .

$$g_2(t) = 95|t - 0| - 50|t - 0.8| + 45|t - 1.8| - 41$$

$$g_2[t_] := 95\text{Abs}[t - 0] - 50\text{Abs}[t - 0.8] + 45\text{Abs}[t - 1.8] - 41$$

8. Sketch the graph of g_2 for the journey.

```
Plot[g2[t_], {t, 0, 2.8}, PlotRange -> {0, 170},
  AxesLabel -> {t, "g2(t)"}];
```



9. Use this rule to confirm your answer to question 3.
 The bus will be passing Woodend at 2:55 pm as expected.

```
Solve[g2[t_] == 90, t]
{{t -> -1.}, {t -> 1.91111}}
```

Conclusion

The author of this paper is passionate about using CAS to enhance the teaching and learning of mathematics. Her involvement in the VCAA CAS Pilot Study has been an exciting development for her and for her students. The unrestricted use of CAS has led to dramatic changes in pedagogy and assessment in each of the schools. CAS has proved to be a powerful learning tool that allows students to move between numeric, graphical and symbolic representations of a problem. It also allows students to observe patterns and explore concepts. The focus of the algebra in senior mathematics courses previously has been on how to carry out mathematical procedures, such as solving equations or differentiating and integrating functions. CAS has made it possible to increase the emphasis on the understanding of concepts and on helping students to decide when and why it might be appropriate to apply a particular procedure, or a particular technique, be it CAS or by-hand. ‘Unrestricted access to CAS has challenged us, as educators, to start inventing new paradigms for the teaching and learning of senior mathematics’ (Garner, McNamara & Moya, 2003, p. 271).

A report in *The Age* (7 June 2004) states that,

CAS allows students to acquire a deeper understanding of the discipline. Students have a focus on understanding the conceptual framework, instead of rote learning. When confronted with new and unseen problems they can apply what they know to the new situation. (*The Age*, 2004b, p. 6)

This can be thought of as using algebra to prove a mathematical conjecture, and then using CAS to provide the answer. In this context, CAS does not replace the mathematical learning process, but enhances it.

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Drawing sense out of fractions*

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Many students consider that learning fractions has little to do with making sense. Indeed, the arithmetical operations performed with fractions (addition, subtraction, multiplication and division) appear to be ‘a dance’ carried out with whole numbers. Students’ reasoning with fractions is often dominated by arcane whole number manipulations that appear to work best when you suspend reasoning. What kinds of experiences help students to show us what they really believe fractions are? How can we assist them to understand that the two numbers that compose a common fraction (numerator and denominator) are related through multiplication and division, not addition?

Introduction

Although the role of calculating with fractions has changed over the years, fraction instruction frequently remains focused on symbolic computation. If students see two fractions they usually attempt to add some numbers, subtract some numbers, multiply numbers or invert and then multiply some numbers. In each case the procedure that they carry out involves doing something with integers. Perhaps it is not surprising that many students do not have a strong sense of the size of fractions. After all, the operations that they carry out with fractions are usually about counting or operating with whole numbers.

The limits of attempting to memorise rules for manipulating symbols in fraction calculations are well documented in large-scale assessment programs. Moreover, it is clear that a focus on symbolic computation does not ensure that children can connect those rules to their conceptual understanding of fractions. For example, children often conclude that the fraction one-fifth is larger than the fraction one-third because of the whole-number interpretation that five is greater than three.

Clouds in my eyes

I was listening to the weather report recently, when I heard the forecaster describe the current situation as ‘three-eighths cloud cover’. As I had been thinking a great deal about how to best teach fractions, I began to wonder what people would think of when they heard the statement ‘three-eighths cloud cover’. What image does it conjure up for you?

* This paper has been accepted by peer review.

As teachers, we seek to understand how children think, before, during and after we teach. This desire to understand how children might represent fractions in a natural context lead to several Year 6 classes being invited to draw the sky to show three-eighths cloud cover. The students' responses provided quite a few insights into what they thought about fractions.

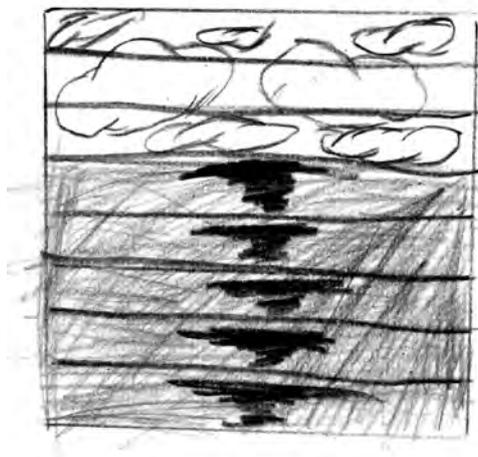


Figure 1. Three-eighths 'high cloud'.

Perhaps, not surprisingly, rectangles featured heavily in the images students drew. Within these classes, students thought of fractions as parts of rectangles.

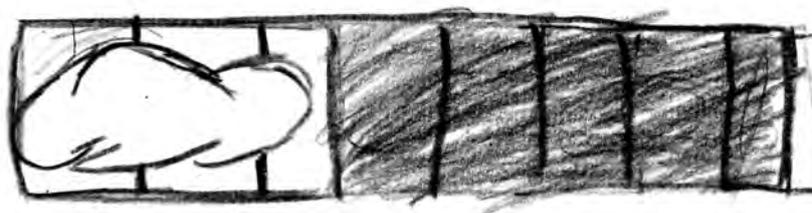


Figure 2. A clouded graph.

In both Figure 1 and Figure 2, the cloud cover appears to be much more graphical than factual. The clouds are used as icons to record that those pieces stand for clouds. Clearly, not all students think about fractions or clouds the same way. In Figure 3, this student's drawing represented clouds using an area model, albeit in a rather stylised arrangement.

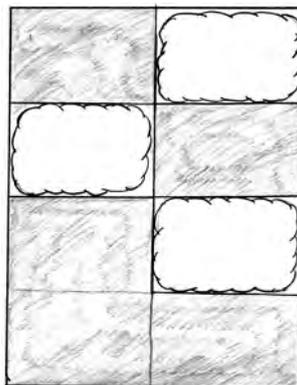


Figure 3. A stylised area model.

Do all students think of three-eighths cloud cover as rectangles divided into eighths? The answer to this question is obviously ‘No’. Although rectangles divided into eighths did dominate the drawings, some students have multiple representations of fractions. Figure 4 shows one student’s drawings where the sky is divided into a 5×8 grid and where the sky is also represented as a sector graph!

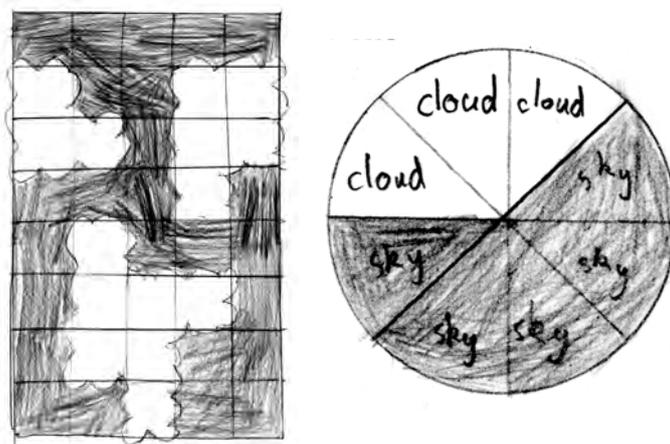


Figure 4. Two different representations from the same student.

Approximately fifteen out of the forty squares are clouds. Does this suggest that this student knows equivalent fractions? The diagrams suggest to me that this student can find equivalent fractions using diagrams and by re-dividing the unit. This is evidence of the Stage 3 expectations of fractions within the new NSW Mathematics K–6 syllabus.

If you asked your students to do the same task, what would it reveal about their understanding of fractions? That is, ‘The weather report said that there was three-eighths cloud cover. Draw what the sky might look like.’ The value of this real-world context is that it forces many students to think about fractions in a different way. Fractions are not simply two whole numbers, one over the other. Look at the way that the student has represented three-eighths in Figure 5.

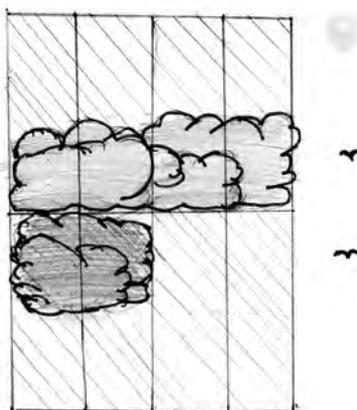


Figure 5. Matching the image and the fraction representation.

In Figure 5, the student has represented the sky with rectangular eighths and drawn clouds that cover approximately half of six-eighths. This is quite a clever use of re-dividing the units.

When I first told others about hearing the weather report describing cloud cover in eighths, several of my colleagues suggested that I was becoming obsessive about fractions. It took me a while to convince them that I wasn't hallucinating. The official unit of cloud cover is *oktas*, or eighths of the sky. When the sky is completely covered by cloud (overcast), the cloud cover is eight *oktas*. When the sky is clear the reading is zero *oktas*.

Why do some students find working with fractions difficult?

Imagine a student encountering the symbols we use to record fractions. She is told that $\frac{3}{4}$ is the same as three out of four. Explaining what we mean by the numerator and the denominator of a fraction might expand this 'definition'. The student then demonstrates her understanding of fraction notation by stating that three people out of four people is the same as $\frac{3}{4}$, two people out of five people is the same as $\frac{2}{5}$ and five people out of nine people is the same as $\frac{5}{9}$. All appears well until your precocious student surprises you by writing $\frac{3}{4} + \frac{2}{5} = \frac{5}{9}$. Now you have a lot of explaining to do!

The rapid transition from modelling fractions to recording fractions in symbolic form, numerator over denominator, can contribute to many students' confusion. The result of this rapid transition to recording fractions is that many students see fractions as two whole numbers; three-quarters is the whole number three written over the whole number four.

Kieren (1988) outlines how the teaching of fraction algorithms can contribute to the development of a superficial understanding as follows.

In work on operations with fractions, the algorithms frequently are developed as an extension of whole number algorithms... an extension of counting (adding with common denominators), or an extension of the powerful syntax of the base-10 numeration system (decimal fractions). Because of this, the curriculum and instruction prematurely emphasise technical symbolic operating rules (lining up decimal points, finding least common denominator, etc.). These extensions usually are not built on the intuitive mathematics of fractional numbers... Children get the form but not the substance of the system. This may result in temporary achievements with fragments of knowledge but not in lasting, useful, powerful personal knowledge. (p. 177)

The history of the use of fractions provides the rationale for why we have traditionally emphasised knowledge of methods of manipulating fractions. The expansion of business and commerce during the industrial revolution led to computation of fractions assuming an important role in the school mathematics curriculum. A time-efficient path to the formal symbolic computation served the needs of society at that time. The role played by fractions in society changed as the day-to-day manipulation of 'common fractions' became less common. Our money system and our measurement system went decimal. Nevertheless, an emphasis on formal symbolic computation has persisted within schools (Behr, Wachsmuth, Post & Lesh, 1984).

Although the need for efficient formal symbolic computation with fractions has reduced in society, fraction learning remains a serious obstacle in the mathematical development of children (Behr, Harel, Post, & Lesh, 1992; Kieren, 1976; Kieren, 1988; Mack, 1990; Pitkethly & Hunting, 1996). Recording fractions in symbolic form needs to build on an underpinning conceptual framework of units (or parts) and collections of parts that form new units. Emphasising numeric rules too soon without underlying meaning discourages students from attempting to see rational numbers as something sensible. Perhaps more time needs to be spent on drawing clouds!

Recording thinking with diagrams

Sharing diagrams provide a good method of representing and calculating with fractions. Not only are they more closely linked to the nature of fractions arising from division than the traditional symbolic notation, they frequently provide access to the images students hold of fractions.

Sharing diagrams are offered to a student as a tool to represent and support his or her thinking. Representational tools are forms of symbolising that support thinking. Students' diagrams should represent fraction problems in the way that they think about the problems. The value of sharing diagrams is in their congruence with the way that problems are interpreted. Standard fraction symbols are dissimilar from both the problem and the thinking involved in solving the problem.

When provided with the opportunity to use diagrams to support their thinking, students often are able to solve problems that normally wouldn't be introduced until after they had been taught formal algorithms. This was clearly evident in students' recordings when we asked Year 4 students questions that required dividing by a fraction amount. Students were asked to draw what would happen if I have 6 cups of milk and a recipe needs three-quarters of a cup of milk. In particular, they were asked to determine how many times I could make the recipe before I run out of milk.

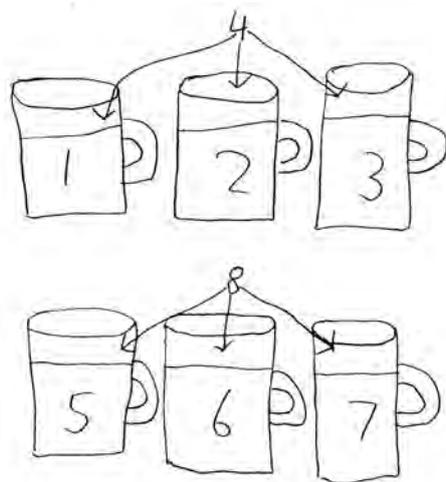


Figure 6. How many 'three-quarter cups' in 6 cups?

Figure 6 shows a very practical solution to this question. Six cups are drawn and three-quarters of each cup is shown. What is also evident is the accumulation of the three one-quarters to make the fourth and eighth quantities. Of course there are times when the context of the problem comes through very strongly.

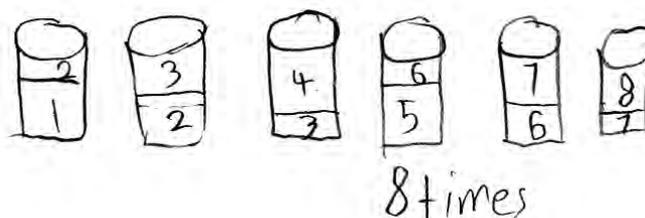


Figure 7. A different way of accumulating 'three-quarter cups'.

In Figure 7, the student appears to have imagined pouring three-quarters of a cup into the first cup. To add the next three-quarters the recording shows one-quarter to fill up the first cup and then one-half of the second cup. For the next three-quarters we have one half to complete the second cup and then one-quarter in the third cup. This process is continued in reaching the answer of '8 times'. As well as demonstrating a clear sense of the size of three-quarters, this sharing diagram suggests that the student recognises ways of seeing three-quarters as composed of one-half and one-quarter. It could even be argued that the student might even understand that eight lots of three-quarters is six, although it is still some years before this will be introduced to the student.

Building on students' informal knowledge of fractions is a sensible thing to do. It can assist us to help students appreciate that fractions are not simply two whole numbers. Using sharing diagrams with realistic contexts is a useful way of enabling students to draw sense out of fractions.

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Improving mental computation skills*

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This paper reports on a teaching experiment, conducted in 2004, which aimed at enhancing students' mental computation performance. Taking advice from the research, this experiment used a program of mental computation skills (devised by the author) to address the needs of a group of Year 9 students. Previously, the mental computation skills these students learned were incidental to classroom teaching. The direct instruction of mental computation strategies was investigated to see if they did make a difference and, to determine if students' ability to compute mentally improved over the course of the program.

Introduction

A review by Owens and Perry (2001) was conducted to assist in the development of a new mathematics syllabus in New South Wales. Essentially, this review was a literature review of pertinent articles and research pertaining to mathematics education generally. In this review, Morgan (1999) is quoted as summarising many of the issues raised about teaching mental computation. Owens and Perry (2001) stated that, 'the intention is to encourage children to develop flexible, idiosyncratic mental strategies, emphasising the mental processes involved'. They suggested that effective teachers need to develop a rich network of connections between different mathematical ideas and to develop proficiency in new approaches to mental calculation.

Taking advice from this research, a program of mental computation skills was devised to address the needs of a group of Year 9 students. Prior to this program the mental computation skills that this class had encountered were incidental rather than planned. The direct instruction of mental computation strategies was investigated to see if they do make a difference and to determine if students' ability to compute mentally improves.

Literature review

Hill and Russell (1999) suggest that the middle years require more emphasis on student-centred approaches to teaching and learning, with clear specification of core content needed by students. In their comments they suggest making time for in-depth learning, having a curriculum which emphasises thinking, problem solving and autonomous learn-

* This paper has been accepted by peer review.

ing, and challenging individual students. Mental computation provides an avenue in mathematics to allow student-centred teaching to become a reality, and further enhance the opportunities for students to build on the knowledge that they have — simultaneously giving students experiences which are enjoyable.

The benefits of developing and using mental strategies for computing have been well articulated, and many primary teachers are now encouraging students to invent and use thinking strategies as a way to facilitate their development of number sense (McIntosh & Reys, 1997). In solving mathematical problems they found that students used mental computation; however, no student used *only* mental computation. The numbers in the problems required some adjustment and some reflection on these adjustments before a solution could be given. In essence, each student used number sense along with estimation and mental computation in formulating a solution (Figure 1).

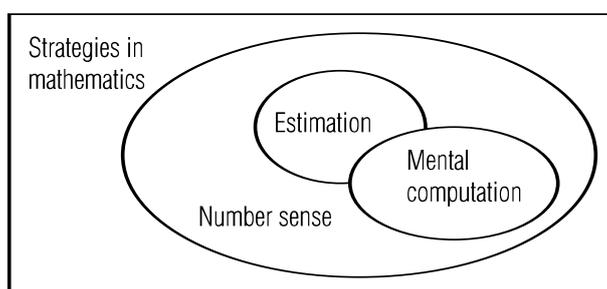


Figure 1

The use of thinking strategies in mathematics involves and overlaps with other aspects of numeracy including mental computation, estimation and number sense. Mental computation refers to the computing of an exact answer without any external tools such as calculator or paper and pencil. The technique used may be an invented strategy or a traditional method. Estimation refers to the production of an approximate answer that allows a decision to be made and often involves some form of mental computation as a preliminary step. Number sense refers to the general understanding of numbers and operations and usually includes both mental computation and estimation.

It is apparent that mental strategies turn a calculation that we cannot do into a calculation that we can do by employing relationships between numbers and operations. These thinking strategies are closely linked to the user's conceptual understanding of numbers and flexibility in decomposing and recomposing numbers (Plunkett, 1979).

Research on mental computation is emerging and is beginning to furnish a profile of student ability and facility in the middle stages of education (McIntosh & Reys, 1997). Some of the findings of the research include:

- students think mental computation is important as it is the skill to be most used out of the school context in adult life;
- students differ greatly in their understanding of what it means to compute mentally, often thinking that to do the algorithm method in one's head is a mental computation;
- students can invent their own strategies although they are often dominated by written algorithms;
- the ability of students to invent strategies varies greatly;
- the format in which a problem is presented stimulates different approaches and different performance levels;
- context influences performance and the strategies employed by students;

- allowing students to describe their thinking strategies reveals interesting and creative thinking;
- the highest levels and use of mental computation are found in students who are confident in their thinking, value alternative thinking strategies, and see the invention of techniques as a powerful result of their understanding of mathematical relationships;
- students processing and thinking about numbers can be changed.

Some important implications from this research that can be addressed by the classroom teacher are:

- indicating to students that developing and using thinking strategies is a valued process;
- asking students to explain how they performed a mental calculation;
- providing a variety of computation settings to encourage students to use strategies different from written algorithms; and
- modeling the behaviour by sharing thinking strategies with students.

Instructional programs have included teaching specific strategies supported by representations/models (Beishuizen, 1993), embedding problems in contexts (Klein, Beishuizen & Treffers, 1998), and development of mental computation through a strategy approach (McIntosh, 1998).

Most estimation and mental computation skills depend on using number relationships, and some depend on numeric patterns. However, such relationships and patterns have rarely been the focus of classroom instruction. It is well known that instruction in the classroom needs to be allocated to calculators, mental computation, estimation and standard written algorithms. Shumway (1994) recommends that allocation of time to these components should be; written computation (10%), mental computation (20%), estimation (30%), and calculators (40%), which is a dramatic reallocation compared with traditional allocation of class time to these aspects of mathematics.

The current allocation of teaching to these areas is inappropriate and that some alternatives need to be investigated. Menon (2003) concluded that focussing on instruction, then on the number relationships for mental estimation and calculation seems pedagogically sound and should bring about greater success in, and understanding of, computation. In his research, Menon (2003) investigated shortcuts that students can use for rapid computations. He indicated that some might argue that these shortcuts encourage rote learning and therefore detract from conceptual understanding, however, in his experience he found that the shortcuts motivated students and that these shortcuts could be taught using number relationships in such a way as to enhance number sense and conceptual understanding.

Carroll (1996) found that traditional instruction in mathematics has generally produced students who are poor at mental computation and exhibit a weak sense of number and mathematical relations. Interviewed students indicated that experiences with invented algorithms and discussing alternative solutions had lead to better ability to compute mentally and a stronger number sense.

Significantly better performance on mental computation will not occur until teachers recognise the role of computational alternatives, and give mental computation its rightful place. In preparing students for mental computation through appropriate strategies some of the desired outcomes might be considered (McIntosh & Reys, 1997). These outcomes should include:

- the appropriate use of calculators and an understanding of their shortcomings;
- a good operational knowledge of basic facts — addition, subtraction, multiplication and division;

- a clear, practical understanding of place value and the system of numeration;
- an understanding of simple fractions, decimals, percentage and interrelatedness;
- self-confidence in, and ability to use, a range of techniques for computing;
- the inclination and the ability to use thinking strategies first when checking calculations, estimating results, or performing various calculations;
- the instinct and willingness to think about numbers in natural, comfortable, and flexible ways when computational results whether exact answers or estimates are produced;
- the instinct and willingness to reflect on numerical results so as to judge their reasonableness.

It is apparent in the research considered that improving students' mental computation is indeed a desirable curriculum and instructional outcome.

Teaching students strategies that facilitate mental computation, estimation and number sense and allowing students to investigate, explore and develop alternatives for numerical computations are essentially the suggested avenues to improve student mental computation skills. In this study the teaching of particular strategies was used to investigate the possibility of enhancing student facility with mental computation.

The study

In response to the apparent lack of attention on mental computation skills a short course involving some skills was developed. It was hypothesised that this short course would assist students in their ability to compute mentally. The course was a series of twenty-minute lessons to be conducted over a period of nine weeks. An overview of the program and the skills taught can be found in Appendix A. Prior to the program the students were pre-tested and on completion of the nine lessons they were given a single lesson to review the course of skills and a week later were post-tested on the skills examined in the short course.

The students were in a Year 9 Advanced Mathematics class. The skills taught were chosen specifically to meet the needs of these Year 9 students. The students were an able group of mathematicians who find mathematics enjoyable and are regularly successful in mathematics. In completing this course of mental computation skills, the students were addressing syllabus content and understanding some of the strategies that many currently used, without fully understanding, why they worked, as well as discussing and using other strategies that they had not seen before in their mathematics classes.

Data from both the pre-test and post-test was analysed for the two test variables: number of correct responses and time taken to complete the task to determine whether the program had improved their speed to complete the computations and whether they were able to get more of the problems correct. Data about gender, task items and strategies were also investigated.

Results

The data analysis of the results in this investigation was performed using *SPSS for Windows Student Version 11.0.0* (2001). There were 34 cases present (19 female, 15 male) for both the pre-test and the post-test. The descriptive statistics provide information about the data collected for both the pre-test and the post-test. Means for score and time improved while standard deviations remained steady. This data suggested further investigation of the

mean scores for both the score and time variables.

To determine the degree of association between the pre-test scores and times with the post-test scores and times, Pearson correlations were used. These correlations indicated that the pre-test scores were associated with the post-test scores significantly ($p = 0.012$), suggesting that if a student scored well on the pre-test then they were highly likely to score well on the post-test. More significant was the association between pre-test and post-test times ($p = 0.001$) which indicated that if a student completed the task quickly then they were extremely likely to complete the post-test more quickly.

To determine whether the difference between the pre-test and post-test scores was significant a paired sample t-test was used. The results of these t-tests indicated that the students scored significantly better on the post-test score ($p = 0.000$) than on the pre-test score. The post-test scores were on average 5.15 higher than the pre-test scores. This represented an average increase of 17.2% by the students after completing the program. The t-tests also indicated that the students improved their time to complete the problems significantly faster ($p = 0.000$). The post-test times were on average 5.72 minutes (5 minutes 43 seconds) faster than on the pre-test.

Together, the results comparing scores and times represent a highly significant improvement in the students' ability to compute mentally. The degree of significance on both students' score and time were very high ($p = 0.000$). Gender differences were not significant.

The items used in both the pre-test and the post-test were evaluated to determine which items were found most difficult by the students. Means for each item on the pre-test were calculated and compared with the means of the post-test scores. Many of the items that showed marked improvement were towards the end of the skills program and were considered 'harder skills'.

Each item in the pre-test and post-test were evident in the strategies taught during the mental computation skills program. The items were grouped and scores for each of twelve strategies were obtained. It was found that seven of the twelve strategies improved significantly based on the results of the pre and post-tests. These strategies were those that:

- squared numbers ending with five;
- combined numbers in a series;
- used difference of two squares;
- multiplied or divided by numbers greater than or less than one;
- simplified fractions and percentages (two strategies);
- multiplied numbers near 100 and 1000.

Equally important were those strategies that did not show significance. Although these strategies did not show significant improvement they constituted an important part of the skills program. They reinforced skills that the students already used and provided necessary skills for the weaker students in the group.

Conclusions

The project investigated whether or not a skills program in mental computation strategies could improve the students' facility with number problems without the use of a calculator. In particular, the analyses of this project investigated: scores, times, strategies, gender and items. These aspects enabled a deeper understanding of the skills program and its effectiveness. As well, they provided further information about the program for future development and use in the classroom.

The students' scores were analysed for improvement between the pre- and post-tests.

In this analysis it was evident that there was a significant correlation between the pre-test and the post-test scores and that the students improved their scores by an average of approximately 17%. The improvement in the students' scores was highly significant. This would suggest that the skills program was beneficial to the group of students that were involved and that it would also be of benefit to other students of this cohort.

The students' times to complete the task were analysed for an improvement between the pre-test and post-test scores. In this analysis it was evident that there was a significant correlation between the pre-test and the post-test times and that the students improved their times significantly. Again, this suggests that the skills program benefits are more than just in the improved scores but it appears to improve the students' efficiency in completing problems involving mental computation.

The students' scores were analysed for gender differences on scores and times. It was found that there were no significant differences in the scores or the times between males and females.

The items were analysed to identify items which had improved significantly from pre-test to post-test phases of the study. The results indicate that over 80% of the items improved or remained the same and those items that did not improve declined by a small, insignificant amount. Of the items that improved, 50% were found to improve significantly. This would further validate the finding that the skills program was a significant catalyst for improving students' mental computation skills.

The strategies were analysed to determine those which were useful in a skills program. This aspect of the study resulted from the analysis of the items used in the skills program. When combining items that evolved from the same strategy there were twelve strategies in total that were taught over the course of the nine weeks of the program. It was found that there were some strategies that the students were familiar with and did not really improve; however, there was a significant improvement in 58% of the strategies taught in the skills program. The strategies in the program were suitable to the cohort in the study.

Like Menon (2003) this study found that the strategies engaged and motivated students and that these shortcuts could be taught using number relationships in such a way as to enhance number sense and conceptual understanding.

Recommendations

1. Further attention be given to the transfer of the knowledge to other aspects of courses and other working mathematically situations/contexts. The strategies in the skills program were extended in some instances; however, none gave the students the opportunity to transfer the knowledge to problem situations. This was intentional as the purpose of the program was to teach the skills and not get 'stuck' on decoding the problem situation.
2. Teaching mental computation strategies to students as part of an Enrichment Maths program would also be an option in the future. Many students invited or involved in these programs have a keen interest in mathematics and how mathematical methods work. Investigating the strategies and inventing their own could easily be components of such a program.
3. The skills used in the program were specifically chosen to suit the needs of the students. Additional skills that target earlier year groups or lower ability students could also be developed for similar programs.
4. The final recommendation would be that teachers recognise the role of computational alternatives, model the use of strategies that the students have learned,

provide opportunities for students to explore new strategies and invest some time in learning practical instructional techniques for alternative strategies

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Appendix A: Improving mental computation strategies (IMCS)

Program overview

Date	Lesson & Content
23 Feb 04	Pre-test
01 Mar 04	Skill Lesson #1 — multiplication and division by 10, 100, 1000 etc.
08 Mar 04	Skill Lesson #2 — squaring numbers ending in 5
16 Mar 04	Skill Lesson #3 — subtracting numbers from 10, 100, 1000 etc.
22 Mar 04	Skill Lesson #4 — multiplying by 5, 50, 500 etc
29 Mar 04	Skill Lesson #5 — combining when adding and multiplying by 11
03 May 04	Skill Lesson #6 — difference between two squares
10 May 04	Skill Lesson #7 — multiplication and division by 2, 4, 8 etc. and numbers $>$ and $<$ 1
17 May 04	Skill Lesson #8 — simplifying fractions and percentages
24 May 04	Skill Lesson #9 — multiplication of numbers near 10, 100, 1000 etc.
01 Jun 04	Review Lesson
07 Jun 04	Post-test

Getting school maths online: Possibilities and challenges*

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The World Wide Web (WWW) is making a significant impact in the field of mathematics education. In recent years, several thousand web sites have been created to promote the teaching and learning of school mathematics. It is now time to ponder how this phenomenon should articulate with the school curriculum and how the wealth of resources available in cyberspace can be appropriately integrated at the classroom level. This paper discusses a range of instructional, curricular and organisational issues associated with such a process and provides recommendations for future research.

Introduction

As early as the 1930s, Shoghi Effendi predicted the imminent arrival and expansion of the World Wide Web (WWW) with these words: 'A mechanism of world inter-communication will be devised, embracing the whole planet, freed from national hindrances and restrictions, and functioning with marvelous swiftness and perfect regularity' (1936, p. 203).

Several decades needed to pass before this vision became reality. The WWW can be now considered as a computer network made up of literally millions of users worldwide. The last decade has attested to a substantial transformation in the way knowledge has been traditionally disseminated. Through WWW technology, information now flows almost unrestrainedly throughout all the regions of the globe and easily reaches homes, centres of learning and remote village settings. The speed of this process accelerates as the globalisation processes dramatically unfold. So amazing has been this information revolution that educationalists have been forced to consider its effects on education. What we need to consider, as we progress into the third millennium, is how to harness the potential instructional advantages of the WWW for the benefit of classroom teaching.

Reasons for online approaches in mathematics education

Several arguments can be put forward to advance the cause of teaching and learning mathematics through online approaches. School curricula need to evolve in line with the changing nature of our society. The WWW is similar to other important landmarks in the evolution of the school mathematics curriculum such as the invention of the hand-held

* This paper has been accepted by peer review.

calculator and the personal computer. Given its increasing and indispensable presence in human affairs, the WWW must be reflected in the school curriculum if that curriculum wishes to remain relevant. The nature of the WWW is such that it offers an enormity and diversity of resources that are, for the most part, free for access and download. These resources have the advantage not only of being recently created but also produced and shared by field practitioners such as teachers, academics, instructional designers and even students themselves. The range of resources has the potential to broaden the spectrum of students' learning experiences thus enriching the mathematics curriculum. Resources available online are also suitable for distance education where students work at their own pace and at any geographic location. Online resources can also be repackaged and redesigned as, for example, thematic units linking school mathematics to real-life situations. Moreover, the WWW is a free-platform technology which makes it a friendly-user mechanism going beyond regional technical specifications. It is above all, a wonderful window to the world that can be opened with a single mouse click in the isolation and remoteness of the school classroom.

Categories of online resources

For the purpose of categorisation, resources available on the WWW for teaching and learning mathematics can be grouped into six inter-related categories, namely: drills, tutorials, games, simulations, hypermedia-based materials and tools, and open-ended learning environments (Alessi & Trollip, 2001; Handal & Herrington, 2003). Drill-and-practice websites mostly present exercises for practicing well-structured mathematics operations. Drill-and-practice formats also evaluate the correctness of students' answers once a set of questions have been attempted. Online tutorial applications are one step ahead of drill-and-practice formats in that the former teaches respondents the procedure for reaching the solution. Online instructional games allow students to participate in an entertaining situation that simultaneously demands their engagement in problem solving at several levels. Online simulations are applications that interactively model or fabricate a real-life situation whose enactment in the classroom is impractical or even hazardous. Hypermedia software, in turn, are complex databases of several kinds of mathematical knowledge linked through nodes of information. Hypermedia formats can be compared to electronic encyclopaedia covering a broad range of topics such as history of mathematics or mathematics vocabulary. By providing cross-curricular content, hypermedia resources are an ideal environment for exploration and investigation. Finally, the WWW also offers tools and open-ended learning environments that can be used for representing data graphically, experimenting with geometrical concepts through interactive diagrams, or drawing complex curves given their equations. The sequence of formats outlined above can be considered as a continuum that progresses from an instructional design that favours the independent, transmissive drill-and-practice approaches through to tools and open-ended learning environments that facilitate a collaborative, problem solving approach.

The organisation of resources into a virtual classroom is probably the most fascinating example of the WWW's potential in education and training (Anderson, 2001; Murphy & Collins, 1997). Virtual classrooms place a community of learners together under the leadership of one or more instructors using several online facilities such as video-conferencing, electronic boards or electronic discussion groups. This online learning approach has been known as synchronous learning since students can learn simultaneously regardless of location. They can also make use of asynchronous formats

such as email and downloaded materials. Virtual classrooms are becoming popular in adult training and are progressively making their way to the school environment.

Issues in online mathematics education

The growing presence of the WWW in education will undoubtedly reflect the way the curriculum is conceptualised and organised. An important discussion should focus on how online learning experiences are aligned to a constructivist perspective of teaching and learning mathematics (Lefrere, 1997). Such an articulation must certainly reflect the pedagogical worth of instructional approaches such as group work, discovery learning, problem solving, real-life situations, in-depth discussions, use of manipulatives, field work, and so forth. Vargo (1997) explains that the constructivist approach in computer education also includes the student having more control and access to information, the use of more discovery learning and explorations as well as the introduction of more case analysis. That approach stands out in clearly contrast with an objective or behaviourist model, in which the user only receives and replicates information. Nunan (1996) adds that flexible delivery through online methods fosters a culture of self-learning, problem solving, and activity-based learning. According to Winn (1997) the following principles apply to a constructivist approach in WWW-based teaching and learning (p. 2).

- Access to the Information Superhighway is not a sufficient condition for learning, though for students in distance learning programs, it may be a necessary one.
- The information that we prepare for students comprises data to which we give a structure that is determined in no small part by the medium in which the information is presented. This structure will be influential in how students understand what we tell them.
- The acquisition of knowledge from information requires effort and involves perceptual and cognitive processes that decode symbols, deploy literacy skills to interpret them, and apply inferencing abilities to connect them to existing knowledge.
- The acquisition of wisdom from knowledge requires practice in the judicious application of that knowledge in the personal and social context in which the student acts.
- By implication, testing just to see whether a student has received information, which is not atypical even in higher education, sheds absolutely no light on what students really know nor on whether they can productively use any knowledge they have acquired.

In brief, a constructivist approach to mathematics and online education envisions a learning community in which learning activities are carefully selected to assist students in constructing knowledge. These activities must also take account of students' previous experiences. The constructivist approach also considers the teacher not as the only knowledge provider but as a learning facilitator who supports active learning. This role includes the provision of learning activities through interactive online technology.

The utilisation of online resources in the teaching and learning of school mathematics may come with a number of instructional, curricular and organisational challenges. As the literature on the field is nascent more research is needed to guide online educational endeavours. The following represents a collection of those issues which ideally must be analysed within the context of the six categories of online resources described above.

The WWW is a relatively new learning technology. Certainly, little is known about its cost-effectiveness in comparison to other instructional approaches. Likewise, more research is needed to determine its effects on students' attitudes towards the learning of

mathematics. It is also important to identify instructional practices within the classroom leading to students' gains in achievement as they work with online learning resources. Similarly, there is a need to examine whether current online learning approaches lead to a decrease or an increase in the gap between high and low achievers. Likewise, the type of social interaction, either among students or between teachers and students, that is generated while working with these resources requires further examination (Clark, 2000). Research is also needed on the impact that WWW technology has on broadening students' learning or cognitive styles and teacher's instructional styles.

Literature generated in the last two decades shows that teachers' pedagogical beliefs about educational innovations can sometimes work against the implementation itself (Handal, Bobis & Grimison, 2001). In particular, negative beliefs about the introduction of technology into the classroom have been documented by Newhouse (1998) and Mills and Ragan (1998). Consequently, it is necessary to characterise current teachers' beliefs about the worth of online instruction. Likewise, there is a need to examine parents' and school administrators' perceptions about online instruction so that potential obstacles can also be foreseen. At the same time, students' perceptions of online learning need to be appraised with respect to motivation to learn and engagement in learning.

The WWW offers a variety of applications that can be considered problem-solving tools. They include applications to solve arithmetic, algebraic and differential equations, curve drawing and graphical representation of data. It is possible that educators could see some of these tools as a process of deskilling. Alternatively, assessment items can be written to accommodate the use of these tools in such a way that reflects the measurement of students' higher order thinking capabilities. Assessment criteria would also need to be re-formulated to distinguish measurement of students' actual mathematical understanding from their navigational skills.

In addition, a number of issues will inevitably arise as a consequence of the increased access of the WWW both at home and at school. Should the whole curriculum or parts thereof be aligned to the utilisation of online resources? How should literacy skills, discovery, case analysis, inferring and other higher-order thinking abilities will be taught and developed in online learning approaches? Which novel learning competencies such as navigational skills, webpage design, computer architecture or applet programming, need to be taught to better utilise online resources? Do these new competencies mean that the curriculum could become overcrowded? If that is the case, which current content should be removed from the curriculum?

Current research reveals that boys are more engaged than girls in computer studies and do better in some aspects of mathematics (Kifer & Robitaille, 1992; McDougall, 2000). Hence, it can be argued that these gender-related trends might also be reflected in the learning of mathematics using online resources. Similarly, it is reasonable to question whether children from low-socio economic backgrounds will be disadvantaged due to their lack of access to the WWW, particularly after school hours (Hartmann & Sweeney, 1999). It has been argued (Navin, 2001) that private schools, particularly from more affluent sectors of society, have an advantage over public schools given the disparity in the investment in educational technology. In a futuristic situation, where WWW learning has a greater presence in the curriculum, it is likely that textbooks will be modified and adjusted accordingly. Textbook writers, curriculum designers and classroom teachers will also need to pay more attention to cross-cultural issues such as differences between imperial and metric units, language barriers and international events, among others. For example, learning to add by counting elephants would not sound very relevant to Australian Aboriginal students in the outback or South American children in the Andes highlands.

Not only is there a lack of literature about using online resources in the teaching and learning of mathematics, there is also a scarcity of specialists in the field. Although both human resources and research will inevitably grow hand-in-hand, the urgency of the hour demands that resources and practical guidance be provided to classroom practitioners. This raises a further set of critical questions to be addressed. Should the leadership come from academics, regional or district consultants, or school head teachers? How should classroom teachers' experiences be shared and extended? Teachers' networks, peer mentoring, inservices, teacher education programs, university partnerships, showcase of best practice, all need to be developed and documented to fulfil a comprehensive support structure for professional development.

The use of the WWW enables the use of a further set of related hardware and software technologies such as data projectors, printers, laptops, data loggers; it also raises issues such as having the classroom connected to the Internet, and of course, the purchasing of appropriate computers. Management and maintenance of the school Intranet is another associated cost as well as the school licensing of more sophisticated software. Likewise, the task of searching the WWW for identifying meaningful learning experiences is certainly time-consuming and demanding for teachers (Godfrey, 2000). Setting up either a personal or faculty database of relevant websites adds considerably to teachers' workloads, particularly when such a database is to be arranged by grade, curriculum area or degrees of difficulty.

Despite its unique advantages, the WWW can be depicted as a disorganised and sometimes misleading pool of unrelated websites. When it comes to educational websites many of them are of poor quality in terms of instructional design, being little different to traditional textbook formats, and being developed by people involved in commercial software (Lefrere, 1997). It is therefore vital to focus our education agenda on issues such as: how many of people involved in the production of current online resources are actually educators and capable of developing something for education purpose? How should teaching education programs begin establishing courses on online instructional design? What makes a website educationally sound? What can be learned from the literature on evaluating computer-assisted instruction (CAI) software that can be applicable to online resources? (Hosie & Schibeci, 2001). Which guidelines in evaluating online resources are worth considering? How can teachers be trained to assess the quality of educational websites? Which quantitative and qualitative research methodologies are most appropriate in the evaluation of educational websites? For example, Hosie and Schibeci (2001) and Reeves and Harmon (1994) have called for more context-bound evaluation rather than checklists because the former captures a lot more of the interaction between the learner and the courseware.

On the effectiveness of WWW-based instruction, Scanlon (1997) reviewed online learning in sciences and found that teachers are more inclined to use traditional resources such as books than the Internet. Scanlon noted a number of concerns on evaluating courseware online such as bandwidth, network reliability and integration of the curriculum with technology. Simons and Jones (1999) evaluated the Online Mathematics Enrichment website in the United Kingdom (www.nrich.maths.org.uk) which provides support for gifted and talented students through the publication of problems and other resources. The evaluation of the program with 450 teachers and 199 students was 'judged to be attractive, functional, easy to navigate, and contain high quality materials' (p. 11). According to the evaluators, 'The main impact of NRICH on the more able students was in terms of helping them to gain a wider appreciation of mathematics and raising the profile of mathematics as a subject that could be interesting enough to pursue either within or outside school or for further study' (p. 11). Indeed, more research is needed to explore the implementation of online approaches in teaching and learning mathematics.

Conclusions and recommendations

Not surprisingly, as with any educational innovation, the introduction of WWW resources in the teaching and learning of school mathematics comes with possibilities as well as challenges. The instructional, curricular and organisational issues outlined in this paper, although not comprehensive, serve as a discussion board for future research and planning. The lack of studies in this nascent field is hindering the utilisation of a vast number of online resources that are already available for classroom use; and yet, the WWW is developing at an increasing speed, not waiting for the school curriculum to catch up. Crucial and urgent to this dissonance is the commitment of the academic community to become engaged in more research in this area. Like a third millennium's version of *Pandora's box*, the WWW is waiting for us to open it and to discover its marvels in this era of globalisation. Hopefully, as educators we can harness its potential benefits rather than succumb to possible chaos.

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Evaluating online mathematics resources: A practical approach for teachers*

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This paper describes a teacher-friendly approach to evaluating online mathematics resources. The Alessi and Trollip (2001) evaluation form is recommended as an instrument for assessing the worthiness of online resources from an instructional design point of view. An exploration of nearly 250 mathematics education websites revealed the benefits and limitations associated with using such a checklist. These issues are discussed through screen snapshots of webpages available from the WWW. This exploration also revealed that online resources from professional organisations' websites seem to be better designed, organised, easy to search and more comprehensive than those from individuals' websites.

Introduction

Gradually, World Wide Web (WWW)-based educational resources are making their way into the school mathematics curriculum (Handal & Herrington, 2003). Online resources are potentially useful compared to normal courseware because of their abundance, availability at no cost, platform free accessibility, and their wide-reaching accessibility. On the other hand, a major limitation of online resources is their lack of appropriate pedagogy, coupled with poor instructional design and layout. According to Alessi and Trollip (2001, p. 392), 'The tendency for the Web to be used only for presentation of materials greatly restricts its instructional potential'.

Little research has been done in the area of evaluating online mathematics education resources. As the WWW grows in influence and size there is a need to document the quality of these online resources and those aspects of their design that are inhibiting their implementation. This study reviews a number of online mathematics resources and discusses their drawbacks in terms of the existing literature on courseware evaluation. The instructional design elements embedded in Alessi and Trollip's (2001) evaluation form are used in this analysis.

Evaluating courseware

How do we know that courseware is well-designed and pedagogically sound? There are at least two approaches in the evaluation of courseware. The first approach makes use of

* This paper has been accepted by peer review.

evaluation forms and checklists that assess mostly interface design, navigation and/or control features of a courseware as well as other intertwined pedagogical variables. These features are then compared against a set of ideal criteria appropriate from an instructional point of view. A number of evaluation forms and checklists have been designed in this way (e.g., Alessi & Trollip, 1991, 2001; Reeves & Harmon, 1994; Sharp, 1996). A second approach is to evaluate courseware with respect to learning outcomes and the quality of the interaction with the learner. This second type of evaluation is referred to as context-based evaluation since assessment is carried out as the resource is used by the learner in a specific learning environment (Hosie & Schibeci, 2001).

In either approach, a number of dimensions or criteria are identified for evaluation. Reeves and Harmon (1994) have characterised fourteen instructional dimensions of computer-based instruction which include epistemology and pedagogical philosophy. Haugland and Wright (1997) developed the Haugland/Shade Developmental Software Evaluation Scale (www.childrenandcomputers.com) to evaluate software for children. Their scale is based on ten criteria, namely:

- (a) age appropriateness;
- (b) child control;
- (c) clear instructions;
- (d) expanding complexity;
- (e) independence;
- (f) non-violence;
- (g) process orientation;
- (h) real world model;
- (i) technical features; and
- (j) transformations.

The distinctive feature of this scale is the introduction of a developmental variable. According to the author, only one quarter of existing software can be considered appropriate for children (Haugland & Wright, 1997). In addition, Stubbs and Burnham (1990) proposed five critical dimensions in the developing of electronic distance education systems. These dimensions include:

- (a) time and place independence;
- (b) realism;
- (c) communication paths;
- (d) ease of use; and
- (e) speed or immediacy.

Alessi and Trollip's (1991) quality review checklist focusses on interface design, navigation and user's control of the page and is based mainly on the following features:

- (a) language and grammar;
- (b) surface features;
- (c) questions and menus;
- (d) other issues of pedagogy;
- (e) invisible functions;
- (f) subject matter; and
- (g) off-line materials availability.

Checklists and evaluation forms have been criticised because of their focus on features that are external and easy to measure, not capturing the process of teaching and learning. Indeed, context-bound evaluation tools can actually cover a broader range of pedagogical issues because of the diversity of methodological tools used such as measurement of learning outcomes through tasks and assignments; conducting interviews with students and teachers, participant observation methods, collecting students' work

samples, video-taping student's interaction, analysing students' responses, and administering attitudinal scales (Hosie & Schibeci, 2001; Reeves & Harmon, 1994).

Evaluation checklists

Although, context-bound strategies are powerful tools in bringing about a whole picture of the effectiveness of a courseware, when it comes to evaluating a large quantity of educational material, such as the case of online resources, checklists do a faster job. This is particularly pertinent for teachers because of their job demands and constraints. Qualitative approaches require specialised training and a longer time to implement. Evaluation forms and checklists have been successfully used for a long time in the academic community for courseware evaluation and have informed research and the teaching community accordingly. These instruments are particular useful as 'screening' tests for new software, and are of most use at the point where a decision has to be made about which software to trial. The use of evaluation forms and checklists also decreases the subjectivity factor and provides teachers with structured assessment criteria without necessarily requiring knowledge about multimedia or educational technology. By using checklists, teachers can become aware of issues in designing and assessing educational software. This is particularly true for teachers who have been educated in environments where the only technology was the blackboard.

Alessi and Trollip's (2001) evaluation form builds on the framework of Alessi and Trollip's (1991) quality review checklist which addresses the evaluation of pedagogical features, interface design, navigation and user's control of an online resource. The checklist has been successfully used in other studies as a courseware assessment tool (Noijons, 1994; Rasegotsa, 1999) and for training mathematics and science teachers in evaluating courseware (Handal, Handal & Herrington, 2003). It seems to be indispensable given the poor instructional design of a large amount of educational software available in the market (Schwier & Misanchuk, 1994; Shneiderman, 1998). Alessi and Trollip's (2001) evaluation form is organised in items related to:

- (a) Subject matter;
- (b) auxiliary information;
- (c) affective considerations;
- (d) interface;
- (e) navigation;
- (f) pedagogy;
- (g) invisible features;
- (h) robustness; and
- (i) supplementary materials.

Evaluating websites

This section illustrates important features identified when using the evaluation form with nearly 250 mathematics education websites. Categories were used to analyse the quality in design and layout of the online resources focussing specifically on interface design, navigation and user's control. Although the discussion is not comprehensive, it is useful as a framework for initial exploration and research. In addition, the organisation of these resources was examined in terms of corporate or individual management of the websites. The study also aimed to validate the categories used in Alessi and Trollip's (2001) evaluation form.

Six of the nine categories of analysis in Alessi and Trollip's (2001) evaluation form were used to evaluate the websites. The following three categories were not considered to be relevant: *supplementary materials*, referring to the quality of the auxiliary printed material that accompanies courseware, do not constitute a requirement for online resources and was not considered. Likewise, *invisible functions* of the lesson are related to the keeping of performance records as well as to issues of security and accessibility. Both features are rarely used in online resources and therefore are not discussed here. *Robustness* refers to the capacity of the program to work in different computer environments. Internet applications are generally multi-platform, although some multimedia effects need specific plug-ins and some webpages are designed to work better in either of the two most popular WWW browsers, namely *Internet Explorer* or *Netscape Navigator*.

During the exploration of the 250 mathematics education websites, some limitations were observed when applying the Alessi and Trollip (2001) checklist. These limitations highlighted essential differences in design and usability issues between online resources and normal courseware. Not all the courseware design features are applicable to online resources for several functional and usability reasons. First, there is a diversity of online resource formats, namely: drills, tutorials, games, simulations, hypermedia-based materials and tools and open-ended learning environments (Handal & Herrington, 2003). For example, drill and practice exercises do not provide complete feedback to the users, that is, a complete worked example. Contrary to many games applications, most tutorials do not necessarily require the use of multimedia effects. Tools and open-ended learning environments are not formatted in terms of questions and answers but require exploration and investigation (Alessi & Trollip, 2001). Secondly, online resources differ from normal courseware in that the former do not come accompanied by manual or printed instructions on how to teach with the resource. This omission makes it difficult to evaluate the online resource in relation to an overarching set of pedagogical goals, outcomes or objectives. In other cases, some online simulations and games require the downloading of plug-ins from the WWW. This often makes the application unreliable as well as more difficult for the assessing teacher to run and evaluate. Finally, many online resources are embedded on webpages that are not consistent with other pages of the same website. As opposed to normal courseware, the organisation and sequencing of online learning activities are not well articulated and goal-oriented making it difficult for teachers to choose especially when they are searching for activities supporting a specific curricular topic.

The following sections present a summary of the important features identified through the evaluation of a large number of websites.

Introduction

Presentation of goals and objectives can enhance the understanding and motivational appeal of the subject matter and should be clearly stated and worded at the student's lexical level. Information must be relevant, accurate and complete. Table of contents, indexes and directions must be clear and information must be accurate and related to the curriculum. The screen in Figure 1 provides students with ample information about the task.

Ideas for Pupils

Pupils working in small groups or independently

Use the thesaurus to check the meaning of mathematical terms used in your books or by your teachers. The thesaurus will allow you to look at terms or words linked to your original enquiry. If you find that another term used in the definition you have found is not clear, you can use the *references* link to find an explanation of this term.

You might also wish to find out how the term links to other concepts you have met. The thesaurus tries to map these connections for you.

It is possible to use the thesaurus when you are reading other sites online. If you have installed the *m-Button* in your browser's links bar you can use it to look up any mathematics word you may come across on any site.

How about using the thesaurus to produce a map of your own around an area of mathematics you are currently studying? Are there ideas that you have not met before that look particularly interesting or connections that you had not thought about? You can produce a map of your own for display in the classroom. Use the **PrtSc** key to capture maps.

You can use the thesaurus to help you find material on other mathematics sites. Most sites have a search function, but it can sometimes be tricky to know what to type in. How do you find something when you can't remember what it is called? The answer is to use the thesaurus first! Start from something you know and use the relations to navigate to the term you want. You can copy and paste the term into a search engine - or use the shortcuts to search engines that we provide.

Figure 1. <http://thesaurus.maths.org/mmkb/view.html?resource=guides>

Displays

It is necessary to check whether (a) displays are uncluttered, (b) overwriting is avoided, and (c) attention is maintained to relevant information. In terms of presentation, it is also important to review whether texts, graphics, colour and sound are used appropriately. Figure 2 shows a cluttered screen.

Figure 2. <http://pbskids.org/cyberchase/games/numbersense>

Motivation

A webpage should maintain the user's interest and must challenge the user across different displays. Visual momentum influences the learner's ability to extract and absorb content that is relevant to him/her across successive displays. Features such as zoom, sound or animation must be assembled in unity and be consistent. Figure 3 shows a webpage with a dynamic percentage bar.

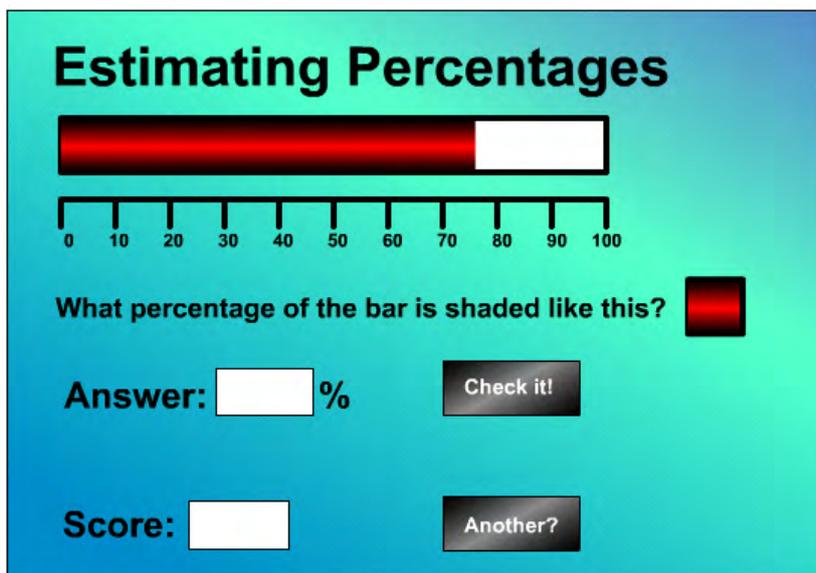


Figure 3. <http://www.hellam.net/maths2000/percent1.html>

Navigation aids

Tools availability should be checked to see whether the tools are active, or if they are present but are not active. Some tools should be removed or hidden from certain places. Otherwise, users get confused into thinking that the webpage is not working properly. For example, the control panel of a webpage might not be active in some sections. Most WWW browsers have sufficient navigational capabilities. Figure 4 shows an easy to follow tool board for selection.

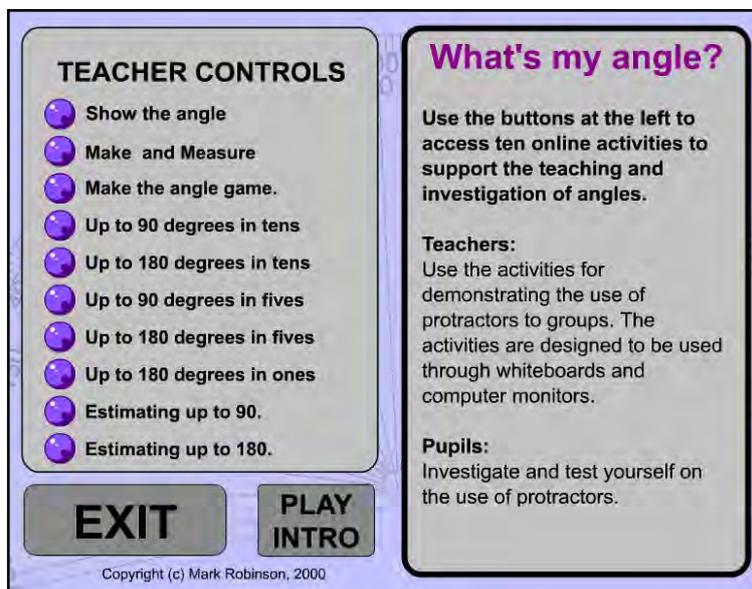


Figure 4. <http://ambleweb.digitalbrain.com/ambleweb/ambleweb/ambleweb/mentalmaths/protractor.html>

Questions

Questions should be relevant and be presented in a variety of formats. Likewise, the webpage must facilitate learner's answering by giving clear choices and the possibility of more than one try. Feedback must be relevant and supportive. Questions should be economical with instructions on answering questions. The activity on Figure 5 shows an activity linking numerical, graphical and symbolic data.

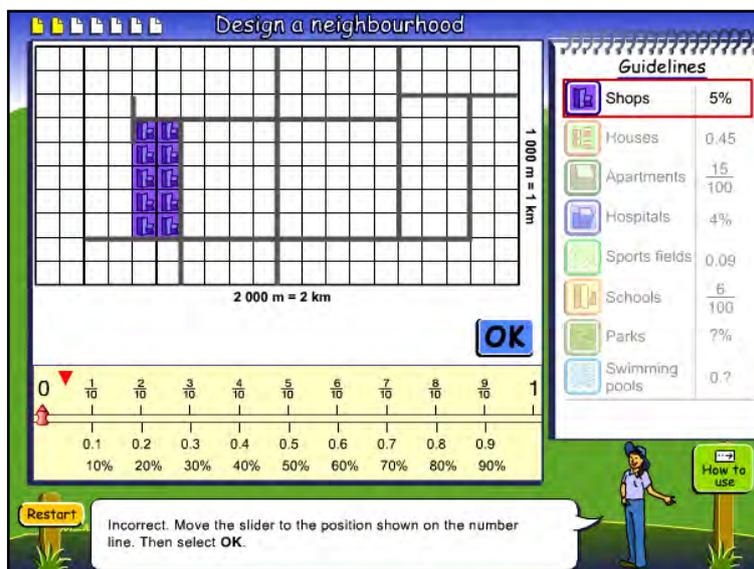


Figure 5. http://www.thelearningfederation.edu.au/repo/cms2/tlf/published/10560/180204_education/L122_design_a_neighbourhood

Format of feedback

Self-evaluation can be achieved by giving the users a sense of accomplishment through acknowledgement or visual cues that indicate their progress. Self-evaluation can be achieved through, among others, self-tests or quizzes which require 'yes or no' or multiple choice answers, or comments on results in simulation activity. The activity in Figure 6 provides continuous feedback on the task.

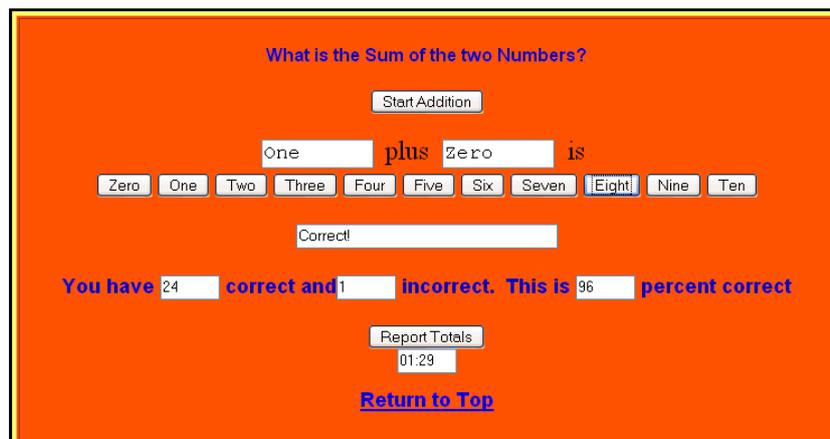


Figure 6. <http://www.aaamath.com/B/addk7ex1.htm>

Content structure

Menus should orient, give the opportunity of making a choice, and also of amending an incorrect choice. A dynamic menu is shown on Figure 7.

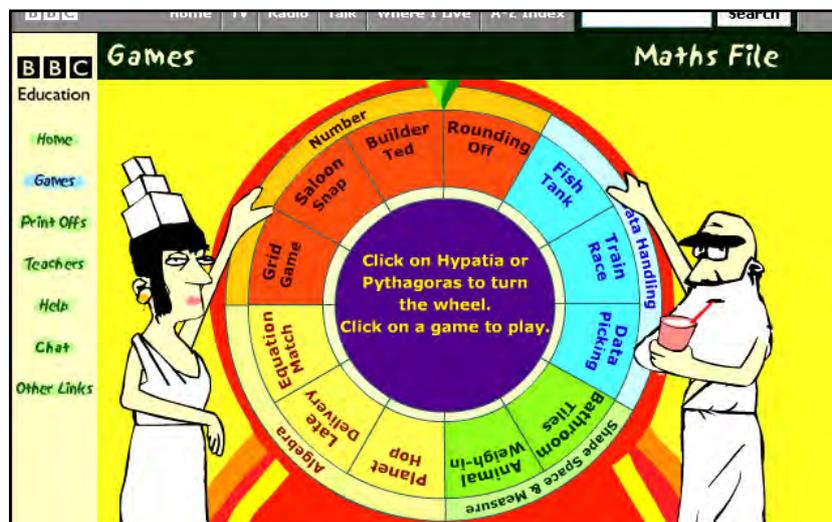


Figure 7. <http://www.bbc.co.uk/education/mathsfiler/gameswheel.html>

Directions

Advance organisers assist learners in finding information. Providing the user with an overview of the topics to be covered and how to access them through hyperlinks in maps or menus is a good start for any webpage. A consistent method of using this information should be presented to the learner in the earlier stages with a on-screen reminder such as instructions. The screen on Figure 8 provides overview information about a webpage on symmetry.

Understanding Congruence, Similarity, and Symmetry Using Transformations and Interactive Figures: Visualizing Transformations



Visualizing Transformations



[Identifying Unknown Transformations](#)



[Composing Reflections](#)



[Composing Transformations](#)

Rotations; translations, or slides; and reflections, or flips, are geometric transformations that change an object's position or orientation but not its shape or size. The interactive figures in this four-part example allow a user to manipulate a shape and observe its behavior under a particular transformation or composition of transformations. In this part, Visualizing Transformations, one can choose a transformation and apply it to a shape to observe the resulting image. In the next part, Identifying Unknown Transformations, the user is challenged to identify the transformation that has been used. In Composing Reflections, users can examine the result of reflecting a shape successively through two different lines. And in the fourth part, Composing Transformations, the users are challenged to compose equivalent transformations in two different ways. Activities like these allow students to deepen their understanding of congruence, similarity, and reflection, and they also contribute to the study of transformations, as described in the [Geometry Standard](#).

Figure 8. <http://standards.nctm.org/document/eexamples/chap6/6.4/index.htm>

Learning metaphor

The presentation of the information should be followed up by students' activity, as students will be more motivated if they participate actively with the webpage. Also, learning experiences, when sequenced, must follow a specific theme or topic. The learning experience in Figure 9 relates to a collection of activities based on the number line bounce.

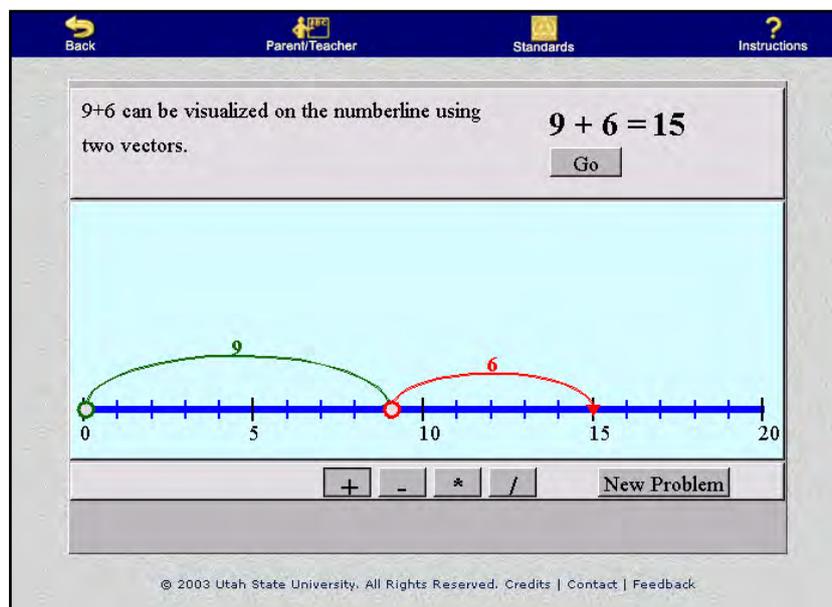


Figure 9. http://matti.usu.edu/nlvm/nav/frames_asid_197_g_2_t_1.html?open=activities

Methodologies

Student's interaction with the webpage should be more proactive than reactive. A proactive interaction emphasises learner construction and generative activity whereas a reactive interaction is an answer to presented stimuli or to a given question. Interaction must be frequent and in a variety of forms. In Figure 10 students are required to draw geometrical generalisations from manipulating objects.

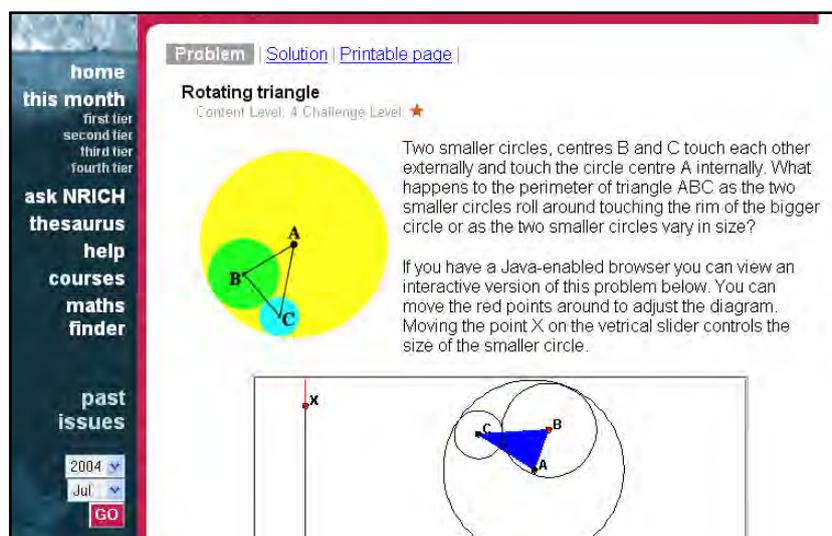


Figure 10. http://nrich.maths.org/public/viewer.php?obj_id=266&part=index&refpage=monthindex.php

Language, style and grammar

Language and grammar should be at the appropriate reading level, technical term and jargon, spelling, grammar and punctuation. Figure 13 shows a high lexical density text.

1.4 And At the Heart of Calculus is ...

The density property described above leads to a concept that is fundamental to calculus, and that is the concept of a limit. If I gave you 2 gallons of milk today, one and a half gallons tomorrow, one and a third gallons the next day, one and a fourth gallons the next day, and so on, and continued in that manner for the rest of eternity, what can I say about how much milk you might get on a typical day? Well, there is certainly a formula for it. If I label the days, starting with today, as 1, 2, 3, 4, and so on, I can say that on day number n , I will give you $1 + 1/n$ gallons of milk. But there is something still deeper I can say about all this. I can say that there is a lower bound on how much milk you'll get on any particular day. That lower bound applies to all days starting with today.

If you tried to argue that 1.01 gallons is a lower bound, I could disprove it by noting that on the 101st day you would be getting less than that. But if you argued that any amount of one gallon or less is a lower bound, I would be unable to find any day in the future on which I would be giving you less than that amount of milk. So all amounts of one gallon or less are lower bounds. But of all of those, the amount of one gallon exactly is special. It is the greatest of all the lower bounds. And even though there is no day that I will give you exactly one gallon of milk, there will come a day on which the amount I give you will be *as close to one gallon as I would like*. Not only that, on all subsequent days the amount will be that close or closer.

In other words, you tell me how close to one gallon I will be giving you, and I can name the day on which the amount I deliver daily to you will be, forever after, at least that close. That is what makes

Figure 13. <http://www.karlscalculus.org/calc1.html>

Help

A *help* function may be available for each task so that the learner has continuous guidance through the learning sequence as shown in Figure 14.

The screenshot shows a web browser window with the URL <http://www.waldomaths.com/>. The page title is "Waldo's Interactive Maths Pages". There are navigation buttons for "home", "email", and "what to do". The main content area is titled "Areas of Triangles - November 2002" and features a green background with a white triangle. The triangle has vertices labeled A, B, and C. The base AB is horizontal and labeled with a length of 20.0. The height from vertex C to the base AB is also labeled with a length of 20.0. A right-angle symbol is shown at the intersection of the height and the base. Below the triangle, there is a control panel with the text "Choose which side is the base" and three radio buttons labeled AB, BC, and AC. A "what to do" help window is open on the right side of the page. The window has a title bar that says "Areas of Triangles - I..." and a "close" button. The text inside the window reads: "What to do", "On the screen you'll see a triangle. You can change this triangle by clicking and dragging the vertices (corner points) with the mouse.", "On the right a calculation of the area of the triangle is done for", and the formula:
$$\text{Area} = \frac{20.0 \times 20.0}{2} = 200.0 \text{ (1 d.p.)}$$

Figure 14. <http://www.waldomaths.com>

Conclusions and recommendations

This paper dealt with issues associated with the interface design, navigation and user's control of an online resource. It indicates how evaluation forms and checklists can be practical tools for teachers to identify positive and negative design features of an online resource. The discussion also showed, in general terms, that the Alessi and Trollip's (1991, 2001) framework can provide teachers with a simple and at the same time meaningful structure to assess WWW-based resources. These abundant resources require professional judgment in their selection and articulation into the school mathematics curriculum.

Generally speaking, it was found that online resources created by professional organisations and organised in inclusive websites such as the Learning Federation (www.thelearningfederation.edu.au), Cambridge University (www.nrich.maths.org), the National Council of teachers of Mathematics (illuminations.nctm.org/imath), York University (<http://www.counton.org>) or the Shodor Foundation (www.shodor.org), have a better instructional design than those created by individuals. These are comprehensive websites whose online resources are more interactive, pedagogical oriented, sorted by grade level and curriculum objectives, thereby constituting a better search strategy for practicing teachers. Additionally, their URLs are also easier to remember! On the other hand, it is estimated that there are 500 individuals' websites — a figure that certainly reflects the growing enthusiasm and commitment of the mathematics education community to produce and share resources using the WWW medium. Eventually some sort of centralised database of online resources by curriculum objective, grade level and/or type of application sought should be designed to facilitate teachers' identification and access to the enormous amount and variety of online resources.

More research is certainly needed to modify courseware evaluation instruments to the nature of online resources. Research is also needed to investigate the process of developing and supporting evaluation skills for practicing school teachers to facilitate the application of these worldwide resources in the mathematics classroom.

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Mathematics and computers as ‘cognitive tools’

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Computers as tools for teaching mathematics, have existed for fifty years. However, not all mathematics teachers in secondary schools have embraced the potential of using this tool. It has been claimed by several authors that computers are not substitutes for teachers, but rather are extension tools or ‘mind-tools’ that mediate learning. Despite the technological advancement on our doorsteps, teachers are still slow in using this technology in the classroom. This study will enlighten the ‘fallacy’ that most mathematics teachers think the use of computers results in an added workload and extra lesson preparation.

Introduction

‘The world is changing, technology is changing, mathematics is changing and mathematics education and society’s perception of, and support for mathematics education must change to meet the needs of the twenty-first century’ (Burke & Curcio, 2000).

Background information on the potential of computer use

Schools and universities need to keep up with the rapidly changing global society. These changes are reflected in growing consumerism, the information revolution, wider use of information technology, and communication in all sectors of daily life, including education, industry, leisure, travel, sports and medicine.

Countries around the world have embraced the integration of technology in their education systems, as young people need to keep up with the technological changes happening globally. Students need to learn computer skills to prepare themselves for the workforce. The 2001 Census in Australia showed that nearly half of all households (49%) used a personal computer at home and 36% of total households reported using the Internet at home (www.abs.gov.au/ausstats/abs). The 2003 Australian Census showed that families are using computer technology, which means that young people are being exposed to increasing numbers of computers. Schools need to keep up with technological change and must integrate computers in teaching mathematics and other subjects.

Positioning myself

Computers have been used in both the secondary and tertiary levels for over three

decades. Yet when you enter classrooms, you will observe teachers in front of the classroom lecturing and talking, using the blackboard to introduce a mathematics topic and writing the content of their lesson on whole blackboard. The students are listening to the teacher, and then opening their books to solve exercises. This is the common scene in mathematics classrooms today, despite the technological advances outside the classroom. There are already several reforms in education using computers as tools for teaching, like the *TIMMS 1999 Video Study* (Hiebert et al., 2003). However, pedagogy has not progressed significantly in fifty years. The majority of teachers are unaware of the potential of using computers to teach mathematics. Teachers should reflect on this and understand that students' learning styles and beliefs have changed and our approaches should cater for their needs. The youth of today is very different from the youth of fifty years ago: they are more critical in their thinking, they question things happening in their environment, they are bored sitting in a normal classroom listening to the teacher at the front of the classroom and then using a book to do their homework and assignments. Today's students want to explore and experiment. This is because of their exposure to new technologies, new information media such as the Internet, chat-lines, videoconferencing, simulations, virtual reality experiences and digital music and movies. Often the students know more than the teachers about the use of these technologies.

In my teaching of mathematics, I have used computers as tools in the classroom. My observations and experiences are:

- integrating technology in mathematics is a powerful tool for students to explore mathematical ideas and real-world problems (the Internet, simulations, databases, programming);
- using technology encourages conceptual learning rather than rote use of formulas and algorithms (use of procedures and flowcharts);
- through the teacher's guidance, students actively engage in the material which encourages student experimentation (spreadsheets, BASIC and LOGO programming, the Internet);
- students learn how to work with others in their groups and learn how to explore questions (group research, interviews, collecting data);
- students are involved in the learning process when they collaborate with members of the class;
- the use of technology in the curriculum enriches students' understanding of mathematical concepts, increases their problem solving abilities and improves their attitudes towards mathematics (formulas using spreadsheets, statistical programs);
- computers can be used as visualisation tools and integrating technology in the mathematics curriculum provides an excellent opportunity to use digital cameras, scanners, word processing and webpage authoring tools.

These observations are supported by Harskamp et al. (2000), Schoenfield (1987) and Yelland and Masters (1997). However, the proper integration of information and communication technology (ICT) requires that teachers are familiar with hardware and software issues (Watson & Tinsley, 1995).

Related studies

The following is a glimpse of studies on the potential of using computers in mathematics in the learning environment.

Dubinsky and Schwingendorf (2001) conducted a long study on *Calculus, concepts, computers and cooperative learning*. The emphasis of the C⁴L program is a pedagogical approach based on a constructivist theoretical perspective of how mathematics is learned.

They pointed out that students need to construct their own understanding of each mathematical concept. They believe that the primary role of teaching is not to lecture, explain, or otherwise attempt to ‘transfer’ mathematical knowledge, but to create situations for students that will foster their making the necessary mental constructions. A critical aspect of their approach is a decomposition of each mathematical concept into developmental steps, following a Piagetian theory of knowledge based on observation of, and interviews with, students as they attempt to learn a concept. The following are the guiding principles of their study and could be adopted when teaching mathematics.

Guiding principles

(From Dubinsky & Schwingendorf, 2001)

1. Research into how students learn is primary (learning styles).
2. Conceptual understanding is the most important form of learning, but calculations play a major role.
3. Technology can be valuable, and some ways of using it can be more valuable than others.
4. Cooperative learning is the right context for a mathematics course.
5. Lecturing should be replaced by a task-oriented interactive classroom.
6. Textbooks and course structure must support the pedagogical strategy.

Another study, conducted by Derry and Lajoie (1993) attempted to clarify how theory influenced practice, and provided an analogy (computer-based learning environments) in an effort to categorise the theoretical position that existed in the field of artificial intelligence and education at that time. The analogy described three imaginary camps: modellers, non-modelers, and middle camp (pp. 2–5). Those in the model camp develop student models that allow the computers to interpret learners’ actions dynamically in the context of a problem solving activity and to provide adaptive feedback based on the students’ actions. In this use of the word modelling, experts’ models of human activity may be automatically generated for the learners when the computer determines that assistance is needed. Student models generate computer models of learning for learners to observe and guide their future actions. The non-modellers believed that is impossible for computer models to be extensive enough to provide the adaptive feedback required. Hence, the non-modellers used technology as tools for learning and often require human beings to serve as modellers or facilitators who used computer tools to enhance student learning.

Computers as ‘cognitive tools’ or ‘mind-tools’

It is important to discuss what computers can do to students’ minds and thinking. Teachers often misunderstand the role of computers in the cognitive development of the individual child. The following is a brief discussion of computers as ‘cognitive tool’ or ‘mind-tools’.

Cognitive tools are both mental and computational devices that support, guide and extend the thinking process of their users. A cognitive tool can be regarded as an instructional technique in so far as it involves a task, the explicit purpose of which is to lead to active and durable learning of the information being manipulated or organised in some way by the task, as asserted by Jonassen et al. (1993).

When using computers as a tool for teaching students are required to present information and organise it in a meaningful way. This means that the students are engaged in deep learning such as critical thinking and analysing problem solving.

How technology tools support learning.

Technologies do not directly mediate learning; that is, people do not learn from computers, books, videos, or other devices that transmit information. Lajoie (2000) argued learning is mediated by thinking (mental process). Learning activities activate thinking, and learning activities are mediated by instructional interventions, including technology. Learning requires thinking by the learner. In order to more directly affect the learning process therefore, we should concern ourselves less with the design of the technologies of transmission and more with how learners are required to think in completing different tasks. Rather than developing ever more powerful teaching hardware, we should be teaching learners how to think more effectively (p. 2). We should focus less on developing multimedia delivery technologies and more on thinking technologies, those that engage thinking processes in mind. The role of delivery technologies should be to display thinking tools — tools that facilitate thinking processes. Table 1 show how computers support learning of students.

Table 1. Computers as cognitive tools to support learning

Some facts about computers	Supports/enhance learning
computer as mental and computational devices	extend thinking process; tools that facilitate learning
computer as cognitive tools	not a teacher or expert but as a mind-extension cognitive tool or ‘mind-tool’
computer as a tool for teaching students	present information and organised them in a meaningful way (databases, spreadsheets, word processing, the Internet, simulation, artificial intelligence, graphics)
computer acts in the acquisition of cognitive skills	problem solving, planning, designing, writing or communication with software
the use of computers has effect in students’ mind	engaged in deep learning, like critical and logical thinking, analysing and problem solving (Mathematics software, Statistical software, Programming languages, artificial intelligence and instructional programs designed by using computers like CAI, CAL)

Technology integration in mathematics teaching

Information technology is now making its way into education. There is a need for a clear integration of new technologies in school, not in addition: integration in subjects, integration in teaching, integration in learning, integration in the school, integration in the profession of teachers (Watson & Tinsley, 1995, p. 5). Yet, this new technology is not widely used in all subject areas. Teachers still think that using computers in the classroom creates an additional workload (e.g., preparation of lesson plans). However, teachers need to understand and be able to use technology in an ever-growing number of ways, consistent with people who use it outside the classroom (Burke & Curcio, 2000).

As Burke and Curcio (2000) pointed out, ‘the world is changing, technology is changing, mathematics is changing and mathematics education and society’s perception of, and support for mathematics education must change to meet the needs of the twenty-first century’. Therefore, teachers must change the way they think about teaching and learning mathematics. Many authors and researchers have already established the potential of using computers in mathematics teaching, the enhancement of students’ learning.

Technology integration, on the other hand, is an additional task faced by educators — particularly classroom teachers — because they are the innovative ‘agents of change’ in education. Therefore, the role of the teacher could change dramatically in the future. In Australian schools, mathematics teachers are not using computers in mathematics. Teachers who are willing to take up technology training could change their teaching strategies. However, there are constraints to implementing technology in the mathematics curriculum, such as the support of the principal.

Role of mathematics teachers in technology integration — what changes?

Shift on teaching methodologies

Increasingly, ‘computer technology’ is a powerful tool for teaching methodologies. The students of today are becoming responsible for their own learning through flexible delivery, online teaching and learning and the notion of the teacher as facilitator. With the introduction of computer-based networks, classrooms could include teachers and students who are working together across long distances.

The introduction and integration of computer technology in the classroom could change the way teachers think and teach. No longer would it be commonplace to see mathematics teachers standing in front of their students and directing them what to do. The mathematics teacher of today could become a ‘busy bee’ going around the classroom, setting up collaborative tasks, and developing students’ critical, analytical, logical and teamwork skills. The teacher as builder derives from early classroom computer innovations in which individual teachers not only select but also redefine learning activities using technology. In turn, significant ideas for revising the technology are generated from such on-site experimentation. This view represents a long-term professional development process of training rather than brief contacts with educational materials. The teacher needs to build new ways of making learning occur in the classroom (Hudson, 1997, p. 4).

The publication *Computer-based Technologies in the Mathematics KLA* (NSW DET, 1997, pp. 7–8) considers several roles of mathematics teachers in implementing and integrating computer-based technologies, such as:

- recognising the diversity of students and group of students;
- creating supportive and challenging learning environments;

- providing all students with experiences in a wide range of computer-based activities;
- providing meaningful activities and learning contexts;
- interacting with students in a range of styles; and
- valuing prior knowledge and experiences.

These roles require changes in the teachers' teaching style and strategies. This task is a very complex one. Teachers are already overburdened with paperwork and monitoring student discipline in the classroom. They can only do their best to implement these roles and changes. Watson and Tinsley (1995) mention various roles with respect to the teacher with pedagogical implications in the IT classroom: acting in turn as manager, task-setter, guide, accompanist, coordinator, explainer, counsellor, leader and even a fellow learner. The teacher now has less control of pedagogy, instead interacting with students as a 'facilitator', 'a learner' and a 'collaborator'. Teachers must accept that in these new rich classrooms their autocratic control is diminished. Some students will bring a wealth of knowledge about the use of computer-technology into the classroom. Students are technologically proficient, know more about computer games, techno-music, cyberspace games, computer animated movies and communication gadgets like mobile phones. This is the utmost reason why teachers' roles in mathematics education need to change.

Technology integration in the classroom calls for a new role for teachers in secondary education, and the successful use of these technologies (computers) depends upon the skills and training of teachers in the use of computers. Technology in teacher education can contribute to successful implementation of technology-based learning in mathematics.

Watson and Tinsley (1995) state that with the influence of computers, 'not only can mathematics be taught differently, but in a very deep sense different, as much greater emphasis being placed on numerical and algorithmic processes and on experimental approach involving exploratory investigations'. Secondary teachers must thus develop a deep epistemological view of the subject. Proper integration of IT means that teachers must know about informatics in general and they must be familiar with the computer itself, both in connection with the hardware and software considerations (pp. 32–33).

Bielefeldt (2001) conducted a survey on information technology in teacher education. The survey was conducted in 1998 and collected information on 416 schools, colleges, and departments of education (SCDEs) in the United States. Respondents were asked to rate their own institutions in terms of a variety of indicators including coursework, technology facilities and support, skills of graduates, and field experience opportunities. An analysis of the survey indicated four groups of items in which the questions were closely related to one another:

- integration of technology into the program (teacher education);
- facilities;
- field experience and
- application skills.

Of these, integration (the actual use of technology in the program) was the best predictor of other aspects of capacity.

Integrating technology into mathematics teaching has great potential. Using computer software (such as spreadsheets, CAD, LOGO, etc.) is an effective method of enhancing students' learning because their thinking skills, management skills, collaborative work with teachers and students, and higher order-learning can be developed, as pointed out by Jonassen et al. (1999). Teacher education should also incorporate technology teaching in their respective area of expertise.

The use of emerging tools of information technology, such as the use of microworlds, programming languages, e-mail, simulations, spreadsheets, CAD in mathematics teaching

offers real opportunities for enhanced teaching and learning. Online tutorials and courseware, as well as Internet-based subject offerings and distance learning can also be used as technology tools.

Summary

The major challenges in the integration of information technology to the mathematics classroom are pedagogy, curriculum content, the organisational structure of the school and the classroom, and the role of the teacher. The teacher is faced by a responsibility to implement and use technology-based learning in mathematics. To support this integration and implementation, head teachers and school leaders need to support teachers in their professional development.

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New York City school mathematics: What are the vital outcomes?

**Brian Lannen
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A group of AAMT members recently worked in New York City on an in-school teacher-training project. The intended vital outcome of the city's innovations in teacher professional development, is to make teaching more student-centred and concept-focussed. However, outcomes at the school level are still strongly influenced by high-stakes testing programs. This paper gives a background of the teacher-training model, the education department's philosophies as reflected through its documents and mandated text programs, a case study of implementation, and a description of the testing practices.

Background: The teacher-training model

New York City has a population almost half that of the whole of Australia. The New York City Department of Education oversees one million students in 1200 schools. Its teachers have varied amounts of pre-service training, many of them young or new to teaching or new to teaching in the USA. The City provides a range of professional development opportunities for its teachers. These include ongoing college courses, regional and city-wide workshop days, new teacher mentors and in-school coaches, and staff development consultants.

The writers of this paper worked as in-school staff development consultants and coaches alongside newly appointed school-based 'math' coaches. Their work involved assisting the math coach with mathematics planning across the school, helping teachers plan lessons and understand new text and manipulative resources, providing demonstration lessons and professional development workshops, and assisting teachers in-class as they implement new ideas.

A pivotal document in guiding the pedagogic practice is *A Comprehensive Approach to Balanced Mathematics* (New York City Department of Education, 2003). Essentially this is to help teachers make their lessons more student-task oriented and concept-focussed. The Chancellor's office produced this document during the summer of 2003 in response to a perception of diminishing standards of achievement in City schools. Along with this, the Chancellor also mandated the adoption of set textbook programs in all schools. The concern over achievement levels comes mostly from student scores in the highly revered City and State standardised tests. Although the research component of *A Comprehensive Approach to Balanced Mathematics* and the philosophies of the mandated texts might suggest otherwise, many teachers still feel compelled to 'teach to the test' and feel that

this is best achieved through an algorithmic approach to mathematics instruction.

A vital responsibility for the consultants has been to gain teacher confidence and steer them towards sound pedagogic practice. This task has been helped by the hands-on nature of the teacher-training model and the generally positive response from students who are being asked to be more engaged in their own learning.

This expectation is spelt out in *A Comprehensive Approach to Balanced Mathematics*: ‘During the mathematics workshop, students are encouraged to question, explore, reflect, explain, and convince themselves and others as they seek understanding’ (p. 9).

In the book *Content-Focused Coaching*, former school District Superintendent Lucy West writes of the need for consultants and coaches to bring about change at a grass roots ‘beliefs and values’ level: ‘Trusting the process and working with teachers in ways they are willing to engage are crucial tenets in effective coaching’ (p. 129); ‘Successful staff development is all about relationships.’ (p. 137).

The coaching process generally incorporated a series of lessons, firstly observing and getting to know the teacher and students, then planning with the teacher for the following lessons. Often the next lesson is taught by the coach/consultant and the consultant then assists the teacher in planning and implementing the follow-up lessons.

The term being adopted for these student-active lessons is ‘workshop model’, where each lesson is expected to contain elements of a mini-lesson (teacher-directed), independent and/or small group work (student-centred) and a summary share session (class community).

Philosophy and intended direction: The text programs

The mandated texts have provided a good source of investigative problems for the student work phase. In middle schools (Grades 6 to 8), McGraw Hill’s *Impact Mathematics* states in its program philosophy that ‘Effective teaching methods for middle grades students are varied and student-centered. *Impact Mathematics* encourages active learning through an assortment of teaching methods — collaborative problem solving, teacher-directed instruction, class discussion, and individual practice’ (Implementation Guide, p. 49). Many of the activities in this program will be familiar to Australian mathematics teachers: the locker problem, fraction strips, constructing a sector graph from a strip graph, crossing a bridge, and more. The text writers acknowledge that many of the algebra investigations are taken from Curriculum Corporation’s *Access to Algebra* (Lowe et al., 1994).

In elementary schools, the text scheme adopted is *Everyday Mathematics*, also by McGraw Hill and originally researched at Chicago University. In the 2003–4 school year, schools were directed to implement this program with Grades K–2, and given the option to further implement it in Grades 3 to 5. The resources provided to each teacher were:

- *A Comprehensive Approach to Balanced Mathematics: A Handbook for Educators* (CAB) produced by the New York Education Department;
- *Everyday Mathematics* program for the specific grade level, a program developed by The University of Chicago School Mathematics Project (UCSMP).

The *Balanced Mathematics* approach (CAB) is to communicate and disseminate information in mathematics for a shared vision. Within the CAB, the workshop structure is outlined for the teachers to complete their lessons. This structure is to support standards-based mathematics teaching and learning by giving students the opportunity to engage in purposeful mathematical activities and conversations. It also enables teachers to meet the diverse learning styles and the needs of the students. The three components of the work-

shop model (mini-lesson, student activities, reflection) are to be presented in both the Literacy and Mathematics teaching and learning blocks. The CAB also outlines New York standards for each grade and the pacing calendar for each month.

Everyday Mathematics (UCSMP) was developed in order to enable the students in elementary grades to learn more mathematical content and become life long mathematical thinkers. It begins with the premise that students can, and must learn more mathematics. The program uses:

- real-life problem solving — numbers, skills and concepts are not represented in isolation, but linked to contexts that are relevant to their everyday lives;
- balanced instruction — whole group, small group and individual activities;
- multiple methods for basic skill practice—choral drills, mental math routines, fact triangles, review problems called math boxes, games and assessment;
- emphasis on communication — students are encouraged to explain and discuss their mathematical thinking;
- enhanced home/school partnerships — daily home links, games, parent letters.

The instructional design is to maximise students' learning and capitalise on student interest:

- high expectations for all students;
- concepts and skills developed over time and in a wide variety of contexts;
- multiple methods and strategies for problem solving;
- collaborative learning in partner and small group activities;
- cross-curricular applications.

At the end of this year there has been some positive progress made in the teaching and learning of mathematics. These are:

- the allocated time for mathematics — 60 minutes for Kindergarten to Grade 2 and 75 to 90 minutes for Grades 3–5;
- kindergarten teachers have commented that the real life aspect of the program has improved the student learning in mathematics;
- cooperative learning in the mathematics sessions;
- student activities and games;
- promotion of manipulatives;
- mathematics literature;
- parent and home program;
- student reference book.

Something good: A case study of positive change

This case study is from a middle school in Brooklyn where positive change was seen to occur as a result of the consultancy intervention and in line with the City's objectives. The school has an enrolment of 600 students in Grades 6 to 8. There is no mathematics coach at this school and the Australian consultant was asked to work with teachers of one grade level at a time, focussing firstly on the introduction of the workshop model, moving teachers away from front-of-class lecturing and equipping them more with student-active investigative tasks. Teachers in New York generally teach only at one grade level per year, so a 'seventh grade teacher' may have five classes all at seventh grade. A vehicle for the professional development, especially at the sixth grade level, was the new *Impact Mathematics* mandated text scheme. Teachers at the school were previously using a mix of textbooks, not necessarily the same text across all classes in a grade level. Teachers have generally 'acquired' a class set of textbooks at some stage and keep them in their room as

the source of exercises for students to practice on. Quite often, however, a lesson would consist of the teacher copying a few exercises onto the board and then talking the students through the solutions. It is rare for teachers to have access to photocopying facilities in the school, so it is not easy for them to develop their own worksheets or merge ideas from a range of printed sources. Some teachers, however, do try this and arrange and pay for their photocopying privately as they feel ‘the text’ does not present material in the way they want. One such teacher is Bill, a sixth grade teacher who is in his third year of teaching and previously worked as a psychologist. At first Bill was wary of the *Impact Mathematics* text, claiming that he already used a range of materials as he deliberately aimed to develop students’ conceptual understanding. He did not want to give up that freedom to fall in line with a set text and drive students through all the exercise work.

This was a good starting point for the consultant. The teacher was already purposeful in wanting to guide his students in developing concepts. Bill’s approach, however, was still very teacher-directed and he did not seem to have an appreciation for how problems could be taken from the text, mixed with and connected to other materials he liked, and used to drive student investigation and discovery. He understood the term ‘constructivist’ but did not seem to probe his students to ascertain their cognitive models for mathematics. ‘Assessment’ was still seen as a regime of formal tests used to produce scores.

The consultant was able to reassure Bill that there is no need to throw out all of the old, while introducing some things that are new. The text scheme in conjunction with the NY City and NY State ‘core curriculum’ documents could be used as the guiding structure and also a source of good investigative and formative activities. The visiting consultant spent two or three lessons each week working with Bill and his classes. The classes were firstly observed, then demonstration lessons given. Bill was pleased with the increased level of student engagement and keen to debrief and plan with the consultant. Within the space of a few weeks, he was eagerly setting up effective student learning experiences. It was clear that this teacher already cared for his students and wanted to help them learn. As the weeks went on, he became increasingly reflective and (unnecessarily) critical of his teaching practice. He used some activities from the mandated text, mixed them with his own ideas and a vast range of manipulative materials. For a topic on fractions, Bill had his students use counters, play money, fraction strips, pizza shapes and calculators. He became focussed on students developing concepts in an inter-connected knowledge framework. Perhaps the most telling example was his spirited report to the consultant of useful information that particular activities revealed about student understanding: ‘I had no idea that they saw it this way. It’s no wonder they didn’t understand what I was trying to teach them. So I modified the activity for the next class and they got it straight away.’

The success of positive change with Bill’s teaching practice is one of many such stories, but it is certainly not the story with all. The reasons it worked here were initially that the teacher genuinely cared for his students and felt the need to teach concepts. The teaching approach and text mandated by the City were steps in the right direction, but the changes in Bill’s classroom are not so much due to the ‘top-down’ directives, but more due to the beliefs that Bill developed about the learning process. Consultancy support with guidance and a building of trust helped this development.

Something not so good: Dominance of the State and City testing programs

Most assessment is still predominantly equivalent to traditional testing. In many classes the curriculum methodologies are constrained by the need to prepare students for the

high stakes testing that determines their progress through school. This diverts effort away from learning for understanding towards a very algorithmic approach which is teacher-centred, based on memorisation of processes and on drill and practice. Test preparation becomes an overwhelming driving force as test time approaches. Additional allocations of time are given to mathematics, both within the school day and extending the school day, and with mandatory attendance. This is augmented by voluntary attendance at free afternoon and weekend tutoring.

In some middle schools four days each week are devoted to teaching with the fifth dedicated to test preparation and testing. Above and beyond this are the monthly tests.

The preparation of tests seems to revolve around ease of marking with an overemphasis on multiple-choice questions. In the most recent Grade 9 state examination (Regents Math A), approximately 70% of the marks were for multiple-choice questions. In teacher-generated tests, the mark allocations often bear little relation to the complexity or difficulty of the questions. In some extreme cases a multiple-choice question was weighted equally with a several part, multi-step word question; i.e., at 10 marks each. The tests are often adjusted to simplify calculation of percentage scores.

In the past there has been little cooperation between teachers in developing tests for identical curricula. There is a growing trend towards using computer-generated tests, either with stand alone software or software packaged as part of a textbook-based kit. At the school level the results of examinations may even play a role in determining tenure of teaching and administrative staff if schools have failed to show improvement in scores.

A student may pass a class without passing the examination. Class success is based on tests (80%) and marks allocated to attendance and homework/class work (total 20%). Project work, thought provoking assignments, portfolios, oral and other 'alternative' assessments rarely played a part in the classes that we observed. However, it is mandated by the City that no student who has attended at least one day per semester, can receive less than 40%. Yet an attendance rate of at least 95% is needed to gain a pass in the class component.

At state examination level, a 65% result is considered a pass. This result is determined by use of a scaling scheme on the raw score, which is varied from time to time. In the most recent Math A (Grade 9) exam, the raw score for a 65% level was deemed to be a minimum of 37 points out of a maximum 84. To obtain a school diploma, students needed a raw score of 28 out of 84, and this was equivalent to a scaled score of 55%.

Few teachers make any detailed analyses of results of tests or examinations for remedial teaching (re-teaching) or even to extend students. The consultants and coaches have had a positive influence in this regard, promoting the use of cooperatively prepared tests at grade level within schools and at regional level.

Conclusion

While it remains that different camps still see different sets of outcomes as being vital, one thing is certain: that it is vital that New York City get their education system right. Too many students are currently disillusioned with an authoritarian system in which they experience little success. The department is making steps in the right direction as it focusses on quality teaching with a philosophy of greater student ownership of their education. The intense consultancy model being employed is a positive thing, and is helping to change the attitudes and beliefs of some teachers and students. However, as teachers become more successful, it is important that they stay and lead in the City system, rather than be attracted to often more lucrative teaching assignments in neighbouring counties.

As for test preparation, the writers of this paper maintain that the best form of test preparation is good teaching in the first place — with lessons that empower students with a real understanding of, and appreciation for, the vital mathematics. As we continue our work in this busy, crowded, diverse and energetic system, it is encouraging that we are more frequently hearing these same sentiments on the lips of the New York City teachers.

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Multiples of three and saving Australia in 1942*

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The start of 1942 was a very frightening time for Australia. The Japanese had devastated Pearl Harbor and much of the US fleet there, they had landed on the Malay peninsula and would shortly capture Singapore, and had begun their island hopping expansion in the south-west Pacific. Had they been able successfully to invade Port Moresby via a seaborne attack from Rabaul, our supply lines would have been cut off. Had their plan to trap the remnants of the US fleet off Midway succeeded, Japanese naval control of the western Pacific would have been complete. I shall explain one essential component of the story as to why this did not happen!

Introduction

In December 1941, Japan launched a surprise carrier-borne attack on the main US Pacific naval base at Pearl Harbor in the Hawaiian Islands, destroying many of the major warships then in port. Fortunately, the US carriers usually stationed there, were at sea at the time and so not destroyed. Simultaneous with this attack, Japanese forces landed on the Malayan Peninsula and began their drive on Singapore, which was captured by mid-February 1942. By late March, General MacArthur had arrived in Australia, having evacuated the Philippines, and most of the islands of the south-west Pacific, and of the Dutch East Indies (Indonesia), were occupied by Japanese forces. Darwin had been bombed, the British Navy routed from the region, and Australians felt extremely vulnerable.

Yet within three months, the Naval Battles of the Coral Sea and Midway had restored hope that a successful struggle would see the tables eventually turned. How did this happen, and what role did Australia play in this short period? The big picture has been documented, but not the crucial roles played by Allied codebreakers in achieving these significant initial victories. I will tell you a small part of the bigger story of codebreaking in the Second World War (WW2), involving some unsung heroes, both distant and local.

WW1 beginnings

Two aspects of history from the First World War (WW1) are relevant to my story. First, the British realised that mathematicians could be usefully employed in cryptological (i.e.,

* This paper has been accepted by peer review.

codebreaking) work and recruited some from universities. Second, although Japan was an ally of the British Empire and America against Germany in this war, there were those in Australia, Britain and America who even then were concerned about possible Japanese plans for empire-building in Asia and the Pacific. These people consequently felt it useful to have links into Japanese society and also personnel proficient in Japanese, so as to obtain independent advice (other than through normal diplomatic channels) on directions in Japanese foreign policy at that time.

In particular, the Royal Australian Navy offered its officer cadets an opportunity to study Japanese, and to facilitate this it gained the support of the Senate of the University of Sydney to establish Japanese studies as early as 1917.

Between the World Wars

Both Britain and the USA had made effective use of significant codebreaking achievements during WWI and these also influenced the final negotiations regarding the sharing out of Germany's former Pacific territories amongst the victors, and in limiting the size of Japan's post-war Navy. Japan was unhappy with both, but the American and Australian governments saw no reason to continue closely monitoring Japanese affairs. The British were less complacent and its Royal Navy (RN) decided to continue covert operations monitoring Japan. This was fortunate for young Eric Nave, from Adelaide, who had entered officer training in the Royal Australian Navy (RAN) as a young lad during WWI and realised that he could receive an extra allowance if he volunteered to study Japanese. He did so well in this that James Murdoch, the Professor of Japanese at Sydney University (who was also contracted to teach Japanese to selected Navy personnel), recommended that he be sent to Japan for a couple of years to live in the country and to study for the British examinations for translators/interpreters. Nave obtained excellent results and his name was drawn to the attention of relevant staff in the RN. So, in 1925, when the RAN no longer had need of specialists in Japanese, the RN requested that Nave be seconded to it, which duly happened. Before long, Nave discovered that his task would be to establish interception and decryption facilities for the RN off China, monitoring in particular Japanese naval and diplomatic messages.

Enter the age of radio

Have you given thought to how difficult it was for ships at sea to communicate with each other, or with onshore stations, before radio communication became available? Navies worldwide were among the first to exploit the benefit of radio communication as a means of keeping in contact (and not just in contact but in instant contact!). But using radio, for all its obvious benefits, had one major disadvantage: anyone who could pick up a radio message could have access to its contents, and it was not difficult to develop interception equipment that DID pick up radio messages!

So Nave came into the business of radio interception at just the right time, and he was a major player in RN work on Japanese codes between the wars, when all seapower nations were busy devising what they hoped would be safe codes for naval communication. The most widely used system for sending radio or cable messages was that based on the Morse code of dots and dashes, so usable secure communication codes had to be adaptable to this means of transmission. I shall briefly describe a coding system that was usually highly secure, and is relevant to my story.

Two-stage or superenciphered codes

Such codes involve a two-stage coding process. Stage 1 uses a *codebook*, in which all the letters and digits, as well as commonly occurring words, phrases, numbers, etc., are each assigned a unique *codeword*, which may be an ordinary word, a sequence of nonsense letters, a set of digits, or some combination of these.

To encode a message (e.g., ‘We will attack Pearl Harbor at dawn on 12/7’), the cipher clerk looks up each word in the codebook and then writes down the set of codewords representing the above message. Now enter the second stage of the process, a stage most easily done if all the codewords are actually numbers of the same length; e.g., of five digits. The cipher clerk now picks up a second book, called an *additive table*. This book might have 100 pages and on each page there is printed a square or rectangular array of numbers, each the same size as the numbers appearing in the codebook. The clerk opens the table at random, and, again at random, puts a finger on one of the entries in the array (which can be uniquely identified by the row and column it sits at).

The clerk, having written down the set of numbers obtained from the codebook, now writes down under each such number, in order, the numbers in the table, starting with the chosen first one. Here is an example. Suppose the six codewords corresponding to the original message are:

31625 23418 71624 33001 01499 62222

and that the 6 consecutive entries in the table are:

21324 44556 06290 66529 12413 88374

The clerk puts one row below the other:

31625 23418 71624 33001 01499 62222
21324 44556 06290 66529 12413 88374

and then adds them in columns, without carry (oh joy!), obtaining

52949 67964 77814 99520 13802 40596

These are the actual ‘groups’ (known as GATs — ‘groups as transmitted’) that the cipher clerk will send. Of course, the clerk will also have to send information on date and time, on the sending station and the intended recipient, and, most importantly, on the exact starting point in the additive table used in composing the message.

The intended recipient, having picked up the radio message and correctly transcribed it, will then have all the information needed to know where to start in the additive table. ‘Stripping the additive’ from the GATs reveals the original codewords, whose meanings are then looked up in a ‘reverse codebook’.

Even without using additional tricks to disguise, for example, how the starting point in the additive table is hidden in the full message, trying to decrypt a message of this kind is no mean feat!

Since the Japanese language does not use the standard Roman alphabet and is also constructed differently, Japanese radio clerks used a system called *Kana Morse*, based on one form of representation of the language. So, as a first step in intercepting Japanese radio messages, our interception operators had to become proficient in *Kana Morse*, a non-trivial task.

Breaking a code of this kind

The simplest and best way to break such a code is to obtain, without it being discovered, copies of both codebook and its additive table, preferably also with a list of instructions on how to compose a message sent using the code. Without this last bit, a codebreaker has first to work out how the various bits of formatting information are included among the actual GATs of the intercepted message, and so discover what part of the additive table was used, before being able to decrypt it.

In practice, without benefit of access to stolen goods, the codebreaker hopes that a number of intercepts will be made, so that their actual forms may be compared in case it is possible to guess which bits contain the critical formatting data. Even so, the task is unlikely to succeed, unless one is led to suspect that a couple of the messages were encoded using the same starting point in the additive table, or at least a common section of it. Another helpful clue might come from guessing that some of the intercepts contain exactly the same underlying message, or have similar structures that suggest an underlying formality of style. Fortunately for us in WW2, both the German military and the Japanese military found it hard to escape formality with regard to message construction, modes of addressing recipients, etc., which all helped in the unravelling of the underlying codes.

There are several techniques available for systematically trying to obtain some new information about two-stage codes from a set of intercepted messages, once one has been able to make a start. A key process for this is called ‘aligning messages’, or ‘putting messages in depth’, which means finding intercepted messages which one believes have used a common part of the additive table, and working out exactly where some of this common part lies in each message. Doing this will be virtually impossible unless one is able to obtain lots of intercepted messages. Even then, it remains impossible if those sending messages are careful to make random choices of starting points for additives and avoid sending stereotyped messages. Fortunately, again, careless or harassed cipher clerks were prone to use the same page, and even the same starting entry on the page, for more than one message. Other radio interception techniques helped in locating the origin of a message and even identifying the operator who sent it.

I shall not discuss further general methods of attack, as my purpose is to explain why a special simpler technique was able to be used on JN25, the Japanese Navy’s principal operational code used during WW2.

Enter the British: Tiltman and Turing, Bletchley Park and the Far East Combined Bureau (FECB)

Brigadier John Tiltman is regarded now as the best British codebreaker of all time. The Japanese Navy introduced a major new five-digit code, JN25, in mid-1939 and Tiltman was given the task of fathoming its structure. He realised that it bore some similarity to certain Japanese Army codes that he had previously studied, and so suspected that it was indeed a two-stage code. By the end of 1939, he had confirmed this and had also found a truly unexpected feature. He suggested that every five-digit codeword in the JN25 codebook was not a randomly chosen number, as expected, but a multiple of 3. So, for example, 23418 may well have been a codegroup, but 31625 could not be one. This reduces the possible maximum number of codewords from 100 000 to one-third this number, i.e., around 33 000 — a considerable reduction! It was also quickly realised that this feature simplified the finding of codewords and additives, because there is a very quick mental check on

whether or not a number is a multiple of 3.

To obtain the hidden codeword from a GAT in an intercepted message, one has to find the additive used to produce the GAT and subtract it. If the original codewords were random five-digit numbers, this step usually does not help unless it produces a known codeword. But for JN25, unless the step gave a multiple of 3, one knew immediately that the number, suspected of being the additive used, had to be wrong.

Alan Turing and his team of mathematicians at Bletchley Park devised a simple mechanical aid, called the *subtractor machine*, to exploit the ‘multiples of 3’ aspect of JN25. Suppose I have been able to collect several messages that are aligned, so that I can arrange them, one above the other, with GATs containing the same entry of the additive table all appearing in the same column. If I am correct in this, then the same additive, when subtracted simultaneously from all the GATs in the one column, should always produce multiples of 3 in every place. If I know the additive from previous work, then I can check this. If I do not know the additive, then trying possible additives at random is not a practicable strategy. I can perhaps find the correct additive by guessing that one of the GATs in this column hides a commonly occurring codegroup that I already know.

If, for example, I suspect that the codegroup 23418 is hidden in one of these GATs, then I take the first GAT in the column, and work out what I must subtract from it to obtain 23418. Suppose this is 34184. I then subtract this from every other GAT in the column and check if each result is a multiple of 3. If most are not, give up and try out 23418 on the next GAT, repeat the subtractions and check again for resulting multiples of 3.

If none works, then I sigh and try the process with another commonly occurring codegroup suspected to lurk behind a GAT. If, at some stage, I end up with every subtraction a multiple of 3, my confidence that I have found the correct additive goes up steadily with the length of the column being used. I am pretty well certain if this length is 8 or more (although in practice, very experienced operators probably settled for fewer GATs in a column).

This machine was built and in operation during 1940. The British codebreaking unit in the Far East, associated initially with Captain Nave, had been expanded into a unit called the FECB, initially based at Hong Kong. It was moved from there to Singapore once Japanese attacks in China and Indo-China threatened its position. It became progressively more involved with the effort to build up the JN25 codebook and additive table, and to ‘read’ individual messages in this code, because the Bletchley Park group was obliged to focus more intensively on the war in Europe.

Other machines were subsequently built and put to use to exploit the flaw in the JN25 codebook; for example, to find ways of aligning intercepted messages.

Enter the Americans: Fabian and Station CAST

The British and American codebreakers began regular exchange of information early in 1941. At that time, none of the main American cryptographic centres had placed much emphasis on JN25, nor had made much progress with it. Lieutenant Rudy Fabian, in charge of signals intelligence Station CAST in the Philippines, then sent one of his staff to FECB, and he returned with full information on how the British had ‘solved’ the construction of JN25 and were using that to break, little by little, into individual messages. Fabian decided that all the codebreaking activity at CAST would focus on JN25 and encouraged its study elsewhere, but Station HYPO on Hawaii was required to continue work on other codes. It is ironic to record that the first really useful JN25 message com-

pletely solved by the Allies occurred just as Pearl Harbor was attacked.

This event immediately gave much more importance and intensity to this task. FECB and CAST worked on it, under attack, until both had to be evacuated in February 1942. Some FECB staff, and most of Fabian's team were relocated to Melbourne, while most of FECB went to Colombo in Ceylon (Sri Lanka).

Re-enter Sydney University and Captain Nave

Early in 1940, four academics at Sydney University — two mathematicians, Professor T. G. Room and 'Dickie' Lyons, and two classics scholars, Professor Dale Trendall and 'Aps' Treweek — with encouragement from the military at Victoria Barracks, Sydney, decided to meet together to learn some Japanese and to study cryptology. By October, their work had been officially noted by the military. Captain Nave returned to Australia in 1941 and was asked to help set up a cryptographic intelligence unit in Melbourne. He learned of the Sydney group and after meeting them, recommended they be joined with some military intelligence personnel to create this group. Later in that year, Room and an officer from the unit were sent to FECB to gain information on their current work and it is likely that they learnt something about JN25 as well as other codes being studied there.

So, when Fabian arrived in Melbourne, he found this unit in operation and by mid-March had set up a new operational station, later called FRUMEL (Fleet Radio Unit Melbourne), continuing the attack on JN25.

Coral Sea, Midway and the Kokoda Trail

Throughout March, April and May 1942, the stronger Allied attack on JN25 began to produce important operational information, usually with only partial decryption of messages, combined with other data obtained from direction finding, traffic analysis and radiofingerprinting.

FRUMEL made important contributions to the intelligence that enabled the Allies to learn of the planned Japanese seaborne invasion of Port Moresby, coming across the Coral Sea from Rabaul. The US Navy staged a successful counter attack (the Battle of the Coral Sea) and forced the invaders to retire, having lost an aircraft carrier in the process. This battle was also the first naval engagement in which warships did not directly fire at each other: the battle was fought out by carrier and land-based aircraft and established completely the dominance of aircraft carriers in all future naval battles.

This was the first significant defeat of the Japanese in the Pacific area. Fearing that a further seaborne invasion attempt would fail because of uncertainty over the provision of air cover, the Japanese decided instead to attack Port Moresby overland from the north coast of New Guinea, leading to the later extensive fighting on the 'Kokoda Trail'.

FRUMEL picked up the message conveying this decision not to use the seaborne route. This information was significant for the imminent Battle of Midway, as it enabled the US Naval commanders to withdraw carriers stationed in the Coral Sea back towards Pearl Harbor and to include them in their plans for this battle.

Admiral Yamamoto, the leader of the Japanese Navy, planned to invade the strategically important small island of Midway, confident of luring the remnants of the US Pacific Fleet, with its all-important carriers, into a trap where his superior forces would destroy them. JN25 intelligence revealed enough detail of these plans for US Admiral Nimitz to plan his own surprise attack on the main Japanese naval force. The result was the Battle

of Midway, 4–6 June 1942, in which four Japanese carriers were sunk, the invasion force again withdrew, and after which the Japanese no longer could claim naval supremacy in the Pacific.

These two crucial victories were tremendous morale boosters for Australians and saved Australia from the real threat of total isolation from all Allies, and possible invasion. Neither would have occurred without the enormous help provided by Tiltman's discovery of the 'multiples of 3' flaw in the design of the Codebook of JN25, which, astonishingly, the Japanese kept in place with all subsequent changes in this code!

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Mathematical knowledge of some entrants to a pre-service education course*

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Standards for teaching emphasise the need for teachers to have deep content knowledge. To assess the mathematical knowledge of students enrolling in its B.Ed. program, the University of New England has introduced a mathematics diagnostic test. This work is the first stage of an ongoing research project into the numeracy needs of students entering the B.Ed. program. The test is a pen-and-paper test that replaces previous on-line, multiple-choice tests. This paper reports on the test results, discusses some common errors made by students and outlines the future direction of the research.

Background

A three-year research project into the numeracy needs of students enrolling in the Bachelor of Education (B.Ed.) program at the University of New England was commenced in 2004. The decision to introduce a mathematics diagnostic test for students enrolled in the B.Ed. program was made in recognition of the fact that it is important that people entering the teaching profession have a deep understanding not only of the processes of teaching and learning but also of the content from their discipline area (Hill, Rowe, Holmes-Smith & Russell, 1996; Thomas, 2000, 2002; Darling-Hammond, 2000; Brown, 2002; Ingvarson, 2002; Buckingham, 2003, Committee for the Review of Teaching and Teacher Education, 2003a, b). This, in turn, has been acknowledged by the introduction of standards for the teaching profession (Interim Committee for a NSW Institute of Teachers, 2004).

Diagnostic testing is concerned not only with assessing a student's level of competency but also with providing suitable follow-up support/remediation. Such tests allow educators to engage in curriculum planning based upon improved knowledge of individual students and of trends in performance. Previously, students enrolling in the B.Ed. program undertook a series of compulsory, online, multiple-choice tests called *personal maths profiles* (PMPs). Students who made errors on the tests were directed to a set of textbooks for the purpose of self-remediation. This system did not provide academic staff with any information regarding student performance and cheating, in particular having someone else take the tests, was also a major issue.

To overcome these problems, B.Ed. students are now required to undertake a compulsory pen-and-paper mathematics diagnostic test. Calculators are not allowed. The test

* This paper has been accepted by peer review.

replaced the PMPs, which, although they were conducted and marked online, did not lead to any data collection or analysis. The pen-and-paper test has the disadvantage of being slower to mark and analyse than online tests, but has the major advantage that academic staff have access to students' hand-written working. This paper provides an overview of the analysis of the student responses on the test.

Data source and methodology

The B. Ed. program typically has enrolments of about 170 students each year. At the time of writing, 159 students had completed the diagnostic test, including 51 males and 108 females. Of the intake, approximately 62% had successfully undertaken some mathematics at Year 12 level since 2001, with another 26% undertaking mathematics at Year 11 level. The remaining students had either not taken mathematics in their senior years or had done so more than three years earlier.

Despite the high proportion of students who had successfully completed a senior secondary mathematics unit, it was decided that all students should be tested, using an instrument that would expose any misconceptions regarding mathematical content that the students would be expected to have mastered before teaching primary-level mathematics. Details of the test instrument are provided in the next section.

The Mathematics Diagnostic Test (MDT)

The test comprised five mental computation questions and thirty items taken from the 1999 Trends in International Mathematics and Science Study (TIMSS) test (TIMSS USA, 2000). The maximum possible mark was 37. The rationale for using these items was two-fold: the test items have been previously evaluated and validated, and the test is an instrument that has been used to assess the mathematical performance of Year 8 students from around the world. Benchmark data are available for each question, enabling comparisons to be drawn between the performances of B.Ed. students with those of students who completed the 1999 TIMSS test.

The TIMSS items covered five content areas: fractions and number sense, measurement, algebra, geometry and data representation, analysis and probability. Similarly, the items covered five cognitive domains: knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communication. The MDT included a selection of items to ensure coverage of the five content strands and the one process strand from the NSW K–6 Mathematics syllabus. However, questions were left open-ended on the MDT, whereas the TIMSS items included four choices of answer. This alteration was made to reduce the chance that students would guess answers and to provide staff with access to all the resultant errors.

Test results

The test results have been analysed in several ways including summary statistics for the entire cohort of students, question analysis by strand and common errors. These are now detailed.

Summary statistics

The level of mastery required by students was set at 80% (or a mark of 30/37). This level was considered to be appropriate to meet the standard set by the Interim Committee for a NSW Institute of Teachers. Only seventeen of the 159 students taking the test achieved this level of mastery. The breakdown of scores is shown in Table 1. The scores were normally distributed with a mean mark of 22.2 (60%) and standard deviation of 5.68 (15%). The lowest score was 8/37 (21%) and the highest score was 34/37 (92%). Forty students (25% of the intake) scored less than 50% of the total marks available.

Table 1. First year test scores for MDT 2004.

Score /37	Score %	Frequency	Cumulative %
0–4	0–10.9	0	.00%
5–9	13.5–24.3	3	1.89%
10–14	27.0–37.8	14	10.69%
15–19	40.5–51.4	31	30.19%
20–24	54.1–64.9	58	66.67%
25–29	67.6–78.4	36	89.31%
30–34	81.1–91.9	17	100.00%
35–37	94.6–100	0	100.00%

Question analysis by strand

The types of questions that were well handled by the cohort included reading information from graphs (but not generating their own graphs), basic computation involving whole numbers (except for multi-digit subtraction) and completing numerical patterns (but not using algebraic rules). All of these questions had a success rate of 80% or higher and a baulking rate no greater than 8.2%.

The question types that were least well handled included conversion between metric units, calculations with fractions, decimals and ratios and the use of algebraic rules to extend a pattern. None of these questions had a success rate greater than 35%. Baulking rates varied between 6.3% and 40.3%.

Overall, the baulking rates were reasonably low (the mean was just below 12%). The main exceptions were the question on triangular numbers, which had a baulking rate of 40.3%, the question on the average weight of a salt crystal, which had a baulking rate of 27%, and questions on substitution of a pronumeral into a linear equation and the repeated use of fractions, each of which had a baulking rate of 22.6%. All of these questions were calculation-intensive and required several steps to solve. Low baulking rates were associated with low-complexity questions that required little computation, including the mental computation questions and questions involving interpretation of diagrammatic or graphical information.

Low baulking rates were not always associated with high success rates, particularly for the mental computation questions. Of these, the multi-digit subtraction question had a baulking rate of 7.5% and a success rate of 47.8% (i.e., an error rate of 44.7%), the question on adding fractions had a baulking rate of 10.7% and a success rate of 30.8% (i.e., an error rate of 58.5%), and the question on multiplying decimals had a baulking rate of 7.5% and a success rate of 22.6% (i.e., an error rate of 69.9%). Similarly, the student-professor problem had a baulking rate of 10.7%, a success rate of just 4.4% and an exceedingly high error rate of 84.9%.

In summary:

- students were proficient at identifying numerical patterns but poor at deriving and using algebraic rules;
- students were proficient at reading graphical information;
- students had a poor background in probability;
- students demonstrated a reasonable grasp of equivalent fractions but lacked ability to use fractions in problem solving and in conversion between different units of measurement;
- geometry questions were well handled;
- students seemed to have a good knowledge of the basic facts of measurement but were unable to apply this knowledge to the solution of problems;
- students lacked computational skills, particularly in the areas of multi-digit subtraction and operations involving fractions and decimals.

The main areas identified as requiring attention were the use of metric units, patterns and algebra, fractions and number sense and mental computation (particularly multi-digit subtraction, addition of fractions and multiplication of decimals). The cohort appeared to have good recognition of basic facts but lacked the ability to apply this knowledge to the solution of problems. This situation was compounded by the cohort's poor computational skills. Some of the common errors that were identified are discussed next.

Common errors

The identification of common errors has allowed us to tailor classroom materials to the needs of the particular cohort. This section focuses on some of the common errors made by the students.

Multi-digit subtraction

One error was demonstrated by 4 students (2.5% of the cohort). In this technique, students treated each 'column' of digits as a separate problem and subtracted the smaller digit from the larger digit, resulting in the working:

$$\begin{array}{r} 8006 - \\ \underline{2993} \\ 6993 \end{array}$$

Another error was made by 10 students (6.3%). Here, students showed a lack of understanding of 'borrowing from zero'. Whilst students actually 'borrowed' to enable them to perform the subtraction, there was no 'pay back'. The resultant working was:

$$\begin{array}{r} 8006 - \\ \underline{2993} \\ 6113 \end{array}$$

Operations with fractions and decimals

a) $\frac{1}{2} + \frac{1}{3}$

The most common error (frequency of 47 or 29.6%) was for students to add the numerators and the denominators yielding the working:

$$\frac{1}{2} + \frac{1}{3} = \frac{1+1}{2+3} = \frac{2}{5}$$

A further 12 students (7.5%) multiplied the numerators and added the denominators giving:

$$\frac{1}{2} + \frac{1}{3} = \frac{1 \times 1}{2+3} = \frac{1}{5}$$

Both of these errors represented a lack of understanding of the structure of fractions, combined with 'misremembered' details of the algorithm for adding fractions.

b) Place in ascending order: $\frac{1}{2}, \frac{2}{3}, \frac{7}{10}, \frac{3}{5}$

The most common error was made by 28 students (17.6%). Here, students focussed on the denominators of the fractions and showed some understanding of how the magnitude of the denominator affects the numerical value of the fraction. Fractions were placed in order of decreasing denominators:

$$\frac{7}{10}, \frac{5}{6}, \frac{3}{5}, \frac{2}{3}$$

Similarly, a further 18 students (11.3%) focussed on the value of the numerators and placed the fractions in ascending order by numerator:

$$\frac{2}{3}, \frac{3}{5}, \frac{5}{6}, \frac{7}{10}$$

c) Shade $\frac{3}{8}$ of the grid (6×4)

Students needed to identify that there was a total of 24 cells in the grid and that

$$\frac{3}{8} \times 24 = 9$$

Two common erroneous answers were to shade 8 cells (11 students or 7%) or 3 cells (8 students or 5%).

d) 0.3×0.3

The most common erroneous answer was 0.9, which was given by 98 students (61.6%). This arose due to a lack of understanding of place value and led to other difficulties on the test, most notably for questions that required students to convert measurements in one set of units to another set of units. Another 4 students (2.5%) added the decimals, yielding an answer of 0.6.

Writing algebraic expressions: The student-professor problem

This question was the most poorly handled on the test; only 7 students (4.4%) correctly answered it, while a further 84.9% attempted it unsuccessfully. The most common error was to write a numerical ratio, either 1:16 (20 students or 12.6%) or 16:1 (56 students or

35.2%). A further 34 students (21.3%) reversed the order of the pronumerals in the equation, giving the answer $P = 16S$. This result is well recognised and can arise from confusion over the use of letters to represent units rather than variables; e.g., $1 \text{ m} = 100 \text{ cm}$. It shows a basic lack of understanding of the concept of variables, resulting in an inability to work with algebraic rules.

Problem solving

The elevator problem

Students were required to determine the number of floors travelled by a lift and then to convert this into the distance travelled by using the fact that the floors were 3 m apart. The main source of error here was arithmetic (counting the number of floors travelled). 104 students (65.4%) made this error, which is one of process rather than a misconception and could have resulted from misreading the problem.

A bag of marbles

This question had a high baulking rate (36 students or 22.6%). Students needed to identify the effect of reducing a quantity by a fraction and then to reverse the process to calculate the original amount. One correct version is:

$$\begin{aligned}\frac{3}{4} \times \frac{2}{3} \times x &= 24 \\ \Rightarrow \frac{1}{2} \times x &= 24 \\ \Rightarrow x &= 2 \times 24 = 48\end{aligned}$$

The most common error was to ‘undo’ the parts of the problem by multiplying by the denominator. So ‘gave away a third’ led students to multiply 24 by 3. Similarly, ‘gave away a quarter’ led students to multiply the result by 4, giving:

$$\begin{aligned}x &= 24 \times 3 \times 4 \\ \Rightarrow x &= 288\end{aligned}$$

A total of 8 students (5%) produced this solution. Other errors arose as a result of incorrect calculations.

Ratio of width of a rectangle to its perimeter

The question stated that the width of a rectangle was half its length and asked students to calculate the ratio of the width of the rectangle to its perimeter. This required students to recall the formula for perimeter and to replace length by twice the width. The most common error (70 students or 44.1%) was to write the ratio of width to length, which could have resulted from simply misreading the question.

What is left if I spend five eighths of \$240?

Students were required to either calculate five eighths of \$240 and to subtract the answer from \$240 or to realise that if five eighths of an amount is spent then three eighths remains. However, 17 students (10.7%) calculated five eighths of \$240 and returned the answer \$150. The source of the error could either be misreading the question or omitting the final step of the solution process.

Club membership

This problem could be solved by using the method of simultaneous equations. Alternatively, students could recognise that half of the number of members is 43. Adding 7 to this value gives the number of females (50) and subtracting 7 gives the number of males (36). The most commonly used method was the latter. However, 35 students (22%) added 14 to 43 to obtain the number of female members (57) and subtracted 14 from 43 to obtain the number of male members (29). There was no written evidence that any student checked their answer to ensure that the constraints had been met. Very few students attempted to apply the method of simultaneous equations to this problem, reflecting a lack of understanding of the power of algebra in problem solving.

Mathematical notation: Write in simplest form: $n \times n \times n$

This question is linked to the measurement strand of the NSW syllabus, where Stage 3 students are required to use correct notation for units of area (e.g., m^2) and volume (e.g., cm^3). A total of 64 students (40.2%) gave the answer $3n$.

Conclusions and follow-up

This paper has presented the results of a diagnostic test used with students entering a pre-service education course. The test involved five mental computation tasks and thirty items taken from the 1999 TIMSS test. The main areas of weakness identified by the test were:

- mental computation;
- structure and value of fractions and decimals;
- operations with whole numbers, fractions and decimals;
- use of algebraic rules to express numerical patterns;
- applying algebraic rules to solve problems;
- reading and using correct mathematical notation;
- appropriate use of problem-solving techniques.

The identification of common errors was useful for two reasons. Firstly, it provided direction for workshops in the core unit, where pedagogical knowledge is used as a vehicle to develop mathematical understanding (Callingham & Mays, 2004). Secondly, probable sources of error were identified. By addressing these in workshops, the aim was to improve the undergraduates' own understanding of mathematics and to develop appropriate problem-solving techniques (including reviewing answers and developing appropriate checking rules).

The work is part of a larger ongoing research project. Students who did not achieve 80% or better on the retest were encouraged to enrol in a new elective unit in mathematical misconceptions. The unit was designed to help students to identify and remediate their own mathematical misconceptions in order that they will, upon graduation, be able to help primary students in the same tasks. The evaluation of the second stage is to be undertaken in 2005.

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The rewards and difficulties of working mathematically*

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This paper summarises the experiences of several teachers who taught a number of Working Mathematically activities using workbooks designed by the authors. The teachers felt that, because of the requirement to communicate their ideas with each other, students had gained a deeper understanding of the mathematics. The activities also enabled students to make stronger connections with their everyday experiences and with other mathematical topics. However, many teachers reported difficulties due to time constraints and several had problems integrating Working Mathematically activities with their regular textbooks. The results point to issues which need urgent attention if the rewards of Working Mathematically are to be realised in practice.

The new NSW 7–10 Mathematics Syllabus

For the past year, NSW schools have been teaching mathematics to students in Years 7 and 8 (students mostly at Stage 4) from a new syllabus (Board of Studies NSW, 2002). There is little change in the mathematical content of the new syllabus compared with the previous one. The major change is in the teaching process. What was previously a separate Problem Solving strand has been replaced with Working Mathematically as a process strand. This strand is to be interwoven with five content strands. To this end, the syllabus lists a large number of illustrative Working Mathematically activities for each content strand.

‘Working Mathematically’ appears to be a uniquely Australian expression used to describe how mathematics is used in practice. It includes the processes of questioning, apply strategies, communicating, reasoning and reflecting. The objective is that:

Students will develop knowledge, skills and understanding through inquiry, application of problem-solving strategies including the selection and use of appropriate technology, communication, reasoning and reflection. (Board of Studies NSW, 2002, p. 12)

The implication is that teachers are expected to follow a process-based approach to mathematics, treating mathematics as a set of interesting challenges or problems rather than as a series of methods and formulae to be learnt for examinations.

* This paper has been accepted by peer review.

Only a few of the Working Mathematically activities listed in the syllabus can be found in textbooks. This is particularly true for Working Mathematically within the Space and Geometry strand. The following Stage 4 examples from the syllabus are typical:

- interpret and make models from isometric drawings (communicating);
- use dynamic geometry software to investigate angle relationships (applying strategies, reasoning);
- recognise that similar and congruent figures are used in specific designs, architecture and art work; e.g., works by Escher, Vasarely and Mondrian; or landscaping in European formal gardens (reflecting).

To implement the new syllabus effectively, teachers need to prepare lessons that use the Working Mathematically processes to teach the mathematical concepts in the content strand. This requires considerable time, thought and creativity.

The Working Mathematically workbooks

To make the Working Mathematically approach of the new syllabus easier for teachers to implement, we wrote two student workbooks (Part A and Part B) for the Stage 4 Space and Geometry strand (McMaster & Mitchelmore, 2003). Activities are included to cover every content outcome of the Space and Geometry strand.

Wherever possible, workbook activities relate to real life or other key learning areas (physics, chemistry, biology, geography, surveying, sport, history, art, design and religion). The activities are also carefully sequenced. Students begin by looking at a simple case where a concept is used, then investigate similar cases and special cases before arriving at a generalisation. Questions in the workbooks are deliberately designed to help students make linkages with existing knowledge and experience.

Students are provided with visual stimuli and a variety of tables, grids, space for geometric constructions, and paper for 'cut-out' activities. Computer software used in the workbook is free to download from the internet and additional manipulative materials and activities are inexpensive and readily available. The workbook also serves as notes for study and future reference.

In 2004, five schools trialled the workbooks. In addition, about seventy teachers trialled individual activities. The purpose of the trial was to gain feedback and share experiences so that the workbooks could be improved. Teachers were given privileged access to the website www.workingmaths.net which includes comments on the workbook activities and a discussion forum.

This paper reports primarily on the results of informal discussions held half way through the year with teachers from three of the trial schools (an independent school for girls (Years 7 and 8), a comprehensive state school (Years 7 and 8), and a state primary school with two classes for gifted and talented students (Year 6)). Some students in the last school, as well as some individual teachers from other schools, were also interviewed. The aim was to obtain insights into the following questions:

- Which types of activity are most worthwhile and why?
- What difficulties are experienced with the Working Mathematically approach?

Experiences of teaching Working Mathematically

The independent school for girls

There are four streamed mathematics classes in each of Years 7 and 8 in this school, and they all trialled the workbooks.

The teachers selected as the most worthwhile an activity, called ‘Travel Tests’, in which students discuss the following questions in small groups.

1. Andy and Bill take a ride together on a see-saw. Andy sits on one side of the see-saw, 2 m from the centre. Bill sits on the other side, 1 m from the centre. With each tilt, whose side travels through the greater angle?
2. Andy and Bill take a ride together on a merry-go-round. Andy chooses to ride a horse that is near the outside edge of the merry-go-round. Bill chooses to ride a horse that is nearer to the centre. Who travels faster?
3. Andy and Bill travel with their families around Australia. They both travel along the same roads, but Andy’s family travel in a clockwise direction while Bill’s family travel anti-clockwise. Who travels further?

This activity generated much argument and discussion among the students. The teachers noted that they did not have to lead the discussion but merely facilitate it — quite a change from their normal role. As a result, students communicated with considerable conviction and authority: ‘This is my diagram. This is how it works.’

A second activity considered particularly worthwhile, required students to answer the following four questions:

1. Describe the pattern of the ripples made when you throw a small pebble into still water.
2. When you pour pancake mixture into the centre of a smooth, flat pan, why does the mixture spread out in a circle?
3. Why is an archery target round?
4. If you squash a circle (e.g., the circle of a roll of cardboard) so it is half of its original height, what shape does the circle become? What happens to the amount of space inside the circle as you squash it?

Teachers liked this activity because it caused students to think about what they had recently learned. In particular, they made connections between abstract concepts and real life situations and saw the value in using mathematical language to communicate ideas.

A third activity which turned out to be valuable, requires students to cut up plastic straws to specified lengths, thread three pieces of straw onto a long pipe cleaner, then twist the two ends of the pipe cleaner together so that (if possible) each straw piece forms the side of a triangle. Questions guide the students to explore the relation between the apex angle and the base, and then to explain why the length of a side of a triangle can never be more than the sum of the lengths of the other two sides.

Some of the students in a third stream class insisted that they could make a triangle out of the three pieces that were 12 cm, 6 cm and 6 cm long. They showed the teacher their ‘triangles’. He explained to the students that these straws placed together would not make a triangle if the pipe cleaner was not threaded through them. The teacher’s first reaction was that he would avoid this argument by omitting these three lengths in future.

On the other hand, the teacher of a top stream class had decided to omit this activity

because she believed her students did not need the practical work. She wanted to use the newly bought textbook and decided to teach the same concepts using it. However, when she analysed the results of the subsequent geometry test, she discovered that the third stream class had achieved a better result than her class on a question related to this activity. It appears that students in the class who had argued with their teacher had understood and remembered the concept better than those who merely used diagrams.

The major difficulty found by teachers at this school was a time issue: They spent longer than intended on the Space and Geometry strand. Nevertheless, they felt the time was well spent because students had gained a deeper knowledge of the subject through the connections they had made.

The state comprehensive school

This small central school (Years K–10) is classified by the NSW Department of Education and Training as geographically isolated, and it receives a socio-economic allowance. There is only one mathematics class in each of Year 7 and Year 8, both taught by the same teacher, and both trialled the workbooks.

The teacher commented that all the ‘hands-on’ activities were worthwhile for her classes. The two most worthwhile activities were one in which students cut out possible nets to see whether they formed cubes, and another in which polygons made out of cardboard were stapled together to make Platonic solids. The students were excited about discovering these solids for themselves and were interested in the historical connection between the Platonic solids and the five ‘elements’ of the ancient Greeks: fire, water, earth, air and the universe. One student, who had difficulties in numeracy and literacy, found the topic particularly fascinating. The teacher commented, ‘He could not spell a word like “school” but he could spell “tetrahedron” and could tell you about each of the platonic solids in great detail.’

The ‘Travel Tests’ activity, described earlier, was also rewarding. In this case, the teacher assigned the questions as homework. One student discussed the second question with his father, who used a circular saw analogy to convince him that Andy and Bill travelled at the same speed. The teacher could not shake the student’s conviction by argument or drawing a diagram, so she took the class into the school yard and had two children run around in concentric circles. The puffing and panting of the student who ran around the outer circle convinced the others that he had run faster. Class discussion of the other questions in this activity revealed several other misconceptions about distance, time and speed which the teacher was able to address.

The only difficulty this teacher mentioned was that she did not have similar workbooks for the other strands of the syllabus. Some of her students do not work well from textbooks, but like the scaffolding provided by the workbook. Also, she commented that most of her students have no interest in trying to remember anything with which they cannot make real life connections.

Selective classes in a state school

Gifted and talented students in the ‘opportunity classes’ in this school are selected from public primary schools in the area, based largely on a test given in Year 4. The two Year 6 opportunity classes, who had both completed Stage 3 mathematics, trialled the workbooks.

The teachers said they enjoyed teaching from the workbooks because they fit with their use of small group work and their intention to promote higher order thinking. They

found that the workbooks benefited students by requiring them to give precise explanations. Students who grasp mathematical ideas quickly seem reluctant to express their understandings verbally. One example of this occurred in an angles activity. Having drawn vertically opposite angles and found that as one angle becomes smaller, its vertically opposite angle also becomes smaller, students are asked why this happens. One of the students commented, 'It's easier to just know it'. Explaining such relationships deepened students' understanding beyond a superficial, procedural level.

One teacher commented that gifted students often ignore the practical aspects of a problem. For example, in the third question of the 'Travel Tests' activity, no student considered the fact that the two cars travel on different sides of the road. Making such connections helped students realise the value of non-mathematical knowledge in helping to solve mathematical problems.

The students particularly liked an activity that enabled them to be artistically creative while learning about congruence and similarity. The workbook reproduced two historic Spanish Islamic designs, and students were asked to use grids provided to create their own geometrical designs. Their designs were mounted and displayed in the classroom. Figure 1 shows two of their many beautiful designs.



Figure 1. Designs by two Year 6 students.

One teacher noticed that all of his students did well on a subsequent Space and Geometry test and that there was less spread of results compared with previous years. The only difficulty he perceived was that his students would be bored if they had to repeat the mathematics they have already learnt the following year in high school.

Other teachers

Six teachers who had tried out individual workbook activities were interviewed. These teachers had chosen a wide variety of activities, the most popular ones being the 'Travel Tests' activity described earlier, activities that explored geometrical shapes using pipe cleaner and straw models, and activities that involved drawing and art. These were viewed as 'fun types of extension activities', and had served this purpose well.

Three teachers reported particular incidents where a workbook activity had helped students understand a mathematical concept. In all cases, the students were not in the top classes. Two teachers reported using selected activities with students in Years 9–11.

All the teachers mentioned 'time pressure' as a major difficulty, since they had to

follow the same teaching programs as other teachers not using the workbooks. Four of the teachers explained that they had not used the workbook as much as they had hoped because they were using new textbooks and were more focused on trialling these. A teacher in her first year of teaching felt she was more willing to try out new ways of teaching than more experienced teachers.

Discussion

Despite large differences between the trial schools, teachers reported the same two main reasons why the Working Mathematically approach is worthwhile:

- because Working Mathematically activities require students to reason and to communicate mathematical ideas to their peers, they achieve a far deeper understanding of the topics studied than they would otherwise do;
- because many Working Mathematically activities are set in realistic contexts, students are able to establish stronger connections between mathematics and their world. This realisation not only promotes greater understanding, but is also highly motivating.

In all of the trial schools, students were confident enough to ask questions and communicate mathematical ideas based on their own investigations and understanding. At two of the three schools there was an indication that the Working Mathematically approach may be reducing the spread of test results. This result is consistent with the findings of studies in England (Boaler, 1998) and the United States (Schoenfeld, 2002) that a process-based approach to mathematics teaching can narrow the performance gap between students while raising performance as a whole.

Research has shown that when students learn mathematical concepts in real world contexts, as opposed to the traditional approach where learning is abstract or 'context-free', they demonstrate important differences in attitude and interest towards mathematics (Boaler, 1998). These differences are particularly important because they are likely to affect the students' subsequent learning of mathematics. The teachers in this study seem to have been reporting a similar effect.

The major difficulty in implementing a Working Mathematically approach seems to be a perception that investigative activities take too much time. The time required could be reduced somewhat as teachers and students become more familiar with the teaching approach and resources become more streamlined. However, there is no doubt that Working Mathematically activities do take time; but the question must be asked: is it not time well spent? If students achieve a deeper level of understanding, benefits could come by reducing the time necessary for revision when the topics are extended in later stages of the syllabus. Also, although the time spent teaching Space and Geometry may be greater, there may be savings in other strands. The Space and Geometry activities, for example, make links with Measurement, Number, and Patterns and Algebra. If this is happening, the overall time required to teach the syllabus may not be so greatly affected. Hiebert (2003) provides some evidence to support this claim.

Many teachers view the Working Mathematically approach as being better suited to higher ability classes. There is no evidence from our enquiry that this is the case. High-achieving, average and low-achieving students all seem to have benefited but in different ways. Teachers still need to adapt their teaching to the students in their class.

Conclusions

It appears that there is much to be gained from Working Mathematically, as it can give students a greater depth of understanding and a more positive attitude towards mathematics. However, even when resources are available to support this process-based approach, many teachers still perceive that there is insufficient time available to implement it.

Implementation of the Working Mathematically strand in the new NSW 7–10 mathematics syllabus is not going to be easy. It cannot be achieved simply by covering the content in the new textbooks, even if they are supplemented with a few worksheet activities. It is also not enough for individual teachers, or even all the teachers at one year level, to introduce an integrated programme of investigative activities. To take full advantage of the rewards of Working Mathematically, the difficulties need to be overcome by a school-wide collaborative effort. Such efforts may lead to teaching programs which are radically different from the ones currently in use.

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Improving whole class teaching: Differentiated learning trajectories*

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Vygotsky's notion of zones of proximal development (ZPD) is used to introduce one aspect of pedagogy that can be differentiated in order to make teaching and learning more inclusive: planning of learning trajectories. This strategy is illustrated with reference to a case study of one Grade 5 lesson.

Introduction

Lev Vygotsky was an eminent Russian psychologist who lectured on the development of thought, language and social development — among other topics — from about 1920 to 1934. Throughout his short career, and particularly just before he died of tuberculosis, Vygotsky and his followers were under threat because their social development theory did not fit with the prevailing scientific and political ideology of Russia. Although ideas from the Vygotsky school of thought were withdrawn from circulation after his death, they continued to be developed in secret by his students. Later changes in the USSR allowed Vygotsky's own research findings and other teachings to be re-published, and after some time his work in developmental psychology had a profound influence on school education in Russia. Interest in Vygotsky's theories has spread throughout the Western world, and neo-Vygotskian theorists have developed further sociocultural theories such as activity theory and situated cognition. Each of these theories builds on the Vygotskian premise that 'the interpersonal precedes the intrapersonal' (Daniels, 2001, p. 70).

The key to learning, here, is social participation. For Vygotsky, the development of the mind resulted from social-goal oriented and socially-determined patterns of interaction between human beings. An important idea was that each child, in relation to learning of any particular topic, has a 'zone of proximal development' (Vygotsky, 1978, p. 86). This ZPD evolves ahead of the child's development, because it describes learning that could take place with the assistance of adult guidance. With social interaction that provides some mentoring, a child's knowledge develops. This results in a new zone of potential development. In a way, it's like holding the carrot before the donkey in that the ZPD is forever ahead of the learner; but of course the ZPD is unlike the carrot because the zone changes as the child moves forward. Thus a child is never 'in' the zone, and the knowledge and skills of the zone are always proximal (potential) rather than current.

The neo-Vygotskian educational theorists, Wood, Bruner and Ross (e.g., 1976) — and later Bruner (e.g., 1983, 1986) — developed the concept of leading children's learning

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forward through ‘scaffolding’. Bruner (1996) wrote about scaffolding as a logical structuring of ideas to be understood in an order that is likely to lead children to develop further and faster than they would on their own. Scaffolding of learning involves two processes: (a) active guidance and modelling by the tutor; and (b) active construction of knowledge by the learner; but these two aspects are responsive and interactive, not merely sequential.

In essence, scaffolding involves a mentor — perhaps a teacher, parent, or more knowledgeable peer — taking on an instructional role by providing pedagogical pathways and modelling that support children’s movement into new territories. Thus teachers’ intervention provides supportive tools for learners that extend their knowledge and skills, thereby allowing learners to successfully accomplish tasks not otherwise possible. Scaffolding closes the gap between task requirements and the skill level of the learner (Greenfield, 1984, p. 118), but teachers should remove scaffolding as soon as the reciprocal cognitive structure of the child can stand on its own. These ideas form a useful theoretical foundation for exploring ways that teachers can scaffold learning in order to advance children’s mathematical learning.

Managing diversity

Times when all Australian children were expected to achieve to a common ‘pass’ standard before progressing from one grade level to the next have passed. These days, it is more common to aim to have each student achieve, but with varying degrees of success in relation to their peers. Thus having different goals and realistic expectations for individuals, or for groups of students participating in any lesson, is common. Hence many mathematics classes are ‘ability’ grouped, or at least teachers provide remedial teaching and extension. Teachers have different expectations for students and do their best to provide the sorts of instruction and activities that will allow each child to succeed.

This sounds fine in theory, but is problematic in many respects. Catering for diversity in these ways can exacerbate difference: one could say, ‘Those who have it, get it,’ — whether the latter ‘it’ be increased opportunities to learn, more challenging teaching, or higher expectations of teachers, peers, parents, and self. Those who do not achieve so easily usually get more teachers’ time and attention, but much of that is spent repeating explanations and demonstrations — perhaps more loudly or slowly — and working on lower-level tasks. Such teaching may repeatedly make the same cognitive demands or, more commonly, lower the demands progressively so that both children and teachers can experience a sense of achievement and lesson completion. Moderation of expectations preserves the self-concepts of teachers as well as learners. While it is comfortable to lower expectations for children who continually struggle with mathematics concepts and processes, extensive research projects (e.g., Askew, Brown, Rhodes, Johnson & Wiliam, 1997) have shown that effective teachers of numeracy maintain high expectations of all children.

Further, it is clear that no teacher can deal with, day after day, thirty changing zones of potential development related to a wide range of mathematical concepts, terms, skills and activities. If we can move children’s mathematical development forward as a whole class, the question of how best to mentor such development becomes more manageable. In this ideal situation, learning objectives belong to a class and all children can understand and participate fully in vital stages of a lesson, including its introduction, its key learning activities, discussion periods and — most importantly — lesson closure. Again ideally, the closure includes planned opportunities for reflection, metacognition and perhaps higher

level thinking such as abstraction, generalisation, and networking of mathematical concepts. Further, it forms a class-wide platform for planning of the following lesson.

A key question here is how to get *all* children headed along the same learning trajectory. Any learning for a lesson or a sequence of lessons comprises three inter-related components: (a) the learning objective; (b) the learning activities; and (c) the teacher's prediction of how the students' understanding will develop in the context of a lesson or a sequence of lessons (Simon, 1995). Simon used the word 'hypothetical' to suggest that all three factors are likely to be somewhat flexible, with many experienced teachers changing the learning goals and adapting aspects of planned activities in response to their perceptions of students' levels of understanding as well as ongoing evaluations of their performance of classroom tasks. Thus actual learning trajectories cannot be known in advance, but hypothetical ones can be planned.

As noted above, a common reaction to children bringing varied levels of understanding to mathematics lessons is to plan quite different learning trajectories. While usually working on the same general topic, some children will experience lower starting points, less demanding activities and easier-to-achieve learning outcomes (see Figure 1, representing so called 'ability' grouping) while others will be extended to higher-than-expected learning outcomes. Another way of coping with difference is to 'ramp up' expectations, starting with relatively easy work but recognising that the students will reach different points of achievement along a common hypothetical trajectory (see Figure 2). Here, starting points are assumed to be the same, activities progressively more demanding, and students attain various levels of engagement, success, and challenge.

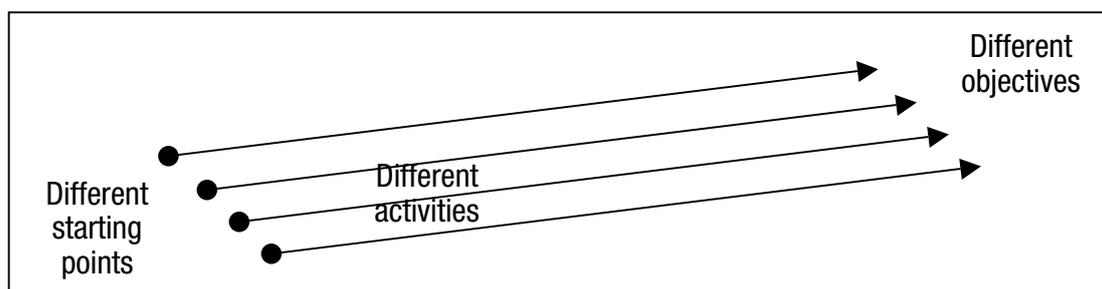


Figure 1. Hypothetical trajectories for different 'ability' groups.

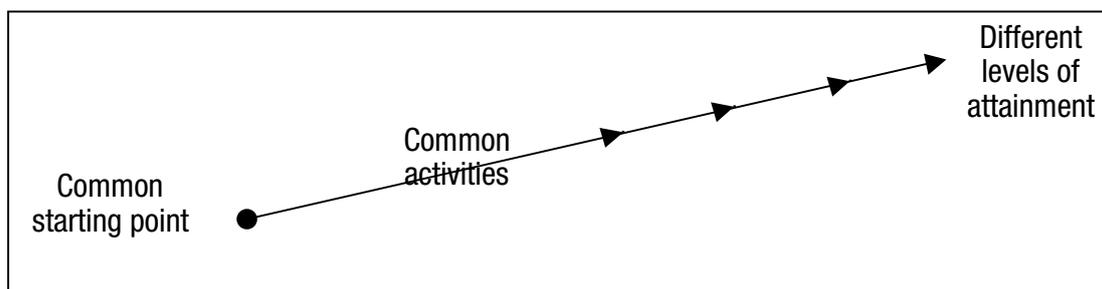


Figure 2. A hypothetical learning trajectory for the whole-class.

Of course, actual outcomes are never as neat as these models: children rarely learn only what teachers intend to the planned level of competency. They fall behind or exceed teachers' expectations, and actual outcomes tend to cluster around or parallel intended trajectories.

It is important to note four weaknesses of the above models. First, neither one sets up a climate for whole-class discussion and group reflection on what has been learned.

Second, both have the potential to disenfranchise some students, weakening self-concepts and publicly displaying levels of lesser achievement. Also, neither model forms a strong whole-class foundation of shared experience and understandings that can be used in teachers' planning for further learning. Last, the models do not address the question of equity: students who come into lessons with less competence leave the classroom less competent — and perhaps with an even greater differential. International comparisons indicate that Australia is failing to offer all students appropriate educational opportunities (see Lokan, Greenwood & Cresswell, 2001), and there is plenty of evidence that children from lower SES backgrounds are the ones who typically get left behind.

Maximising success in mathematics for disadvantaged students

In a three-year research project, we are exploring how to improve outcomes for Indigenous pupils and those from lower SES backgrounds by making relatively minor changes in mathematics classrooms. We are using teachers in schools where there is a mix of SES student backgrounds, and are researching mathematics teaching in upper primary and junior secondary levels because results may be transferred to both lower and higher levels. In recognition of research findings about the negative effects of achievement grouping as well as a concern that mere ramping of tasks has the potential to exacerbate difference, our focus is on improving whole-class teaching.

Our research is exploring ways that whole-class pedagogy can be differentiated to meet the needs of individuals while moving students forward through communal experience. We are researching whether teachers find specific inclusive strategies easy to use, what additional demands they may place on teachers, what teachers must know and be able to do to put them into effect, and ways in which the strategies change the learning experiences of and outcomes for students in mathematics classrooms. In particular, we wish to identify ways that such differentiation in pedagogy may improve outcomes for Indigenous pupils and those from lower SES backgrounds.

The data collection is being guided by a framework, developed from Clark and Peterson (1986), that has teachers' beliefs and understandings interacting with opportunities, constraints, intentions, and actions. The research approach is a combination of (a) interpretive analysis of teaching and teacher development, and (b) broader quantitative data collection. Currently, teaching experiments using the processes established by Sullivan, Bourke and Scott (1997) are being carried out, involving modelling and coaching (see Guskey, 1986, and Clarke, 1984, for a rationale for this approach). In each classroom, a researcher is working with the teacher. Observation notes are being recorded, using a format developed as part of the investigation, and some children's work is being collected. Short pre-lesson and post-lesson interviews and/or written reflections by the teachers are being used to collect evidence of teachers' planning and reactions as well as specific case examples.

Differentiated learning trajectories

The strategy being reported in this paper is the use of 'differentiated learning trajectories'. These are whole-class trajectories, with the added feature that teachers invite students experiencing difficulty to solve a similar problem with reduced cognitive demand, but only as a scaffolding step that provides immediate access back to the class' hypothetical learning trajectory (see Figure 3). The alternative pathways are shaped by

‘enabling prompts’. The whole class still works on the basic tasks, but some students have differentiated experiences that enable them to undertake those tasks in order to reach the same levels of attainment as their peers. Thus the class remains a coherent learning community based on common mathematical experiences and ideas. This notion is easiest to understand through an example.

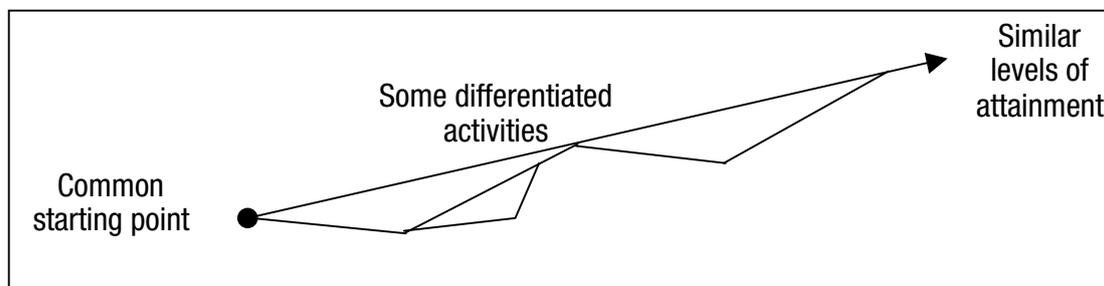


Figure 3. Differentiated learning trajectories.

Roma’s lesson: Areas of rectangles and triangles

The teaching of the area concept starts with the idea of area as covering (with tiles, feathers, wrapping paper, or whatever) in the first year of schooling. As skills of describing, measuring, and comparing areas are introduced, the underpinning idea is ‘area as squares’. Activities like counting the number of squares enclosed in a shape drawn on squared paper, and/or coloring squares to create shapes, are common in primary classrooms. The next stage — still in primary mathematics education — is to build a foundation for understanding area formulae; usually the idea of the area of a rectangle as being ‘length by width’ is tackled formally, often with children gaining little understanding to underpin that knowledge. Less commonly, teachers use activities to explore basic ideas about the area of other straight-sided shapes such as the triangle and the rhombus. Children have the potential to get left behind in the teaching and learning of each of these stages, so junior secondary teachers are often faced with the need for remediation — or perhaps ‘ability grouping’ — as students move to formal calculation and comparison of areas of more complex straight-sided shapes, circles, surfaces of solids, etc.

Roma is a Grade 5 teacher who, when presented with a worksheet that asked children to work out the area of some given solidly-colored shapes such as the one shown in Figure 4, thought that only about half of the class would be able to complete the sheet.

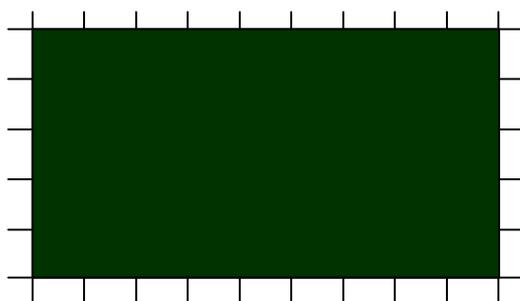


Figure 4. How many squares?

However, an alternative worksheet was provided, for children to help themselves to if they wished. The ‘enabling prompt’ on the sheet had rows drawn in (see Figure 5). A further prompt worksheet that they could help themselves to had squares drawn in (see

Figure 6). Square counters were also available, but were not used by the students. These graphic prompts, only used on the first example on each sheet, were enough to enable all students to complete the worksheet successfully.

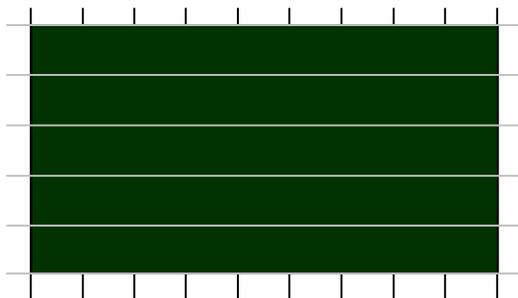


Figure 5. First enabling prompt.

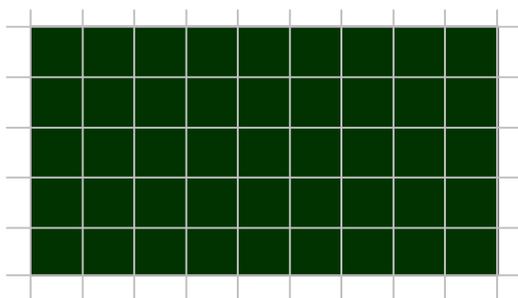


Figure 6. Second enabling prompt.

It is important to note several points:

- it was the children’s decision whether to use those prompts, and whether they needed one or both to enable them to progress to the original task;
- picking up an extra sheet was not a very public act;
- the conceptual essence of area — as rows (lengths) and a number of these (width) — was the focus of the prompts;
- the cognitive demand was removed only one step at a time, with a return to the full demand once children had engaged with the prompt(s);
- there was certainly no sense of their being a remedial group needing extra teaching; in fact, the children taught themselves, with the assistance of pre-prepared scaffolding tools.

All of the children were able to participate fully in mid-lesson discussion about their work, reporting a variety of ways of thinking about area that all came back to the ‘length by width’ idea, even if those words were not used by everyone. The differentiated learning trajectories had enabled the class to have enough communal experience to proceed as a whole group.

The next stage of the lesson involved finding the areas of three triangles that were drawn on squared paper (see Figure 7).

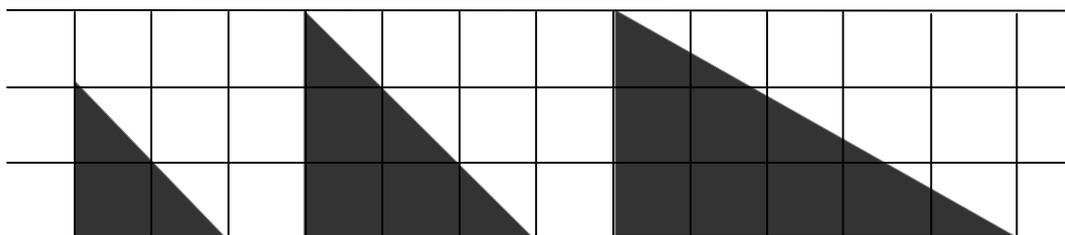


Figure 7. What is the area of each triangle?

As a result of the children's communal experience with rectangles, it only took one suggestion from the teacher to set some children who had been drawing in and counting squares and half squares onto the right track: 'Think about what you did with areas of rectangles'. Those children were soon drawing rectangles around their triangles and calculating their areas (and half areas) using what they had learned earlier in the lesson. Again, they all seemed able to participate in a lesson-closing discussion that involved some abstraction, generalisation, and networking of mathematical concepts. While individual testing was not undertaken after this lesson, it appeared that all students reached a common learning objective even though there had been some differentiation of learning trajectories. The observer noted that there was a sense of communal knowledge that could be used as a platform for further development.

Conclusion

Differentiating learning trajectories in this way is just one form of differentiated pedagogy. Results from our early work in this broader endeavour are promising. What we have found to date is that:

- through a research process of modelling and coaching, teachers soon learn to employ such strategies effectively;
- the strategies are broad enough to be useful in most mathematics lessons and indeed some other lessons;
- children who have the potential to be left behind in the teaching and learning processes are usually able to participate fully, sometimes at a surprising level of engagement and success.

It is useful to think of what is happening in the classrooms in the light of Vygotsky's theory about zones of proximal development. Children in the observed classrooms have certainly achieved more with mentoring than they would have on their own. However, the mentoring provided has not been repeated teaching by the teacher, peers, or parents. It is self-tutoring, using pre-planned scaffolding that is supplied by the teacher, but its use is under the control of students. In the case presented above, enabling prompts took the form of drawings that provided just enough assistance to scaffold thinking so that students could quickly re-join the class' learning trajectory. The prompts merely presented the same problem, but with a slightly lower conceptual demand.

Over the next two years of this project, we will focus more on gathering empirical data to find out how the use of such strategies moves children's mathematical development forward.

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Algebra outcomes for your K–9 class

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For too long formal algebra has been taught in Australian secondary schools without an acknowledgement of the necessary pre-algebra understandings that underpin it. What are they? Is basic pattern work and sound understandings in number sufficient?

Background

For many teachers of mathematics, both primary and secondary, algebra is ‘that brand of mathematics where you use letters instead of numbers’. This definition is also used, I believe, by the wider community, and is often a broad reflection of the way algebra is taught.

For too long algebra has been considered something to be taught in high school. I personally believe that formal algebra is frequently taught too soon to many students, even in a high school. The achievement of algebra outcomes is dependent on the understanding of ideas that underpin formal algebra, or pre-algebra. Too often these understandings are assumed to be developed through work in Number and other strands during the primary years. Formal algebra often just ‘appears’ in the curriculum, usually at the start of secondary schooling. Since it doesn’t seem to connect with other learning is it any wonder that the above definitions and misunderstandings arise?

Algebra outcomes are desirable for all children, not just those wanting to study higher mathematics. It is the responsibility of all teachers to ensure appropriate learning experiences for the children they teach in order that their students achieve these outcomes. A K–12 approach is essential and as a consequence a K–12 understanding is required by all teachers of mathematics.

What is algebra?

The *Collins Australian Dictionary* (1981) on my desk defines algebra as, ‘A mathematical system used to generalise certain arithmetical operations by using letters to stand for numbers’, and that is exactly what it is, but the problem lies in the ‘using letters to stand for numbers’ part of the definition. Many misconceptions arise when we start taking that literally. For example, if we write ab as ‘letters standing for numbers’ we may be tempted to think that a and b have place value and that therefore means that a is in the tens position. So if a

has a face value of ‘3’ then in the number represented by ab it must stand for ‘30’.

Hence, we are not able to take the definition from the dictionary literally with respect to ‘using letters to stand for numbers’. The most important part of the definition, I believe, is the part about ‘used to generalise... operations’. It is this generalisation that gives algebra its power. Unfortunately many teachers emphasise the ‘using letters to stand for numbers’ part at the expense of the ‘used to generalise... operations’ part, with the result that lots of students in schools (who later become adults like you and me) connect school algebra with the use of letters in mathematics and frequently have unfavourable recollections of their ability to succeed with algebra at school.

Let us have a closer look at the ‘generalisation of operations’ aspect of the definition. We know that $5 + 3 = 8$ and therefore $3 = 8 - 5$. We might then consider the same with another set of numbers: $2 + 7 = 9$ and therefore $7 = 9 - 2$. So we might ask ourselves, ‘Is there something general happening here that does not depend on the numbers being used?’. We try it again with another set of different numbers such as $1 + 3 = 5$ and notice that yes, $3 = 5 - 1$. So what seems to be happening (in words) is ‘when we add two numbers together to get a total then the first number we added seems to always equal the total subtract the second number’.

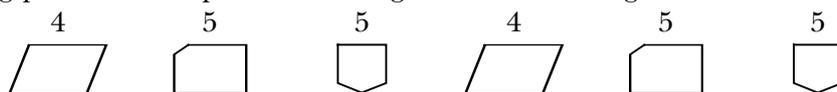
If we let the first number be represented by m (whatever the number actually is) and the second number be represented by z (whatever the number actually is) and the total be represented by k (whatever the total actually is) then we could say that $m + z = k$ means that z must equal $k - m$ (that is, $z = k - m$). So if $m + z = k$ it follows that $z = k - m$.

We might test this out with another set of numbers just to check: $23 + 42 = 65$ so if our theory is correct then $42 = 65 - 23$. We are correct. We can see here then that what is happening with the numbers is a very powerful operation fact that works regardless of what numbers we are using. This fits in with the dictionary definition stated above: ‘A mathematical system used to generalise certain arithmetical operations by using letters to stand for numbers’.

When should algebra be taught?

Historically, algebra has been taught in high schools. There is a very good reason for this. As we learn more and more about arithmetic the need to be able to generalise its properties and learn more about the patterns produced through arithmetic operations increases.

There are some aspects of algebraic thinking that do, however, occur in very young children. Indeed, basic patterning is developed at a very early age and the ideas of ‘in general’ and ‘unknown quantity’ are implicit in each of the mathematics strands of Space, Measurement, Chance and Data, and Number. Children who pair counting numbers with a repeating pattern of shapes are working with and creating a function. For example:



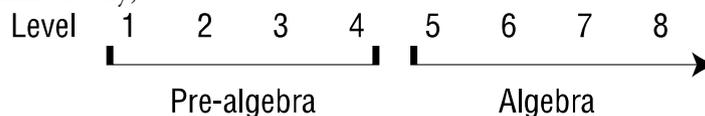
‘Change’ is also a very important algebraic idea with which young children are very familiar. They understand that most things change over time and when they measure something changing over time they can describe it with words (e.g., ‘It is hotter today than it was yesterday’) and including numbers (e.g., ‘My plant grew three centimetres yesterday’).

Clearly there is a difference between developing algebraic ideas and teaching formal algebra. I can recall my first algebra lessons when as a student I was plunged immediately into the formal syntax and semantics of algebra; using letters in equation-solving and

so on. This should be avoided at all costs. Teachers in the early years and middle and late primary years should avoid teaching and using the formal language of algebra and be sure that their students do not equate algebra with symbol manipulation, as many of us did. Of course, if we are certain that our students do understand these early algebra ideas of generalisation then it makes sense that we continue to build on them.

The teaching of the early ideas of algebra as outlined above however, should begin in the early years of schooling. This is so that when the time comes for introducing formal algebra, it is part of a seamless transition and students understand and appreciate the need for symbols and algebraic expressions resulting from ‘where they are at’ with their work in arithmetic.

We need to understand what this means for children who are working towards achieving these outcomes but for whom we may be unable to describe their progress with the language of formal algebra. We are able however, to describe what it might look like without the language of formal algebra. Mathematicians describe the algebra children can do prior to their demonstrations of formal algebra achievement, as outcomes in pre-algebra. Diagrammatically;



Pre-algebra and algebraic thinking

We have seen that algebra is to do with making generalisations. For example, many children (and adults) are able to make statements about numbers and number operations such as $3 + 4 = 4 + 3$ and similarly $6 \times 7 = 7 \times 6$. For them to realise however, that there is something general happening here that has nothing to do with the specific numbers used is an example of when they are using algebraic thinking. To formalise this thinking and use variables to represent these situations, such as $a + b = b + a$ and $a \times b = b \times a$ (and knowing that a and b represent any numbers) is a demonstration of an outcome in algebra. Hence, the algebraic thinking that underpins the algebra outcomes is not algebra but nevertheless is essential to the achievement of the algebra outcomes. This algebraic thinking is often called pre-algebra.

Let us consider the three desirable algebra outcomes for all children listed above, in the context of pre-algebra.

Variation

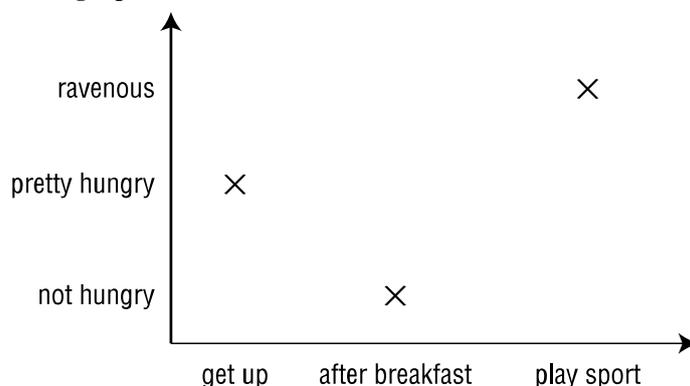
Students need to be able to:

- understand that nature of variation (or change) and represent it in different forms.

Another word for ‘variation’ is ‘relationship’. We get the words ‘variation’ and ‘variable’ from the word ‘vary’ and so variation is about things that vary or change over time. For example, our height changes over time and we can collect some data representing our height at different times in our lives, show this data in a table, and draw a graph of those changes. We can also write an equation to describe the height change over time. So we have many ways of showing and describing (including with words) the variation in our height over time.

For some things that vary, it is difficult to collect exact data. An example of this might be our hunger levels as the day goes by. We know that this ‘variable’ changes because we can describe it in words with statements such as, ‘When I get up I’m usually pretty hungry

but after I have breakfast I'm not hungry until about mid morning, unless I play some sport and then I'm ravenous afterwards.' In this statement there are three different levels of hunger described in the words: 'pretty hungry', 'not hungry', and 'ravenous'. We might plot these points on a graph as shown:



Having drawn these points we would then probably join them with some curvy lines since we know that our hunger increases or decreases gradually over time (sometimes at faster rates than others) and that we do not just suddenly feel really hungry unless we simply have not been thinking about it for a while, but this does not mean our hunger level has not been increasing.

Very young children can describe change using words only; e.g., 'I'm colder than I was yesterday'. As children get older they need to learn how to represent this change using tables, graphs, and symbols as well as being able to describe it verbally. They also need to be able to interpret the changes that other people represent in tabular, graphical and symbolic forms. Teachers should help them do this.

Children learn the early ideas of 'function' through early work on proportion. For example they know that there are two fifty cent pieces in every dollar and so can read and make sense of a table of proportionate values such as:

Number of dollars	1	2	3	4	5
Number of 50c pieces	2	4	6	8	10

So they are developing the early understandings of 'dependence' where one quantity depends on another. In this example, the number of 50c pieces depends on the number of dollars you have. Children may develop these understandings through trading activities where they trade five white cards for one red card, for example.

Generalisation and symbolic expressions

Students need to be able to:

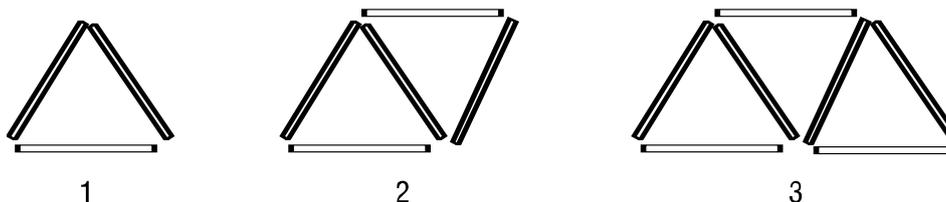
- generalise to some extent and use symbolic expressions

A 'symbolic expression' might be something like $2k + 3$, and expressions like these are the building blocks for algebraic representation. For children to understand a symbolic expression, they need to understand what a variable is; i.e., what the ' k ' is in the above expression. There are many misconceptions that children have about the use of variables in algebra (see Perso, 1993), so we must be extremely careful to ensure that we, as teachers, understand what a variable is in the formal algebra sense.

Students learn to express generality primarily through working with patterns. They can describe in an oral or written form, the repeating patterns they see in objects, shapes, colours, letters, actions and in numbers. They can also hear patterns in music, sounds and

rhythms. Students can skip count by sixes for example and then colour the numbers they have said on a hundreds chart. They can then ‘see’ the pattern in their numbers in the colours of the squares. This will then help them work out what the next number will be because of where it is, as opposed to what the number is in the sequence. They can see beads threaded on a necklace in a repeating pattern of colours such as red, red, blue, green, red, red, blue and predict the next colour. Children can represent the pattern in the colours using numbers: 1, 1, 2, 3, 1, 1, 2, 3... or using another number pattern such as 3, 3, 6, 4, 3, 3, 6... They can also represent the same pattern using physical actions such as claps and stamps by first saying what each action will represent in the pattern: for example, if a clap is for the red, a stamp is for the blue and a flick is for the green, the pattern will be ‘clap, clap, stamp, flick, clap, clap, stamp, flick’. This is helping them to be able eventually to generalise. A student in a Year 4 class, for example, might be able to go one step further and say, ‘The twentieth action in this pattern is a “flick” because the pattern repeats with a sequence of four different things; so every fourth action will be a “flick”’.

As students develop they are able to recognise ‘growing’ patterns and distinguish them from repeating patterns. They can determine the nature of the growth and suggest rules such as, ‘It goes up by two each time,’ and they can use their rule to predict what the next elements will be. For example, consider a pattern represented with matchsticks:



By setting up a table as follows students can see that they are adding two matches to the previous shape each time. So they would know that the fourth shape in the pattern would have the ‘same number of matches as the third shape plus two.’

Pattern number	1	2	3	4
Number of matches	3	5	7	9

Students who are further developed in understanding patterns will then be encouraged to see if they can connect the pattern in the pattern numbers with the pattern in the number of matches. They might see that the two patterns are connected in some way, identifying a connection between the elements of the pattern and their position in the pattern. They are then eventually able to verbalise this connection and say for example, ‘If we double the pattern number and add one to the answer, we can tell how many matches are needed for the next one.’

Facility with the symbolism of algebra will then enable them to write this statement using symbols; i.e., ‘If we let the number of the pattern be represented by p and the number of the matches used be represented by m , we can show this as $m = 2 \times p + 1$.’ Being able to generalise this rule using symbols is formal algebra. The essential understandings described here that underpin this facility and the understandings that it depends on, are pre-algebra. You can see that the more experience children have with visualising the patterns in the shapes, translating the patterns into numbers, and then understanding and describing the patterns in the numbers in words, the smoother the transition into formal algebra. Many curriculum documents and textbooks include pattern work prior to formal algebra but few explain to teachers (or students) what the connection is and why it needs to be emphasised and developed.

It is very important for the understanding of the use of variables in formal algebra that

children have lots of experience to say or read their expressions. In the example above, for instance, children need to read $m = 2 \times p + 1$ as ‘the number of matches equals two lots of the number of the pattern add one’. This verbalisation is critical. If students are unable to do this than it is likely that they will never move beyond a simplistic understanding of why letters are used in algebra.

If students do not understand m and p as each representing ‘the number of’ something, they very quickly start to think of m as standing for the word matches and p as standing for the word pattern. This belief that a letter in mathematics stands for a word is held by many students studying formal algebra and is the root of many problems (see Perso, 1993).

To write, solve and understand equations and inequalities (the latter part of the outcome) requires a deep understanding of the equivalence relationship.

Equation solving and equivalence

Students need to be able to:

- write, solve and understand equations and inequalities

To solve an equation (an expression where one part is equal to another part) or inequality (an expression where one part is greater or less than another part) is to find all the values that make the expression true. Children need to be able to write equations and inequalities using the symbolism of algebra, having been given the constraints of a situation, and then solve these equations and inequalities. They then have to be able to make sure their answer/s make sense in context.

Clearly, being able to write an equation is a lot more demanding than simply being able to solve one. This outcome is a clear statement by curriculum writers that symbolic manipulation and being able to use equation-solving strategies (which was once one of the main foci of learning algebra in school) is insufficient on its own, but is a part of a bigger process. Being able to write an equation using symbols means understanding the syntax and semantics of algebra, and knowing how to identify variables, as described previously, from a written context.

It is clear that to understand what an equation is, we need to understand what ‘equals’ means. If we write $2 + 4 = 9 - 3$, we are describing an equivalence relationship between two expressions. If we carry out the operations of addition and subtraction indicated in these expressions, we observe that their answers are equal: i.e., $6 = 6$. We can see that there is a difference here: two expressions can be equivalent while their answers are equal. The expressions are not equal; in one sense it is not correct to say that $2 + 4$ equals $9 - 3$ because they are not the same expression. Technically they are only equal after you perform the operations that are indicated.

This might seem like splitting hairs but this is a very important point if children are to be able to work with algebraic expressions and solve equations. It is important that the idea of equality is taught as a relationship, not as an operation. Many children learn that the equals symbol is telling them to ‘do something’. They need to learn that it is alright to have an expression such as $3 + 4$ and just leave it like that without feeling the need to ‘find the answer’. Research has shown that this drive to ‘close’ an expression is very strong in students and comes from a sort of ‘conditioning’ (Collis, 1974). This conditioning is due to experiences in their classrooms where, whenever they see an equals sign, it is in the context of having to find an answer; hence, they inadvertently learn that ‘equals’ is a mathematical operation and that the ‘=’ symbol means ‘do something now’ or ‘find an answer now’.

Children need to have lots of experiences with the idea of ‘equivalence’ both with

objects and numbers. Lots of balancing experiences with beam balances and with visualisation for example, are important.

They can see when the two sides of a beam balance are not level and can decide to put more ‘things’ on the side that is ‘lower’ in order to maintain the balance or to take something off the side that is ‘higher’.

Students also need to develop compensation strategies using objects and manipulatives to give them a sense of the ‘balance’ that the relationship of equivalence embodies. For example, to the question, ‘If you have three books and I have five books, what could we do to each have the same number of books?’, children should be encouraged to respond with things like, ‘You could give me one of yours’, or ‘You could go and get two more’, or ‘We can put all our books together and then each take half of the books’. Children should learn to physically show and draw these responses as well as to verbalise them. As they get older, they should also be able to represent them on a calculator.

Students similarly need to be engaged in writing equivalent number sentences such as $4 + 1 = 7 - 2$ or $4 + 1 = 2 + 3$ and then being asked to write down as many other equivalent statements as they can for $4 + 1$. This partitioning of numbers is frequently done in the context of ‘calculation’ but usually students are working with one operation only; i.e., the idea of doing partitioning is so that students learn all of the ‘parts’ that make up a number. For example,

$$3 + 8 = 11$$

$$4 + 7 = 11$$

$$5 + 6 = 11$$

$$6 + 5 = 11$$

so $3 + 8 = 4 + 7 = 5 + 6 = 6 + 5$ and so on. This is not the same as using different operations in equivalent expressions as described above, because what is happening here is that the students are continually comparing with the answer, that is, closing their expression, in order to compare. They need to learn to ‘hold that fact’ and change one side of the equivalence without changing or closing the other. For example,

if	$3 + 4 = 9 - 2$
then	$1 + 2 + 4 = 9 - 2$

and so learn to work from one line of working to the next without closing (or finding the answer) of either side. This should not be left until students are working with variables (as needed by formal equation solving in algebra) but must be developed strongly in the context of numbers first.

It must be pointed out that in order to solve an equation like $3 + m = 12$, the techniques of formal algebra are not required. These types of equations can be done intuitively and students should be taught to do these using intuitive methods. Asking children to translate statements of this type using language such as, ‘What do we need to add to three to get twelve?’ is extremely helpful for children learning to work with unknown quantities. It is this verbalisation combined with an understanding of inverse operations that will lead children to go on from this, to say things like, ‘We can work this out by saying twelve take three’.

Many questions with a variable on either side of the equals symbol can also be done intuitively, for example $3 + m = 11 - m$. Some students will be able to work this out without using formal algebra but only if they feel comfortable working with these expressions and also if they are able to appreciate and understand the idea of variable and know that the value of m must be the same on both sides.

All of these methods help children to develop skills and understandings that are essential for using formal algebra to solve equations. Clearly there is an enormous amount of learning that underpins success with formal equation-solving and enables children to be able to write equations, which is what the desirable outcome is about.

Concluding remarks

Often, due to the poor teaching experiences that we had as students and the resulting lack of understanding of the subject by ourselves, we pass on that fear and ‘aura of the unattainable’ to our students. Indeed, the values attached to facility and understanding of ‘algebra’ by our society have a lot to answer. The subject is almost revered by many institutions, particularly universities, and educated ‘snobs’ who believe that facility with ‘algebra’ is unattainable by the masses. This, I believe, is nonsense, since every child, provided with appropriate learning experiences, is able to achieve algebra outcomes to some extent.

I have attempted here to draw attention to the enormous responsibility that we have to this end. If we want all students to have success with algebra that is beyond a superficial facility with algebra syntax and symbols, we must first ensure their success with pre-algebra.

(Note: For further reading and examples of classroom activities and detailed lesson notes that support the approach to teaching pre-algebra understandings as described in this article, see Perso, 2003)

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Box plots: Issues for teaching and learning*

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In this paper we discuss reasoning to do with box plots. We review findings in the research literature on upper-secondary students' understanding of box plots, refer to relevant comments in the examiners' reports for tertiary entrance examinations in Western Australia, and describe our own observations in a Year 12 class. A common theme is that students' comparison of properties of two or more sets of data using box plots tends to be limited to quoting statistics. It seems that developing strategies for comparison is a challenge for teachers. Appropriate strategies will be explored in the conference session.

Introduction

Research by Biehler (1997) and Pfannkuch, Budgett, Parsonage and Horring (2004) indicates that students are generally able to construct box plots accurately after instruction, but interpretation of the plots tends to be narrow, especially when comparison is involved. The examiners for the Tertiary Entrance Examinations (TEE) in Western Australia report similar outcomes, namely that students are usually competent with constructing the plots, but interpretation and comparison tend to be limited (e.g., Curriculum Council, 1998b, 2002b, 2003b).

With interpretation in mind, we investigated introducing the plots using dot-frequency diagrams and histograms, in a Year 12 class. The teaching approach was in line with Bright and Friel's (1998) recommendation that students can benefit from comparing different graphs of the same data. McClain and Cobb's (2001) description of educational software that allows the user to group data into equal sized groups on dot-diagrams also influenced the approach that we took. Comparison was discussed in class in the context of establishing change of origin and scale effects on data.

In this paper, we summarise the findings of Biehler (1997), Pfannkuch et al. (2004), and the TEE examiners (Curriculum Council, 1998b, 2002b, 2003b), and describe the treatment of box plots in the Year 12 class and the learning outcomes. In addition, we list examination questions that will be discussed in the conference session. The questions call for comparison of data using box plots.

* This paper has been accepted by peer review.

Reported findings

Biehler's (1997) study involved a group of Year 12 students and a group of student teachers. Both groups had undertaken statistics courses in which statistical software was used for analysing data. Students were interviewed in pairs at the conclusion of their courses and asked to work on a statistical problem. They were to decide whether weekly hours spent on homework was the same for students for whom a curfew was imposed and for students without a curfew. Data were provided. Software use was an option.

In general, the students jumped to produce computer outputs without questioning whether they would inform the topic being investigated. When interpreting box plots of the hours of homework, some showed that they did not understand that area and lines indicate the location of data, and they were confused about the property that the number of data and the area of boxes are not in proportion.

In using box plots to compare the hours of homework with and without a curfew, students tended to focus on differences in the medians. They paid limited or no attention to spread, or had difficulty interpreting the measures of spread. They tended to use mainly technical language in their conclusions (e.g., comparisons were in terms of the median, etc.) and had difficulty in stating their conclusions in terms of the homework/curfew context. Their technical interpretations were characterised by imprecise language.

The study by Pfannkuch et al. (2004) involved a class of Year 11 students. A unit of study was specially designed to address known problematic areas in students' statistical reasoning. At the end of the unit the class completed an assessment that required them to analyse maximum summer temperatures for two cities and choose which city to visit for a summer holiday. Students were asked to pose a statistical question (e.g., which city has the higher maximum summer temperatures?), analyse the temperature data, draw a conclusion, justify the conclusion with three statements, and evaluate the statistical process.

All chose to calculate the five statistics for box plots and many drew the plots for the two sets of data. Then, most students compared features of the data as displayed on the plots in a non-discriminating manner and did not justify or explain their conclusions. Most (27/30) compared equivalent statistics (e.g., median values) but did not discuss the statistics in relation to the data sets as wholes, and many (16/30) included range which was not relevant to the question. More than half (18/30) compared non-equivalent summary statistics. Twenty-one mentioned variability but none discussed variability in relation to the medians. Nine alluded to distribution but there was little attempt at defining the shapes of the distributions. Thirteen students presumed that the data sets were not a valid basis for comparison because they were unequal in size.

The 1998 TEE question on box plots (Curriculum Council, 1998a) asked students to compare the distribution of heights of 30 adult females and the distribution of heights of 30 adult males. Data were displayed as box plots and were listed for females. The five statistics (extreme, median and quartile values) for the heights of males were listed. Examiners noted that: 'Many students commented upon the similarities and differences for the individual statistics rather than, as required, on the general features of the distributions such as central location, range and symmetry' (Curriculum Council, 1998b, p. 4). The TEE question in 2002 (Curriculum Council, 2002a) asked students to discuss the similarities and differences in the central tendency and dispersion of three sets of data. The data were displayed as box plots (two sets) and the five statistics were provided for the third set. Examiners noted: 'Far too many candidates made statements that could not be inferred from box and whisker plots... many failed to recognise the similarity in the shapes of the boxplots' (Curriculum Council, 2002b). A similar question was asked in 2003. Fewer students made invalid statements, many based their comparisons on median,

range and interquartile range (as expected), but few mentioned the opposite skew of two of the distributions (Curriculum Council, 2003b).

In summary, weaknesses in students' interpretation of box plots, as reviewed above, were:

- not recognising the convention that the frequency of data for each interval on a box plot is the same;
- using imprecise language; and
- having difficulty stating conclusions in terms of the real contexts of data.

Problems with comparison of data sets were:

- basing comparison on the five statistics that can be read from the plot, without recognising global relationships;
- not recognising variability;
- not integrating comparison of measures of variability with comparison of the medians;
- not recognising distribution shape; and
- wrongly assuming samples must be the same size for valid comparison.

Further, (a) most of these weaknesses with interpretation are not peculiar to the use of box plots but apply to students' interpretation of data generally (e.g., Buffler, Allie, Lubben & Campbell, 2001), and (b) comparison can challenge experts as well as students (Biehler, 1997). In particular, comparison of box plots becomes complex when different measures of spread support different conclusions, and interpretable patterns sometimes do not exist (Biehler, 1997).

Box plots in the Year 12 class

The description below is based on systematic research in the first author's 2004 Year 12 Applicable Mathematics class. The second author attended the class as an observer, during a unit of work on statistics. The lessons were video-recorded, audio recordings were made of students' one-to-one conversations, and students' assessment scripts were photocopied. The claims in this paper are based on analysis of the video and assessment data.

The Applicable Mathematics syllabus specifies that students should: 'Construct and interpret boxplots, noting their use in comparison of centres and spreads of data... for ungrouped data, the lower/upper quartile is the median of scores to the left/right of the median' (Curriculum Council, 2003a, p. 49). In practice, outliers are not distinguished on the plot. The Year 12 class had constructed box plots and undertaken simple interpretation in Year 10.

Box plots were discussed in two separate lessons in the Year 12 class. The night before the first lesson, students were set an introductory activity for homework. A worksheet was provided which asked them to: draw dot-frequency diagrams for four sets of listed data; divide each set of data into four equal groups by circling the dots on the diagrams; and determine the mean, median, quartile and extreme values for each set. The worksheet gave an example of what was expected (see Figure 1). The diagram on the worksheet was produced on *Autograph* 2.10 (Hatsell, 2002).

The lesson started with whole-class discussion on the dot-frequency diagram in the example. The corresponding diagram from *Autograph* was projected onto the whiteboard, without the grouping. Students were asked to describe the graph and they commented that there were 'repeated values', 'more lower values than higher values' and that the graph 'isn't symmetrical'. They were asked to consider how the data could be displayed

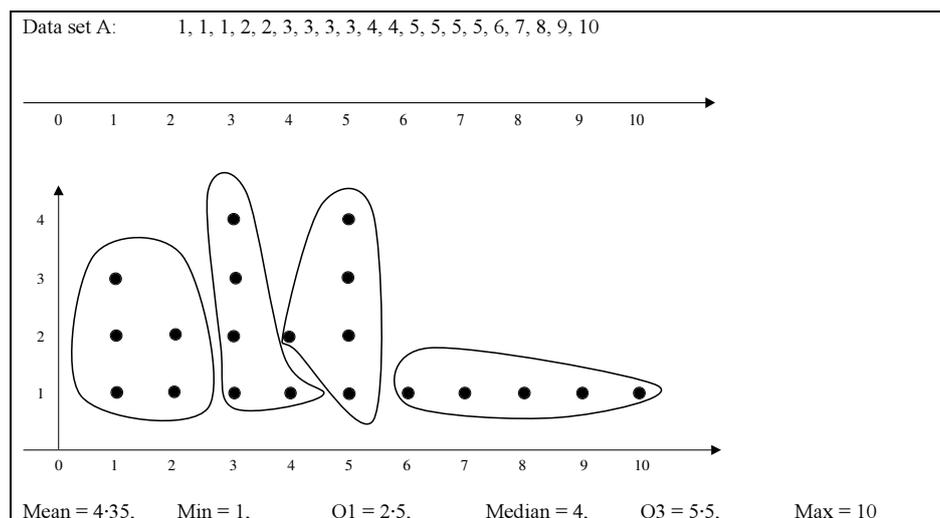


Figure 1. Worksheet example.

in different ways. They suggested a histogram and a frequency polygon. These were hand-drawn on the projected dot-diagram. The hand drawing allowed the processes of construction to be discussed. Then a student suggested a box plot and this was also drawn on the whiteboard, using statistics that students had calculated, and with reference to the grouping of data on the dot diagram. Each student drew a histogram on one of her dot-diagrams, and a box plot on the scale above it (see Figure 1).

Subsequently, students constructed box plots for all sets of data on the worksheet. When they had finished, the box plots were produced with *Autograph* and were displayed on the whiteboard. The class was questioned on the plots. The relationships listed below were identified. Ambiguity and imprecision similar to that observed by Biehler (1997) were evident but are hard to avoid.

- Set A (see Figure 2): there are 50% of data in the interquartile range, the middle 50% of data lie in the interquartile range, the benefit of the interquartile range over the range is that it can take away extreme pieces of data that do not fit, the median is not in the middle of the plot, the median is in the middle of the data;
- Set B: there are double the number of data in the left box as in the right, there are double the number of data because the whiskered quarter is not there so the data are included in the box, the density of data is greatest in the left box;
- Set D: the minimum is the same as the lower quartile, the maximum is the same as the upper quartile, a lot of data are the same, there are repeated values at the quartiles, data are densely packed at the beginning and end, half the data are on one side and half are on the other side, the maximum and minimum are symmetrical about the median;
- Set E: the longer whisker means the gap between the upper quartile and maximum is large.

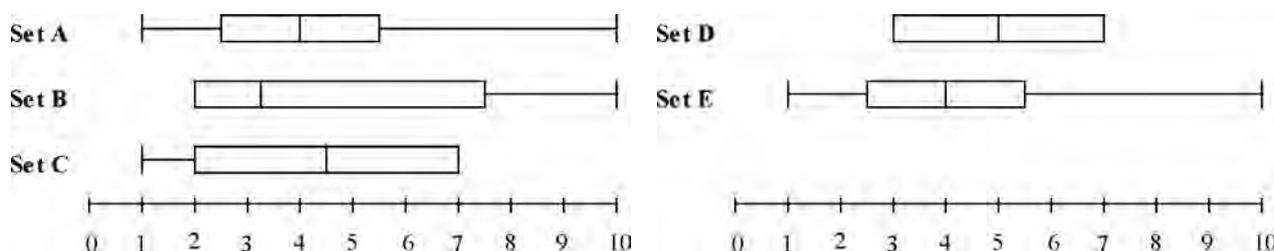


Figure 2. The box plots for the data that were provided.

Other work on box plots was as follows. Students were set homework that required them to calculate summary statistics and produce box plots for data before and after the addition of a constant or multiplication by a constant. The results were discussed in class and, as well, interpretation of box plots was revisited.

There was agreement on the effects of addition and multiplication, for example, that spread among data was the same before and after addition of a constant to all data; and spread increased/decreased if all data were multiplied by a constant because distance of data from zero (not the median) changed. Comparison of the width, range and interquartile range of the box plots, and positions of the medians, featured in the conversation. As well, it seemed from students' responses in class that there was widespread understanding of (a) the convention that the whiskers and boxes on a box plot each represented one quarter of the data in a set, and (b) that the total number of data and distribution of data within intervals could not be inferred from a plot.

The students were also directed to read worked examples in their textbook (Lee, 1999) and complete the textbook questions on box plots. Comparison questions and the solutions that were provided encouraged use of the medians to characterise groups (e.g., to show that one group of students tended to have taller students than another). As well, they encouraged comparison of extreme values, interquartile-range, range, skewness and symmetry, and interpretation in terms of real-life contexts.

In a subsequent assessment, a question was included that involved comparison of weights of eggs produced by free range and caged chickens. Students compared the medians (15 out of 23 students), and/or range (17 students), interquartile range (11 students), extreme values (2 students), and shapes of the distributions (1 student). Five commented about central tendency (not the median), two commented about dispersion without mentioning range or interquartile range, one discussed interquartile range in relation to the median, and four related their answers to the weight context. Hence, students tended to not go beyond technical comparison of equivalent statistics. The outcome is consistent with the findings of Biehler (1997), Pfannkuch et al. (2004) and the TEE examiners (e.g., Curriculum Council, 2000b).

Conclusion

After looking at students' responses in class and in the assessment, we reached two main conclusions. First, targeting the principles that underlie box plots with the dot-diagram approach potentially contributed to students' understanding of the principles. The Autograph software was valuable in that it produced accurate, attractive graphs quickly. These were the focus of discussion. The problem that the construction processes are not evident on computer-generated graphs was addressed with the by-hand construction.

Second, too little time was spent on strategies for comparison of contextualised data represented as box plots. Albeit, any further time on box plots in class would have been at the expense of time on other syllabus components. Furthermore, as Biehler (1997) and Pfannkuch et al. (2004) point out, 'strategies for comparison' is an undeveloped field. Attention needs to be given to developing a framework for comparison.

Therefore, in conclusion, we invite conference participants (or readers of this paper) to pursue this important task and consider strategies for comparing the box plots from the following questions, and to decide what form of questions are appropriate for testing understanding of statistical reasoning related to box plots.

- (a) Advanced Placement Statistics 2000 Question 1, 2003 Question 1. Available from <http://apcentral.collegeboard.com>. Free registration, then follow the links:

- <Exams > Exam questions > Statistics>.
- (b) Western Australian Applicable Mathematics Tertiary Entrance Examinations, 1998 Question 4, 2003 Question 6. The 2003 question is available from www.curriculum.wa.edu.au/pages/publication05.htm
- (c) Education Queensland, Box plots worksheet Question 3. Available from <http://exploringdata.cqu.edu.au/boxplots.htm>. Follow the link <Boxplots worksheet>.

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Mathematics: Dead or alive!*

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There is very life in our despair,
Vitality of poison.
(Lord Byron, *Childe Harold*)

We use the word ‘vital’ in different ways depending on the context involved. For example, if we want to describe a person, we refer to their vital statistics, i.e., usually the date of their birth, marriage and/or death. Vital functions usually relate to a person’s heart, liver or lung organs. If we say something is of vital importance, we are referring to a circumstance without which nothing can successfully proceed. So how do we relate mathematics to this? Is a person’s success or failure in mathematics a matter of a statistic? Is it a vital function? Is it of such importance that life cannot go on? What we believe mathematics is about and how it functions will determine the answers to these questions. Should teachers be making mathematics vital or is this the task of the learner alone? These are fascinating questions which need to be explored together with possible pathways for gaining further understanding. How the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* helps in this regard must be considered.

The Oxford Dictionary gives the following definition of the word ‘vital’:

a. & n. Of, concerned with or essential to, organic life, as v. energies, functions, v. power (to sustain life), wounded in a vital part; essential to existence or to the matter in hand, as in a v. question, question of v. importance, secrecy is v. to the success of the scheme; affecting life, fatal to life or to success &c., as a v. wound, error; ... v. statistics (of birth, marriage, death &c.). [OF f. L. vitalis, (vita life, cogn. w. vivre live & Gk bios life, see -AL)]

So, is mathematics vital? Is it alive? Has it died? The theme of this conference, ‘Making mathematics vital’ could be read in several ways. Whether a person is alive or dead is gauged by the existence or not of a heart beat. Is the heart of mathematics beating or has it been stilled in some way? Does the theme suggest that mathematics is not necessary for life?

A teacher told me once that she and her class planned a ‘mathless day’. When she asked them how they knew when to come to school, most of the class said they had looked at the clock at home. When asked how they got to school, they were able to tell in fairly mathematical terms, facts about direction and distance. When asked what lesson they

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would do, they agreed that mathematics always came first. So, right from the start of the school day, they had to admit that they used mathematics and there is no such thing as a 'mathless day'.

On another occasion, primary teacher education students were asked to keep a journal of their experiences in meeting and doing mathematics in their everyday life. At the end of the semester, one comment, among many, was that one cannot escape mathematics: 'Mathematics is everywhere!' This student has never since been able to see mathematics except as a vital everyday necessity. Mathematics is vital for life.

There are some pessimists who observe the lack of interest in doing mathematics as displayed by some students in some classrooms and say that mathematics has lost its meaning and does not enthuse students though possibly never going so far as to recommend that mathematics be dropped from the school curriculum. There are those who recognise the kind of political football that mathematics becomes at times when educators, politicians and bureaucratic officials play off one aspect of learning and teaching mathematics against another; but do these circumstances mean that mathematics is dead? I believe not! So consider the factors that can kill mathematics and those that can breathe life into the subject. Factors that kill mathematics in the classroom include:

1. the acceptance of myths about mathematics such as that which states that mathematics is arithmetic or computation;
2. the lack of understanding of mathematics as a dynamic discipline;
3. lack of knowledge of mathematics;
4. lack of knowledge about how mathematics is learnt;
5. lack of enthusiasm on the part of the teacher;
6. lack of faith in the learner;
7. fear of failure; and
8. overwork.

There are probably others, too, but this list is a formidable one in itself and it is never good for one to dwell too long on negatives. The above factors are prevalent in some schools more than others and some of them relate more to certain circumstances than to others. For instance, teacher education courses for both primary and secondary teachers quite often fall short of what is required to alleviate some of the above factors. This can be due to lack of time, inability to fund mathematics and mathematics education units sufficiently and even lack of understanding on the part of the teacher educators, particularly those who are not mathematicians or mathematics educators themselves. In many cases, policy is determined by such people despite impassioned pleas and explanations by the mathematics educators.

There are also students who apply for university courses in education because they have not done sufficiently well to get into other courses or have been unable to get a job in the area for which they prepared themselves in their undergraduate degree. In many cases this means the student is not committed to teaching mathematics so much as earning a salary or getting a job in the easiest way possible.

Primary teacher trainees have an extra pressure on them in so far as they will be expected to teach a range of subjects, including most that they have not previously studied. This quite often includes mathematics. This could mean that they will have inadequate background knowledge in mathematics for teaching in the primary school where an understanding of fundamental concepts is absolutely vital.

Another possible contributory factor is the curriculum. When emphasis is placed on arithmetic or on passing public examinations, students cannot be blamed for having perceptions that are negative and counterproductive as far as their mathematical development is concerned. This is a most difficult problem to solve but there are two

areas through which some of the deadening factors can be treated. One is from the point of view of the teacher and the other is from the point of view of the student. Then, too, the way in which these relate must be considered. In their report on professional development McRae, Ainsworth, Groves, Rowland and Zbar (2001) stated that the two directions that the issue must be tackled are, 'the ways in which professional development activity is constructed and evaluated, and by supporting the crucial importance of its place in change efforts focussed on outcomes-based education' (p. 18).

Professional learning of teachers of mathematics

Many types of in-service courses and professional development programs have been instituted in schools and by systemic authorities. Just to mention a few, there are the Early Years Numeracy Project in Victoria, Count Me In Too in New South Wales and Australian Science and Mathematics School program in South Australia. One that has been having considerable success in NSW, besides Count Me In Too, is the Lesson Study process. In this project, initiated by the NSW Department of School Education, was adapted from the Japanese experience (Stigler & Hiebert, 1999).

The Lesson Study process involves a group of teachers within the one school with a team leader meeting regularly to plan, teach, evaluate and refine a lesson. The following steps are usually carried out:

- teachers meet and plan lesson
- one teacher implements the lesson, observed by at least one other in the group.
- group meet to evaluate the lesson
- group meet to refine lesson for next iteration.

The process can be repeated until teachers are satisfied they have produced a good lesson.

The particular strengths of this process, very much in keeping with McRae et al.'s (2001) criteria for professional learning programs, are:

- a focus on student performance;
- the practice of teaching and learning;
- enhancement of pedagogical and discipline knowledge;
- a focus on active, collaborative learning.

An evaluation carried out by White and Southwell (2002) found that participating teachers:

- deepened their understanding of how students learn mathematics;
- enhanced their mathematical content knowledge;
- developed skills to teach mathematics more effectively;
- provided more meaningful classrooms for mathematics; and
- worked collaboratively in teams.

The last of these points was the one most emphasised by most teachers. The value of working collaboratively seemed to be a surprise for some and a definite benefit for most. The value of observing other teachers and being observed themselves was also considered a great asset.

The implementation of this Lesson Study program contributed greatly to alleviate factors 2, 3, 4, 5, and 6 as listed above. The question remains, however, as to the effect if continued long term and although, logically, the method should reduce the time involvement because of the shared preparation, not all teachers agreed that it did.

While this particular implementation of the Lesson Study process is considered unique because of the individual school's ownership and authority, there are similar programs that differ mainly in the extent of outside expert involvement.

Of a different type of program for professional learning is the Australian Science and Mathematics School, the first of its kind in Australia (www.asms.sa.edu.au). It is a joint venture between the South Australian Department of Education and Children's Services and Flinders University and is dedicated to the learning and teaching of mathematics and science. Besides a strong emphasis on mathematics and science in the curriculum, ongoing professional development of staff is also central to the School's focus. This emphasis on professional development involves a wide range of strategies.

The AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* lists personal professional development as one of the subgroups in the strand Professional Attributes. Lesson Study meets the requirements of this section but also those in the other two strands.

The investigational approach to learning and teaching mathematics

In approaching this aspect of teaching and learning mathematics, the assumption is made that the emphasis on outcomes-based learning is not based on a student's cognitive achievements only or even on cognitive achievement that is equal to all other students in the same age level but also on the appreciation of mathematics and other affective criteria that may be considered as contributing to achievements in mathematics. For instance, if a student reaches a certain level of achievement in a computational process, the outcome achieved may be admirable but even more admirable is that the student has gained a sense of confidence in his or her ability and enjoyment in that confidence.

For many years, teachers of small children have utilised their amazing curiosity to explore mathematical situations. In that way, mathematics has come to life for the children and teachers have felt a sense of satisfaction that the children's enjoyment and enquiry have been achieving positive results for them all. In the past two decades, teachers of older students have also discovered that teaching through investigations not only arouses interest and achievement for the students but also gives them as teachers a sense of being a facilitator and encouraging the mathematical development of their students (Southwell, 2004).

An investigational approach is one that can be utilised at all levels. Ollerton (2003) maintains that

because mathematics is essentially a collection of ideas used to describe the world and a set of tools for solving problems, students need to experience mathematics in problem solving ways. Each module, therefore, needs to be based upon exploring ideas and using and applying mathematics. (p. 102)

Students can be encouraged to devise different methods and come up with different options. This is one of the real advantages of investigations. The creativity and independent thinking of learners is sometimes overlooked but in investigations, they are encouraged to develop novel ways of doing mathematics. This can have valuable outcomes as far as everyday life is concerned. Sometimes students lose interest in a mathematical situation or problem because it is not 'real' to them.

There are some important implications of working through investigations and problem solving. These include the following:

- work in this area is best related to the students and not imposed by pre-conceived ideas of what the teacher thinks is required;
- work in investigations must proceed from the student's point of view;

- the onus is on the teacher to be aware and to notice how students respond;
- working mathematically embraces all aspects of problem solving and includes more, e.g. materials;
- working mathematically can encompass routine tasks;
- thinking mathematically involves abstractions, generalisations, argumentation, going beyond a particular problem or investigation.

These implications need to be considered more fully in relation to the outcomes that are desired as a result of mathematics teaching and learning. The benefits of such an approach include:

- collaborative learning groups are most suited to investigations;
- the student can set his/her own goals and work at his/her own pace;
- it encourages creativity and independent thinking and working;
- the students can work on a situation of their own choosing and hopefully commit themselves to resolving the situation; and
- students are more highly motivated.

There are many detractors from this investigational approach to teaching and learning mathematics. Time is an issue when students are expected to sit for public examinations within a certain time frame. This can, indeed, be a problem but not an insurmountable one. Another argument that is sometimes put forward against an investigational approach is that it is haphazard and unfocussed. This is where the awareness and skill of the facilitator teacher is needed to ensure that the outcomes achieved are linked to the students' existing body of knowledge and not be left as an isolated fact, no matter how exciting and satisfying that might be. Also the question must be asked: 'Ultimately, is the outcome of proficiency or high score beneficial if the student becomes anxious and subsequently a non-performer?'

The links between teachers' professional learning and the students' achievement

Causal relationships of this kind between teachers' professional learning and students' positive outcomes are difficult to establish. There are so many factors impinging on a mathematics classroom and as well these may affect students in different ways and at different times. One or two examples are pertinent, however. Manouchehri (2003) reports a study in which a mathematical inquiry environment was established through mathematical modelling. She particularly sought to develop the ideas of argumentation and a learning community and as a result claimed a richer understanding of the power of groups and opportunities for professional development that arose.

Reid (2002) described a study investigating reasoning patterns of Grade 5 students. They were given a counting squares problem and asked whether they could prove they had found them all. From their responses which included the investigational processes of conjecturing and testing their conjectures, Reid (2002) was able to identify the reasoning patterns of the students for different situations. In this way he not only gained a greater understanding of the way in which the students were thinking but he was gaining professionally himself.

While AAMT (2002) has concentrated on professional standards for teaching, the resulting impact on student learning cannot be ignored. The strands and substrands of the document are interrelated. If teachers set their sights on high standards for their own performance, there is a stronger than otherwise chance that students' outcomes in terms of achievement in performance, attitude and application will improve.

Conclusion

It is not assumed that if the above actions are implemented that mathematics will spring back into a vital life experience for all students. These are only two of the many factors that operate in this complex situation. There are many other factors that are outside the scope of our considerations and, indeed, outside our control: a student's health, for instance, family circumstances, and socio-economic considerations are generally outside a school's influence.

Several recent reports have suggested that the role of the teacher, the importance of professional learning and the establishment of an environment of innovation need to be considered (Department of Employment, Science & Training, 2003; Lovat, 2003; Australian Council of Deans of Education & Australian Council of Deans of Science, 2004). These are all for the purpose of enlivening mathematics and science or teachers in general. Lord Byron's words are very apt. The very fact that we sometimes despair of what is happening awakens and enlivens us to do something about it. Mathematics is not dead: it is alive and with help will become the force it has always been.

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Action research for improving professional standards in mathematics*

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Teachers of all subjects are constantly being asked to reflect on their teaching in order to develop greater understanding and skill in teaching and to enhance students' learning outcomes. It is suggested that this reflective practice is one component of teaching and learning mathematics that will revitalise mathematics in Australian schools. With ever increasing demands on teachers' times, it is necessary to find ways in which this reflective process can be implemented without taking additional time. Some of the difficulties that teachers might encounter in implementing the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* will be discussed and an action research strategy recommended for overcoming them.

Excellence is something all thinking people hope to achieve, though sometimes it seems a long way off. The old adage, 'The more we know, the more we realise how little we know,' is a timely reminder for teachers that the will to learn is a life time passion. Teachers in general can be described in terms of Chaucer's clerk: 'gladly wolde he lerne and gladly teche'. Striving for excellence in teaching mathematics must inevitably involve learning and this is emphasised in the AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools*.

The question is therefore raised as to methods that can be used to provide appropriate professional learning programs for teachers. For some time, teachers have been asked to become reflective practitioners (Schon, 1983; Boud, Keogh & Walker, 1985) and to enable their students to also be reflective in the work they do.

The process of reflection is sometimes confused with metacognition or simply going back over an experience. These are involved but real reflection goes a stage or two further and leads to commitment to action as a result of the reflection. In mathematics, this action could be in terms of the links that are made between different areas of knowledge or between different methods or strategies; or having worked on some realistic problem or investigation, the class or just the student could decide to take the kind of action that could best be described as social action. Unless one acts on the outcome of the reflection, the process could become counterproductive.

To reflect on a mathematical experience, the learner needs to return to that experience, remember the positive aspects of the experience, deal with negative issues such as anxiety or fear of failure or unwillingness to take risks, then re-evaluate the experience.

* This paper has been accepted by peer review.

This re-evaluation should lead to a new perspective on the situation, and perhaps a change in behaviour, but certainly a commitment to action. In the re-evaluation phase, the four processes of association, integration, validation and appropriation are relevant. The experience or situation on which reflection is taking place needs to be made meaningful through association with some previous learning or idea. The learner then needs to integrate any new learning into an existing body of knowledge, validate it in some way to see if it makes sense and then appropriate it as personal knowledge.

Figure 1 is adapted from Boud, Keogh and Walker (1985).

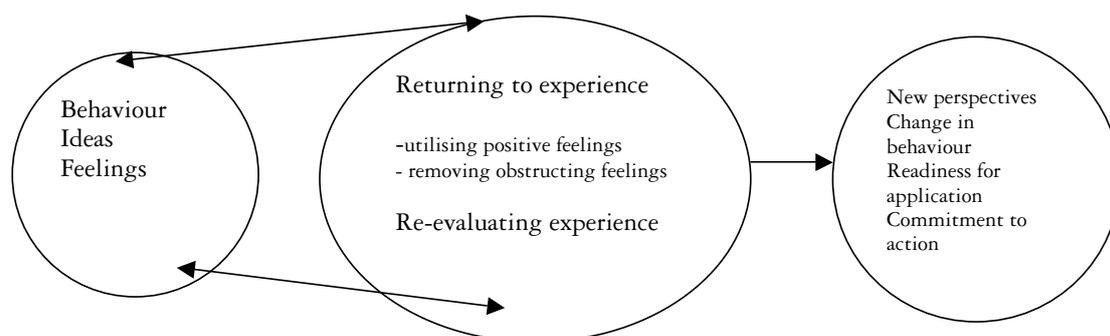


Figure 1. The reflection process in context (adapted from Boud, Keogh & Walker, 1985, p. 36).

Reflection is an aspect of professional learning for mathematics teachers that can be subsumed by others, e.g., action research. Action research seems to have been known to teachers for some time and is a process that is accessible to most teachers. It is a key method of being involved in one's own professional learning.

Action research

Burns (1990) describes action research as, 'the application of fact finding to practical problem solving in a social situation with a view to improving the quality of action within it' (p. 252). He also suggests that there are four main characteristics of action research, namely, it is situational, collaborative, participatory and self-evaluative.

Further to these characteristics, Arhar, Holly and Kasten (2001) list the elements of ethical commitment, cycle of reflective practice, public character and collaboration (p. 39). Commitment to professional practice is almost a prerequisite to engaging in action research. The cycle of reflective practice is given as observe, act, reflect. The process might start by the practitioner observing something in their professional practice that intrigues them or concerns them. A plan is then formulated to gather data which lead to reflection for further observation and action. For action research to be true research, it has to be shared with others and so has a public character (Arhar et al., 2001). It also has to involve others and therefore is collaborative, either between the researcher and the subjects or between researchers as well.

Arhar et al. (2001) identifies several types of action research and makes a distinction between them and teacher research in general. Teacher research can be described as any research carried out by a teacher or a group of teachers. Similarly, classroom research is research carried out in and about the classroom by a teacher or anyone else. Action research is

Research undertaken by individuals or groups which is founded on an active ethical

commitment to improve the quality of life of others, is critically reflective in nature and outcome, is collaborative with those to be affected by actions undertaken, and is made public. (Arhar et al., 2001, p. 47)

It follows then that teacher action research is action research carried out by a teacher or others and classroom action research is action research that takes place in and is concerned with life and practices occurring in the classroom. Collaborative action research is undertaken by individuals or groups who have a common focus.

Another definition of action research is that of Carr and Kemmis (1986) who wrote:

Action research is simply a form of self-evaluative enquiry undertaken by participants in social situations in order to improve the rationality and justice of their own practices, their understanding of these practices, and the situations in which the practices are carried out. (p. 162)

They suggest that the two essential aims of action research are to *improve* and to *involve*. In a later paper, Kemmis and Wilkinson (1998, cited in Atweh & Heirdsfield, 2003, p. 56) identified five additional characteristics of action research to the three usually accepted as planning, action and reflection. The five were participatory, collaborative, social, critical and emancipatory. It is participatory if it involves people within a practice in the process of research; it is collaborative if it involves individuals or groups of people both from within the situation and without; it is social in that it is part of a social context and critical in that it sets out to answer concerns that have come as a result of examining some situation or practice. It is emancipatory if it enables all participants to control the way in which their practice and their knowledge of their practice is improved.

Issues in action research

Crawley (1998) finds that action research recognises the complexity of the classroom situation and the large number of factors that impinge on the classroom. He and others (Noffke, 1994, cited in Raymond & Leinenbach, 2000, p. 302) see the relationship between theory and practice being enhanced through the action research process. Academics in general are considered to be more interested in the theoretical bases for learning and teaching mathematics whereas the classroom teacher is very much concerned with improving the learning and teaching situation in the classroom.

Noffke (1994, cited in Raymond & Leinenbach, 2000) identifies several issues in action research. The first is whether action research is real research. Arhar et al. (2001) claims that for research to be real, the results must be made public. In the cases that follow, this has been done in varying degrees. In one case the results have been taken overseas (Osler & Flack, 2002) and all have been shared within the classroom and the school concerned.

Another of Noffke's concerns has to do with the politics of knowledge production and refers to the establishment of appropriate relationships between researchers and practitioners and the breaking down of barriers between them. The valuing of researchers by teachers and of teachers by researchers is a key element in research partnerships. Atweh and Heirdsfield (2003) hint at this issue when they discuss authorship and voice in research (p. 57).

Raymond and Leinenbach (2002) ask the question as to whether collaborative action research is just another name for reflective practice. If one accepts the process put forward by Boud et al. (1985), then action research is more than reflective practice. In

the re-evaluation of an experience, the learner has to be able commit to action and in action research the commitment must turn into actual action. Another critical aspect is that in action research, the individual is developing him or herself within and through his or her practice.

Some of these issues are developed further in relation to the three cases described below.

Learning from others

Case one

Raymond and Leinenbach (2000) report a collaborative action research project in which Marylin, in reflecting on her algebra classes, was constrained to ask questions in relation to her teaching and her students' learning of algebra. She was teaching a Grade 8 in a school that had decided that all Grade 8 students should learn algebra. She was concerned as to whether what she was doing would help her students to understand algebra better and whether what they learned in her class would help them for future learning. She sought the assistance of a university lecturer, Anne, who was interested in other questions as well as those Marylin had formulated for herself. She was interested in exploring how the involvement in a collaborative action research project would affect Marylin as a mathematics teacher. They worked as teacher and researcher initially but as the project progressed, Marylin became more and more involved in the researcher role.

Marylin was using a particular manipulative approach in twenty-six lessons, preceded by a nine week period when she taught using a non-manipulative approach and a textbook. For the study she had 120 subjects from five classes. Data were gathered using an end of year survey, weekly student reflections, student work samples, test scores, teacher reflections and teacher observations and a whole class interview conducted by the university partner. Some students were also videotaped doing algebra problems. The second phase took place the following year when the students were mailed a survey to complete to test the durability of the Eighth Grade program. The response rate was low (19 out of 90 mailed). Of these, eight agreed to participate in an individual interview.

In the first year of the study, it was found that the students' scores were higher during the manipulative phase than during the textbook phase and on a standardised test, Marylin's students performed better than expected. In the second year, most of the class entered classes that were very different from the one they had enjoyed with Marylin. From a follow-up survey, it became evident that most of the students could not succeed in the classes in which they were first placed. Despite some concern that the transition was not proceeding smoothly, Marylin was able to reflect on her own beliefs and enabled students to ask questions about their own learning. As a professional development experience for Marylin, it had proved fruitful.

Case two

The use of action research in a study on the professionalisation of beginning women teachers was reported by Atweh and Heirdsfield (2003). Three women in their first year of teaching in schools at a considerable distance from each other formed an action research network with the university staff. The study focussed on the concern of the teachers to ensure that their teaching of mathematics was inclusive and bridged the gap between the university and the employer's responsibilities. Data were collected through

regular teleconferencing of the action research cell, access to email, and reflective journals. University team members also contributed reflections based on all the documentation available, including records of the teleconferences.

Analysis of the data resulted in perceived learning in relation to inclusive curriculum and professionalism in teaching. At the beginning of the year, all three teachers had found it challenging to find a pedagogy culturally appropriate for their students who came from diverse backgrounds. It was found that over the course of the study, the nature of the comments made by the three teachers changed significantly to convey a change in the way in which they thought of the situation. From a deficit model of describing the class, it became for two of the teachers a concern for the appropriateness of the curriculum. In terms of their professionalism, all three gained in their confidence in teaching mathematics, in the development of reflective practice and in developing supportive professional networks.

Case three

Two primary teachers relate their Poppy legend as an explanation of what they learnt over a period of time and how their collaboration affected their teaching practice. After an initial jolt caused by a conversation in the classroom and the realisation that the Grade 5 students with whom they were working had little understanding of what schools are about, the two teachers listed the conclusions they had drawn from the conversation and set about redefining their own beliefs first and then gathering activity ideas to assist students redefine theirs. They discovered the Project for the Enhancement of Effective Learning (PEEL) and were able to enter into a fruitful relationship with academics who were able to encourage them to believe that what they were doing was very important and other outcomes such as an awareness that they were:

- encouraging students to develop strategies to remain ‘on task’ for extended periods of time;
- developing and using teaching strategies and procedures that encouraged students to make links in their learning;
- valuing what students brought to class and provided opportunities for them to access and use their prior knowledge; and
- doing research related to that of others who were exploring their theories of teaching and learning.

(Osler & Flack, 2002, p. 229)

While this project began as a partnership between two teachers with a concern about the way in which their students view school, the subsequent research and collaboration with academics resulted in outcomes with very definite benefits to mathematics teaching. The teachers involved developed a deeper understanding of their own teaching practice and how that relates to their students’ learning. Particular aspects such as the linking of ideas is of relevance for mathematics teaching.

Conclusion

The AAMT *Standards for Excellence in Teaching Mathematics in Australian Schools* (2002) advocates three domains of excellence with relevant substrands in each case. These substrands are not mutually exclusive so the interrelationships between them need consideration. In

the examples of action research above, teachers gained greater understanding and knowledge of their students as Osler and Flack (2002) did, and a greater understanding of the way in which their students learn as the three teachers in Atweh and Heirdsfield's (2003) study. All the participants reported above were highly committed to maximising their students' opportunities to learn mathematics and to reach high standards of performance. All the examples also reported professional growth in varying degrees, including personal professional development. Osler and Flack in particular reported the effect of their growth on their colleagues though there were elements of this in the other two examples. Certainly, all three examples portrayed the characteristic of action research of making public the results of their studies, thus fulfilling community responsibilities. The third AAMT domain, that of professional practice, was exemplified to a degree and was to be found in each the three examples.

From the above examples, researchers have used a number of different strategies to collect data. These have included group discussions, surveys, individual interviews, action research cells, reflective journals, observation and videotaping. Others that might be added to the list are: photographs, field notes, rating scales and check lists, focus groups, and life history. The importance of keeping records and the final report writing are key issues that need to be considered at some early stage of the research. Unfortunately, it is outside the scope of this paper because of length and timing.

The value of action research to improve the learning and teaching of mathematics is without question (Crawford & Adler, 1996). In so doing, it has the potential to enhance the professional learning of teachers and strengthen their understanding of their practice as well as improving their practice itself. It is probably not possible for all mathematics teachers to engage in action research in its fullest sense but there are many opportunities for collaborative action research in all levels of schooling.

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Relational thinking about numbers as a bridge to algebraic reasoning*

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Some students choose to solve number sentences, expressed as an equivalence relation, by first identifying relations between numbers on opposite sides of the equal sign, while other students use calculation from the outset to simplify and solve the expressions involved. The first kind of thinking may be described as relational and appears to embody thinking which is quite different from the second approach. Using a study of Japanese students, this paper illustrates relational thinking and identifies some key features. Students who solved number sentences relationally were more likely to apply this thinking to algebraic expressions.

Researching relational thinking

The *Standards for Excellence in Teaching Mathematics in Australian Schools* (AAMT, 2002) point to the importance of teachers having ‘a strong knowledge base to draw on in all aspects of their work’. The research reported in this paper is directly concerned with the third aspect of professional knowledge, namely ‘knowledge of students’ learning of mathematics’. The teaching of number in the primary school should provide opportunities for learning about number properties and the structure of number sentences — much more than being able to calculate. Looking ahead to the early years of high school, students need an understanding of structure and relationships, based on their experiences with number, as they are introduced to algebra. Many students, however, appear to find algebraic thinking new and puzzling, whereas others are able to make clear connections with their knowledge of number patterns and relationships.

These issues provided a rationale for the current study which was carried out in two stages: the first stage in Tsukuba University’s attached schools in Tokyo (Grades 3, 5, 7) from February to March 2004 with one class at each Grade level; then a second stage completed in Tsukuba City schools (Grades 4, 6, 8) during April and May 2004, with three classes involved at each Grade level. In all, 379 students were involved.

The study aimed to address claims raised in three previous studies, and to answer two further research questions. Kieran (1981) claimed that many children in elementary school think the equal sign is a direction to find the answer (and not as an indication of equivalence or balance). Do children in Japan think of the equal sign as indicating an equivalence relation or a balance? Given the further claim by Carpenter and Franke (2001) that the proportion of children in the elementary school, prior to instruction, who

* This paper has been accepted by peer review.

can think relationally, is quite low (probably no more than 10%), it was important to ask how accurate is the estimate for children in Japan. In addition, Behr, Erlanger and Nichols (1980) argued that there is no evidence that children change their thinking about equality as they move through the elementary school. Is there evidence of such change for children in Japan?

Two further research questions were asked. From students' responses, is it possible to identify some clear identifying features of relational thinking? Finally, if students in Grades 5 to 8 exhibit clear instances of relational thinking for number sentences, does this thinking transfer to some expressions involving literal symbols?

Questions involving number sentences

Different questionnaires for Grades 3–4, Grades 5–6 and Grades 7–8 contained three groups of common items involving number sentences. These were based on a shorter questionnaire prepared with colleagues, George Toth and Loretta Weedon, of the Catholic Education Office in Melbourne. The three groups were as follows:

- Group A

Consisting of four addition questions, such as

$$26 + 39 = 23 + \square$$

$$\square + 17 = 15 + 24$$

- Group B

Consisting of three subtraction/difference questions such as

$$39 - 15 = 41 - \square$$

$$99 - \square = 90 - 59$$

- Group C

Two questions involving balancing, such as $746 - 262 + \square = 747$

Children were not prompted in any way to opt for a particular method of solution. They were, however, asked to explain their thinking for each question. Clearly, the numbers were chosen such that 'relational thinking' was a feasible method of proceeding. In Group A, for example, since 26 and 23 are three apart, the missing number is 3 more than 39. In Group B, equal differences require a different logic. Being able to apply relational thinking for the expression $746 - 262 + \square = 747$ depends on seeing the last number as one more than the first number, so recognise that $-262 + \square$ is equivalent to adding 1.

Children could solve this question by calculation. For example, some children first calculated $746 - 262$ to give a new expression $484 + \square = 747$, and after that calculated $747 - 484$ to find 263. By reducing the initial expression to a simpler form, they used the difference between 747 and 484. Such calculation-based thinking rests ultimately on simple arithmetic relations. Nevertheless, clear differences between calculation-based (arithmetic) thinking and relational thinking emerged from children's responses. These will be discussed later.

Questions involving multiplication would have proved difficult for younger students, and required them to be familiar with order of operations. The use of addition and subtraction was seen to give all students an opportunity to attempt all number questions with confidence.

Questions involving literal symbols

For Grades 5 to 8 only, additional questions involved using literal symbols. One question was included for Grades 5/6. Two questions were included for Grades 7/8. Literal terms and number terms had to be placed in a relation true for all values of the unknown. The link question for Grades 5–6 and Grades 7–8, asked students to place $n - 1$, $n + 5$, 7 and 1 into four ‘boxes’ of the expression $\square + \square = \square + \square$ such that the resulting expression ‘is always true’. It was possible for a correct response such as $(n - 1) + 7 = (n + 5) + 1$ to be written in the reverse order and without brackets. In addition, students in Grades 7–8 were asked to write a similar expression involving the terms $n - 1$, $n + 1$, $m + 3$ and $m + 1$.

In order to deal successfully with these expressions involving literal symbols, it was hypothesised that students needed to use thinking similar to that required for thinking relationally about numerical expressions.

Students’ responses to numerical expressions

Scoring scheme

Individual questions were not scored separately. An overall or ‘holistic’ scoring scheme was applied to each group of number sentences, using the following scores:

- 0 arithmetical thinking (no evidence of relational thinking in any question)
- 1 relational thinking shown in some questions, but not successfully executed
- 2 relational thinking shown in some questions and successfully executed
- 3 relational thinking shown in most questions and successfully executed
- 4 relational thinking shown in all questions and successfully executed

This multi-point scale allowed a single score to be given to each group even where the thinking may have been different across sub-questions. Scoring advice was prepared for each group, and a benchmark sample, illustrating the five-point scale, was prepared for graduate students who then worked in pairs to mark children’s written responses. Children’s responses were all ‘double checked’, and any discrepancies noted. There were high levels of agreement between pairs of markers with discrepancies easily resolved.

Responses to Group A (addition) questions

The percentage of calculation-based responses (score = 0) was about 45% for Grades 3–4, and remained between 30% and 40% for the other two grades, with a slightly smaller proportion of calculation-based responses in Grades 7–8 compared to Grades 5–6.

The proportion of those using relational thinking (score 1, 2, 3, 4) increased steadily by grade level, with the proportion of those relying completely and successfully on relational thinking (score = 4) increasing from about 22% in Grades 3–4 to about 42% in Grades 5–6, and to about 40% in Grades 7–8.

Responses to Group B (difference) questions

The percentage of calculation-based responses was slightly more than 60% in Grades 3–4, suggesting that younger students were less inclined to use relational strategies for the Group B questions than they were for Group A questions.

The proportion of calculation-based responses fell to nearly 40% in both Grades 5–6 and Grades 7–8. Across all grades there was a steady increase in the proportion of students who use relational strategies on all questions (score = 4), rising from about 10% in Grades 3–4 to over 30% in Grades 7–8.

In both Grades 3–4 and Grades 5–6, about 20% of students scored 1 on Group B questions. Their most prevalent mistake was to apply to difference questions the same relational strategies that had been used successfully to deal with addition problems. By Grade 7–8 the proportion of students using this ‘failed relational thinking’ fell to just over 10%. For Group A and C questions, the proportion of correct responses was so high, regardless of strategy used, that score = 1 was rarely applied.

Responses to Group C (balancing) questions

For Grades 3–4, the proportion of students using a calculation-based approach was nearly 80%, dropping to 65% for Grades 5–6 and to about 55% for Grades 7–8. Nevertheless, the proportion of fully relational responses to Group C questions by Grades 3–4 students, while less than 10%, showed that some quite young students were able to solve these more complex balancing questions relationally. By Grades 5–6, the proportion of fully relational responses (score = 4) to Group C questions doubled to about 20%, and was over 30% for Grades 7 & 8.

Summary

Successful relational thinking (score ≥ 2) was most readily applied to addition questions (Group A). It was applied less readily and sometimes with lack of success to difference questions (Group B), and less readily, but still with evident success by some students, to the balancing questions (Group C). These results are summarised in the following table:

Score ≥ 2	Grades 3–4	Grades 5–6	Grades 7–8
Group A	50%	60%	63%
Group B	15%	36%	46%
Group C	20%	32%	42%

Disposition to apply relational thinking seemed to peak for Group A in Grades 5–6 with little extra growth in relational thinking in Grades 7–8. However, for the other two Groups of questions, there was steadily increasing growth in disposition to use relational strategies across the three grade levels.

Even by Grade 7–8, there remained a proportion of students (ranging from 30% for Group A to over 50% for Group C) who relied entirely on calculation-based approaches.

The estimate by Carpenter and Franke (2001) based on a USA study that only about 10% of children, prior to instruction, were able to think relationally was not supported by this study, with a much higher proportion of children at all grade levels showing a capacity to engage in relational thinking.

Contrary to the claim by Behr et al. (1980), disposition to use relational thinking increased by grade level for all groups of questions as children moved through elementary school and into junior high school.

In contrast to Kieran’s (1981) claim, many children in this study interpreted the equal sign as indicating a balance or equivalence between both sides of an expression, as the next section illustrates.

Illustrations of relational thinking

The study provided rich and varied illustrations of children's relational thinking in regard to the three groups of questions used. Despite these differences, some underlying characteristics of relational thinking can be identified.

Student 5.19 (Grade 5, student #19) completed the number sentence $23 + 15 = 26 + \square$ by saying, 'Comparing 23 and 26, since 26 is three more, so 15 has to become three less.'

Student 5.38 analysed the number sentence $43 + \square = 48 + 76$ by placing the two pairs of numbers directly under each other: $43 + \square$ and $48 \uparrow_{-5} + 76 \uparrow_{+5}$.

Student 5.21 solved $73 + 49 = 72 + \square$ by connecting 73 and 72 with a line under which was placed -1. Student 5.21 then drew a line joining 49 and the box on the right hand side. The student said: 'The box has to contain an opposite number [to -1] that produces zero.'

Student 5.37 analysed the number sentence $23 + 15 = 26 + \square$ in the following specific way: 'Because I have to make $23 + 15$ equal to a "three larger" number and a "three smaller" number, I get 12. Left side and right side are balanced.'

Student 8.10 re-expressed $99 - \square = 90 - 59$ as $(90 + 9) - \square = 90 - 59$ and then rewrote this as $(90 + 9) - (59 + 9) = 90 - 59$.

Student 7.32 transformed the initial difference sentence to $39 + (-15) = 41 + (-\square)$ and then applied the procedures that had been used to deal with 'addition type' sentences: 'An increase in the first number has to be balanced by a [corresponding] decrease in the second number. In this way (-15) becomes (-17).'

Student 7.31 transformed the number sentence $99 - \square = 90 - 59$ to $99 - 90 = \square - 59$ obtaining a correct result of 68.

Student 7.21 took $23 + 15 = 26 + \square$ and generated two sub-relationships, $23 + x = 26$, and $15 - x = 12$, where $x = 3$ gives a missing number of 12.

Key features

Students who think relationally about number sentences appear to:

- focus on a number sentence, viewing it as a whole;
- treat the equation symbol as standing for equivalence or balance between the numbers and operations involved, and not simply as an answer to a calculation;
- identify — or impose — a structure depending on the nature of the numbers and the operations involved;
- impose different structures, and describe these structures in different ways;
- do not work from left to right and do not solve by calculating sub-totals;
- use structural relationships between different terms, depending on the numbers and the operations involved.

On the other hand, calculation-based or arithmetic thinking:

- simplifies a numerical expression through calculation to the point where a simple relation can be applied to solve a problem;
- produces a new numerical expression as a result of calculation (unlike relational thinking which typically deals with all terms in the one expression or line);
- is clearly appropriate in some situations where relations between numbers are not evident;
- was preferred by some students in some questions who had used relational thinking in other questions;
- was the only strategy used by some students;
- cannot be used successfully with expressions involving literal symbols.

Performance on expressions involving literal symbols

The previous five-point scoring scheme was modified to grade responses to questions involving literal symbols. This scoring scheme was applied to responses from students in Grades 5–6 and in Grades 7–8 to questions involving use of literal symbols:

- NR used for no response to the question
- 0 no evidence of relational thinking: incorrect or inadequate relation
- 1 relational thinking shown: correct relation shown but nothing else
- 2 correct relation shown, and successfully illustrated with one or more numerical examples
- 3 correct relationship shown, and explained by indicating some structural balance with respect to the number terms, arguing for example that ‘we can ignore’ n or m terms
- 4 correct relationship shown, and explained by explicit and clear reference to structure showing that all terms ‘balance’ on both sides.

Grades 7–8 responses

Question 1

How did those who used relational thinking for the number sentences respond for expressions involving literal symbols?

- 33 students (out of 144) showed high level relational thinking (score ≥ 3) on all three groups number problems: Group A, Group B, and Group C.
- Of these, 30 scored ≥ 3 on the questions involving literal symbols, showing that those who used high relational thinking on number sentences had a very high probability (91%) of dealing successfully with expressions involving literal symbols.

Question 2

Did those who used relational thinking for expressions involving literal symbols respond at the same level for number sentences?

- 77 students (out of 144) scored ≥ 3 on one or more question involving literal symbols.
- 60 of these students responded at ≥ 3 on one or more groups of the numerical questions, showing a strong reverse trend (78%). Still, there were some students who dealt successfully with expressions involving literal symbols and opted to use arithmetical thinking on the numerical expressions.

Question 3

How did those who did not think relationally on number sentences perform on expressions involving literal symbols?

- 37 students had score = 0 on *each* of Group A, Group B, and Group C. Their responses to the literal expressions were scored as follows:
 - (score 0) 7 students
 - (score 1) 11 students
 - (score 2) 6 students
 - (score 3) 4 students
 - (score 4) 9 students
- 18 students of 37 appeared to be experiencing difficulty (score = 0, 1) with literal expressions. The other 19 (score ≥ 2) were able to shift to relational thinking when required to deal with literal expressions.

Grades 5–6 responses

Question 1

How well did students in Grades 5–6 who used relational thinking for the number sentences deal with the expression involving literal symbols?

- Of 23 students who showed clear relational thinking (score ≥ 2) on all three groups of number problems, 19 scored ≥ 1 on the question involving literal symbols. Only two of these 23 failed to respond to the question involving literal symbols, contrasting with a much higher non-response (NR) level for students in Grades 5–6 as a whole.
- 35 students out of 133 did not answer this question (NR = 35). Given that seventy two students (54%) wrote a correct relation involving the four terms (score ≥ 1), students who showed clear relational thinking (score ≥ 2) on all three groups of number sentences had a much higher probability (83%) of dealing successfully with expressions involving literal symbols.

Conclusions

The study showed a clear and strong link between children's disposition to use relational thinking to solve number questions and their ability to read and correctly construct expressions involving literal symbols.

The results, while convincing for students in Grades 7–8, are more encouraging still for students in grades 5 & 6 who have not been exposed to expressions involving literal symbols. Nevertheless, performance of students in Grades 7–8 should not be underrated, since they were given a question involving two 'unknowns', which they had not been taught.

Further investigation is needed to explain why the Japanese curriculum is effective in disposing many students to use relational thinking in dealing with numerical expressions and expressions involving literal symbols. One of the strengths of the Japanese National Course of Study is its emphasis on having students read and interpret number sentences in uncalculated form. This is also evident in textbooks.

The curriculum is not effective for all students. Some were unable to deal successfully with expressions involving literal symbols. However, other students who used calculation-based approaches in dealing with numerical expressions could deal successfully with expressions involving literal symbols. The picture is a complex one and needs to be investigated further.

This research suggests that a calculation-focused curriculum in the primary school is unlikely to provide students with the mathematical capacities needed to understand algebraic structure and reasoning in high school. There are implications also for the way we approach arithmetic in Australian primary schools. Too many Australian students see no link between their study of arithmetic in primary school and their introduction to algebra in high school. It appears that the mathematics curriculum for elementary schools in Japan is able to balance the twin goals of achieving computational proficiency and understanding the structure of arithmetic operations.

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Mathematics for everybody: Implications for the lower secondary school*

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If we take as our starting point the quite reasonable proposition that numeracy is 'having the competence and disposition to use mathematics to meet the general demands of life at home, in paid work, and for participation in community and civic life' (Willis, 1992, pp. 77–84), then the interaction between school mathematics and numeracy becomes critical. The practice of school mathematics focusses on developing student knowledge and understanding of mathematics per se; it also emphasises the applications of mathematics to other contexts. In both of these practices the primary purpose is, appropriately, the development of mathematical understanding. However we would argue that, in the school setting, the context for numeracy lies beyond the mathematics classroom, most obviously as an integral part of students' learning across the curriculum. Here the focus is not on the mathematics; it is on the context. Yet understanding the mathematics is crucial in working within that context. In this paper we propose a 'numeracy framework' as a way of describing numeracy, diagnosing learning issues, supporting teacher planning and for teaching to students so that they can choose to act numerately beyond the mathematics classroom. We use the results of an Australian research project in numeracy across the curriculum in the middle years of schooling, and examine the implications for teachers of mathematics.

Numeracy — More than solving problems with mathematics

It seems that numeracy is finally being taken seriously by education and training sectors and systems around the world. However there still does not seem to be a shared understanding of what 'numeracy' is. People perceive and describe numeracy in many different ways. A wide variety of terms is used almost interchangeably with numeracy. These include, for example: mathematical literacy (e.g., Organisation for Economic Cooperation and Development, 2001), quantitative literacy (e.g., Steen, 2001; Dossey, 1997; Forman, 1997), mathematical skills (e.g., Marks & Ainley, 1997), statistical literacy (e.g., Watson, 1995), critical numeracy (e.g., Yasukawa, Johnston & Yates, 1995) and critical mathematical literacy (e.g., Frankenstein, 2001).

Yet despite this rich and varied discussion of the importance of mathematics in action in practice, a focus on the mathematical concepts, procedures and skills students should know and be able to do is still the dominant paradigm in schools of what it means to be

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numerate (Thornton & Hogan, 2003). Such a focus on the essential aspects of mathematics appears to embody a naïve view of improving student numeracy. It assumes that, ‘mathematics can be learned in school, embedded within any learning structures, and then lifted out of school to be applied to any situation in the real world’ (Boaler 1993, p. 12).

However, this does not appear to be the case. There is a growing literature on the nature of transfer of learning and the evidence suggests that students do not automatically use their mathematical knowledge in other areas. Lave (1988) found that even experience in simulated shopping tasks in the classroom did not transfer to the supermarket. On the other hand, it appears that people use highly effective informal mathematics in specific situations (Carraher, Carraher & Schliemann, 1985).

It would be easy to attribute this lack of transfer of mathematical skills to other contexts to a deficient mathematics curriculum and poor teaching, but the quite considerable debate about transfer of skills shows that even if mathematics were taught and learned very well people would not necessarily apply it to new situations (Griffin, 1995). Researchers in the area of situated cognition argue that cognitive skills and knowledge are not independent of context, and that activities and situations are integral to cognition and learning (Brown, Collins & Duguid, 1989).

In order to respond to these issues there has been an attempt to contextualise school mathematics using contexts which appear to be relevant to the students. It was hoped that this would help students to see the purpose and usefulness of the mathematics they were learning, and that the mathematics would make sense. However, despite teachers’ best efforts many of these ‘real world problems’ appeared contrived rather than real (Willis, 1992); required students and teachers to participate in ‘a wilful suspension of disbelief about reality and mathematics’ (Williams, 1993); and left out factors relevant to the real situation (Boaler, 1993). Further, these attempts still had a primary purpose of teaching mathematics rather than developing numeracy. It would seem that if students are to learn to use mathematics outside the mathematics classroom then that is where they need to experience using mathematics.

There are many examples of the numeracy demands and opportunities across the school curriculum (Hogan, 2000; Hogan & Kemp, 1999). Many of these activities could be performed in ways that make little demand on numeracy and indeed they could be structured explicitly to avoid mathematical demands. However, done well and fully, these tasks are likely to make considerable call upon students’ capacity to apply mathematical ideas in context. Doing mathematics well is integral to doing geography (or history, or language, or technology, or science, etc.) well. Thus teachers in all curriculum areas need to take seriously the numeracy demands of their curriculum and the strategies they use for students to learn. It is everyone’s business (Hogan, Jeffery & Willis, 1998; DEETYA, 1997); it is crucial to student learning within the subject.

A numeracy framework

Over the past few years we have been working with teachers in primary and secondary schools to explore the numeracy demands and opportunities that occur across the curriculum. The research methodology was based on *Research Circles* (Australian National Schools Network (ANSN), 1999), in which teachers came together for periods of time to discuss their work, to observe and evaluate classroom incidents, and document these case studies. The teachers undertook to record, in as much detail as possible, the circumstances in which students encountered mathematical ideas, the problems they had in

understanding the mathematics and/or the context, the action taken by the teacher and what the students did next.

The research provided a rich array of examples of teachers observing student numeracy, and a constructive forum through which others could provide feedback. It became apparent that the teacher-researchers had begun to look more closely at the students' responses to numeracy demands across the curriculum. They had begun to see that a student's numeracy problem might not be simply a matter of not knowing the mathematics, but might relate to the context, or the student's inability to continue work on the task once confronted with something they could not do. When it was seen to be an issue with the mathematics, the teachers were more sensitive to what the mathematical problem might be.

The examples provided support for the idea that numeracy requires more than routine facility with basic mathematical procedures, and more than the capacity to apply mathematics to a set of applications within the mathematics classroom. Numeracy requires the purposeful use of mathematics beyond the mathematics classroom.

We claim that being numerate requires a blend of mathematical, contextual and strategic know-how. However the blend of these three 'know-hows' needed for a particular situation will be determined both by the context and by the orientations, skills and knowledge of the person, and their capacity to take up three key roles. These roles have been described (Hogan, 2000; Willis, 1998) as the *fluent operator*, the *learner* and the *critic*. These six aspects of being numerate form the basis of a *numeracy framework* that we have been using with the teachers.

A NUMERACY FRAMEWORK

Being numerate within a context involves a blend of three types of know-how

Mathematical

Contextual

Strategic

and three roles

The fluent operator

The learner

The critic

(Hogan, 2000)

Three types of 'know how'

Mathematical 'know-how' involves knowing, understanding and using the mathematical ideas which typically comprise the school mathematics curriculum in measurement, number, geometry (space), algebra, probability and statistics (chance and data).

Caroline is an art teacher working with a small group of somewhat disengaged 14 or 15 year-old boys. She set them the task of making a 'tag'¹ mural containing their school's initials. In designing their tag, the boys had to call on several mathematical concepts such as ideas of scale, cost and location. When ordering wood for their mural they discovered that measurements were given in millimetres, and they needed to convert their calculations from centimetres.

Contextual know-how involves understanding the contextual features of the mathematics in the situation — what terms mean in the context, and what interpretations make

1. A tag is a piece of graffiti usually spray-painted onto a large public surface. It acts as an identifier and symbol of the graffiti artist's identity.

sense. This requires more than a familiarity with the context. It requires an understanding of how the mathematics in the situation is shaped by the context.

The boys in Caroline's art group used their mathematical knowledge of scales to calculate the desired final size of their mural. They then made the decision, on economic grounds, to paint the mural onto plywood. However when they rang building suppliers they found that plywood came in sheets of varying thicknesses, and they had to decide on a thickness that would be economical, manageable, and that would not buckle. Their choice of the final size was shaped by the context.

Strategic know-how involves the orientations and strategies to manage one's way through routine or non-routine problem situations. These include almost habituated ways of getting going when stuck, selecting key information and representing and organising it in models, diagrams and lists, breaking a task into component parts or identifying and working on related problems or sub-problems, and organising the approach in a systematic way. It may also involve making one's assumptions explicit to decide whether a particular procedure is appropriate, posing questions for oneself in order to come to grips with the essence of the task, and knowing to check that the solution makes sense in the context and fits the original specifications or constraints.

The boys in Caroline's art group came across the challenge of how they were going to enlarge their tags from the A4 piece of paper onto three boards of much larger scale. One particular student who did not have a strong background in mathematics, but proved to be a good problem-solver, suggested transferring the drawing onto an overhead transparency, then projecting the image onto the plywood of a large enough scale, and to then trace around the image.

Three roles within a particular context

The fluent operator

Within a particular suite of contexts, people who are numerate will show fluency of use of the knowledge and skills regularly used in those contexts. This is the comfortable, quick and ready, almost unconscious use involved in being 'at home' with their everyday uses of mathematics. Their mathematical actions and thoughts will be smooth (fluid) and almost automatic with the relatively routine aspects of the situations. These are the actions of the experienced administrator undertaking a task done over and over. There is no new learning and any mathematics being done by the reader is almost unconscious. Developing fluency with numeracy tasks in familiar settings is a useful skill.

Julie is a Science teacher working with a class of thirteen year-olds. The students were watching a video on how the blood circulates around the body. The narrator talked about the number of red blood cells in the body, what their purpose is and how they travel around the body. After the video Julie asked the students if they had any questions and what they thought of the video. Straight away one of the students asked some questions related to red blood cells:

'What does 250 million look like? That is a lot of blood cells! How is it possible for all of those cells to fit into our body?'

'If you were fatter, like myself, wouldn't you have more red blood cells?'

It was apparent to Julie even after an initial discussion with the class that many of the students did not know how to represent 250 million using numbers. She had assumed they would all be fluent operators with respect to the number itself. However most of the students were not able to visualise 250 million as a quantity of cells, or indeed as a quantity of anything.

The learner

The numerate person, however, also uses mathematics to help make sense of something new or to deal with new or altered circumstances; that is, to learn. This is the deliberate ‘Can I make sense of this?’ kind of use of mathematics to cope with an unfamiliar task or understand something new.

The young man who asked the questions is acting numerately, by asking the questions. He was using his limited knowledge of mathematics to try and make sense of something new and he was heroic enough to ask the questions on behalf of the class. We can guess that he was figuring that 250 million was a lot of cells: how do you fit that many ‘things’ in a body? He was using his common sense notion that a bigger body might contain more blood and therefore have more blood cells than a smaller body. The mathematics and the context (the science) were interacting to confuse him. Having the agency to ask the questions is something we might want for every young student.

The critic

Finally, the numerate person uses mathematics sensibly and critically, knowing what mathematics is and is not, and what it can and cannot do in order to be able to judge and question the appropriateness of its use.

In looking at information from two different sources, a conflict becomes apparent. The text that students were reading claimed that there are about 5 million red blood cells in a microlitre of blood. This means that there are around 6 trillion cells in a human body. Perhaps the 250 million is in a drop of blood, around of a millilitre. Having the capacity to ask questions about whether the mathematics is appropriate, and to try to reconcile information from different sources, is a key aspect of being critically numerate.

Thus, a person who is numerate within a particular context must, to different levels and varying degrees, assume each of the three roles of the fluent operator, learner and critic.

Implications for the teacher

An awareness of numeracy across the curriculum generates opportunities for engaging students with the numeracy in tasks. The teachers in the projects with which we have been involved, who have begun to explore these opportunities, have been suggesting the following strategies. Teachers and educators not involved in this research have commented that this list just represents ‘good teaching’. This may be true. The essential point to make about this list though is that they are to be applied to numeracy — across the curriculum.

- *Capturing the numeracy in the moment*
Be alert to the numeracy demands that arise in class work. Take the time to notice the students learning needs in the experience. Decide to deal with it — then or later.
- *Being aware of possible numeracy demands when planning*
Take the time to review your curriculum planning for possible numeracy demands. This does not mean ensuring that these demands are always pointed out to the students prior to the experience (indeed if the numeracy demands are always identified for the students they will not get the essential experience of dealing with them themselves) but the teacher can be prepared for the possibility that the students might need extra time, extra support or explicit teaching. Planning also means that teachers can ensure that students are confronted with dealing with sufficient numeracy demands over the school year. The teacher might also identify any

areas where sensible links to mathematics could be made and therefore where there might be opportunities to enhance student numeracy.

- *Allowing students to work it out*
Provide students with both individual and collaborative opportunities to work things out for themselves. Do not rush to do it for them. Be patient and flexible with time to allow students to engage with the numeracy themselves, ask questions, fully understand the lesson and gain confidence in themselves as learners.
- *Supporting student numeracy learning by questioning*
Facilitate discussion and support students' deliberations by asking questions about their handling of the task. Questioning can help students identify the numeracy and then use their mathematics. Try to keep the questions open to encourage a willingness to participate.
- *Diagnosing student numeracy by listening purposefully*
Monitor students' numeracy knowledge and skills by asking questions and listening purposefully as students engage with numeracy in a moment.
- *Debriefing the numeracy*
Ask open questions that encourage students to reflect on the use of mathematics in the situation and the role numeracy played in their learning, understanding and problem solving. Ask the students, 'What is the key mathematical idea that we have used here?' Ask them too, 'Where else might we use this idea?'
- *Practising*
Give the students a different context that makes similar numeracy demands to one they have completed to allow the students an opportunity to practice their capacity to be numerate.
- *Promoting critical use of mathematics*
Discuss with students whether or not mathematics might be able to shed new light on a situation. Where mathematics is being used: is this use appropriate? When they have used some mathematics to learn something, do something, make something: was it the best method?
- *Teaching the framework to the students*
As with literacy models explicitly teach the students about numeracy and the ideas in the numeracy framework. Discuss with students what it means to be numerate. Students can practice the role of identifying any mathematical concepts, ideas, terms, procedures and skills that may exist in the situation. They can find out more about the meaning and sense of mathematical terms and processes as used within particular contexts and how the context might influence the mathematics. They can practice deciding if the group might need to use mathematics in a context. They can ask whether the mathematics needs to be altered, how accurate the group needs to be, if it makes sense in the context and whether they used a good method. They can ask whether it makes sense or not to use mathematics, who is using the mathematics and why, what is their purpose in using mathematics, as there might have been a better way to do the mathematics, and so on. They can practice finding out about the mathematics they might need to use in dealing with a situation.
- *Sharing information on student numeracy with others*
Sometimes a teacher cannot seem to find the way to help a student with a particular numeracy problem. Teachers have found that it helps to share problems like this and get advice from others.

Implications for the mathematics teacher

Given the claim that numeracy, in the sense described above, can only be learned and practised within the context in which it is met, what, then, is the role of the mathematics teacher? How might the mathematics teacher support the development of student numeracy as practical intelligent use of mathematics in context? How can the mathematics teacher acknowledge the traditional goals of the mathematics curriculum, but also take seriously that one part of the job is to support students' numeracy beyond the mathematics classroom?

While we cannot assume that because we have taught something inside mathematics it will be transferred by the learner to new situations, we propose that there are some things we can do, as mathematics teachers, to help. We suggest that the role of the mathematics teacher must include, at least, aiming to:

- *Build confidence, identity and agency*
If students are to act numerately beyond the mathematics classroom they need the confidence and disposition to see themselves as competent users of mathematics, and to acknowledge the agency of the discipline of mathematics (Boaler, 2003). Thus the mathematics teacher's role will include encouraging students to interrogate the mathematics by asking questions such as: what is the key idea here? Could we use this idea in some other setting? Is the solution to this problem mathematically consistent?
- *Build mathematical understanding*
Acting numerately beyond the mathematics classroom will require that students see mathematics as a sense-making enterprise (Flewelling, 2001). The mathematics teacher's role will thus include providing students with opportunities to develop both relational and instrumental understanding (Skemp, 1976), and to encourage students to ask questions such as: does the solution make sense? Is there another way of looking at this problem? What if we changed some of the constraints or parameters?
- *Promote mathematics as connection-making*
If students are to act numerately beyond the mathematics classroom they will require the capacity to draw on a rich array of mathematical skills and understandings, and to appreciate that all areas of mathematics are richly connected. Thus the role of the teacher is one of connection-making (Askew et al., 1997), encouraging students to look for links. Where have you used this elsewhere? Where might we use this elsewhere? Can we draw on other parts of mathematics to help solve this problem?
- *Provide authentic opportunities for students to use mathematics*
The capacity of students to act numerately beyond the mathematics classroom will be enhanced when students are given the opportunity to act numerately within the mathematics classroom. Mathematics teachers should provide opportunities for students to engage with tasks that are both mathematically rich and contextually authentic. Such tasks may not necessarily be practical, nor immediately relevant to the student. However they should provide opportunities for students to make decisions about what mathematics might help, about what information might be necessary or extraneous. It is important to note that we do not consider word problems such as those typically found in textbooks authentic. It is not the mention of a context that makes a problem authentic; rather it is the nature of the reasoning and decision-making that must be undertaken by the student.

- *Acknowledge that not all mathematics is immediately practical*

The power of mathematics lies in its capacity to de-contextualise a situation. Removing the ‘noise’ of a problem by abstracting it into a mathematically concise and tractable form enables the student to see beyond the immediate and seek more generalised principles. The mathematics teacher should thus promote mathematics as a study worthwhile in its own right, provide opportunities for students to see the power of mathematical reasoning, and to see how abstract mathematics can shed new light on situations from the real world.

An example of a mathematics teacher promoting numeracy: furniture removal

Phil is a Grade 6 teacher at a school in the Australian Capital Territory. He was looking at distance-time relationships in his mathematics class. He asked the students in his class the seemingly simple question, ‘If I want to move the furniture in my house from Canberra to Sydney, a distance of 300 km, and the van travels at 100 km per hour, how long will it take to return to Canberra?’ After students had given the simplistic answer of six hours, Phil asked the class to divide into groups and ask questions of the situation. What followed provided a rich insight into students’ numeracy, as they discussed issues such as accuracy and constraints, and made sense of the mathematics in context.

Some groups presented a solution that took into account time taken to unload the van; some discussed to which part of Sydney the delivery was to be made; some discussed the need to take a break from driving; some discussed the difference in speed between a full and empty van. It was clear that the students felt ownership of their reasoning and solution; that they saw mathematics as integral to the problem; that they appreciated the limitations of that mathematics within the context; that they could switch fluently between de-contextualised thinking and an appreciation of the real problem.

Our observation of the interactions within the lesson suggest that two things made this incident one which could help to build student numeracy: the problem itself and particularly its lack of specificity, and the teacher’s role in prompting critical thinking. By not specifying in the problem other aspects of travel or furniture removal that students should take into account, students were given the opportunity to build agency. They were able to decide for themselves what might or might not be important. A more tightly specified problem might immediately exclude some students who could not make sense of the situation or read the words; it might close discussion about what should be considered; it might remove the agency from the students, the context and the mathematics, and place it with the teacher or the textbook. Specifying the problem so that there is a ‘right answer’ might prevent students from examining the reasonableness of answers and discussing and accounting for differences between solutions.

As the teacher Phil saw his role as primarily one of asking questions. He encouraged the students to draw on their knowledge of both the mathematics and the context. He asked students to evaluate their answers by asking ‘What if...?’ and ‘Have you thought about...?’. He encouraged and valued a variety of solution methods including use of a distance-time-speed formula, but also a sense-making approach that built on students’ everyday understanding of speed.

Conclusion

For students in the middle years of schooling the most common context in which they might have the opportunity to develop and demonstrate numeracy is within other curriculum areas. The teacher of those curriculum areas has a crucial role to play in noticing when numeracy is important, and in providing students with the time and opportunity to explore the numeracy in the context.

The mathematics teacher has an equally important role to play. This is not one of teaching the mathematics ‘first’, so that students can then use it in other curriculum areas. Nor is to try to ‘invent’ contexts that give the appearance of making mathematics useful. Rather it is to promote agency and sense-making, and to give students an opportunity to appreciate the power of mathematical reasoning both for its own sake and within an authentic context. This requires teachers of mathematics to value complexity rather than simplicity, fuzziness rather than clarity, and to allow students time to grapple with deep mathematical ideas.

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Lessons from research: Students' understanding of statistical literacy*

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This paper reports on the findings of research into the development of students' understanding of statistical literacy over the years of schooling from Grade 3. Statistical literacy at the school level is associated with the application of concepts from the chance and data part of the school curriculum in contexts that expand with students' experiences over the years and that ultimately require critical questioning skills. Various tasks and examples of students' levels of performance are presented to illustrate the development of student understanding. These focus on language, sampling in context, and questioning claims.

Introduction

'Chance and data' has been a component of the mathematics curriculum in most Australian states for well over a decade now. The model curriculum, embodied in *A National Statement on Mathematics for Australian Schools* (Australian Education Council [AEC], 1991) and *Mathematics — A Curriculum Profile for Australian Schools* (AEC, 1994), was written with the advice of statisticians but with virtually no research on student understanding and its development over time. The documents, however, spawned an interest in educational research in the field, particularly in Tasmania, and several projects have been carried out over the years. The earliest research, although including a range of contexts, was reported in terms that reflected relatively closely the curriculum itself (e.g., Watson, 1999). Awareness of the underlying presence ('omnipresence' according to Moore, 1990) of variation in all chance and data activity, led to an expansion of the focus of research to students' appreciation of the importance of variation as they learned about chance and data (Watson, Kelly, Callingham & Shaughnessy, 2003). Some of these outcomes were reported in Watson (2002). A further awareness of the need to assist all students to acquire statistical literacy skills and document the development of these understandings led to further research with an increased interest in the context within which chance and data activity occurred (e.g., Watson & Callingham, 2003). This report briefly introduces a hierarchy to achieve the goals of statistical literacy, suggests a model of the development of statistical literacy understanding, and provides some examples that may be useful for teaching or assessment purposes in the classroom.

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Models

A pathway to achieve the goal of critical thinking in statistical settings can be put forward in a three-tiered hierarchy (Watson, 1997) that has the following sub-goals:

- understanding statistical terminology (Tier 1);
- understanding statistical terminology when it occurs in various contexts (Tier 2);
- questioning statistical claims made in context but without proper justification (Tier 3).

Although it might be attractive to think that terminology is mastered first, then it is mastered in context, and finally critical questioning develops, both common sense and research indicate that there is likely to be much interaction among the tiers. A limited understanding of the term ‘sample’ as a ‘test’ may enable a student to become involved in a context such as blood testing or testing water from a stream. Considering the idea in the context may assist in building a broader conception including the part-whole relationship and the need for the sample to be representative of the whole from which it is taken. It is likely, however, that the need to assess a claim for validity will require a deep understanding of both terminology and context.

Recent research used responses from 4000 students to 80 items designed to cover the statistical literacy hierarchy, to suggest a potential model for the development of statistical literacy. The analysis was based on considering student ability and item difficulty at the same time and producing a scale including both. Analysing the content of items that were increasingly difficult for students helped suggest the characteristics of increasingly sophisticated understanding. As partial credit was given for items in terms of increased structure and/or appropriateness of responses, it would be expected that the increasing sophistication of responses to particular items would reflect the increasing sophistication on the overall scale. This research suggested that there are six levels associated with the developing construct of statistical literacy (Watson & Callingham, 2003). These are summarised in Table 1.

Table 1. Suggested levels of the statistical literacy construct (adapted from Watson & Callingham, 2003).

Level	Brief characterisation of levels
6. <i>Critical Mathematical</i>	Critical, questioning engagement with context, using proportional reasoning particularly in media or chance contexts, showing appreciation of the need for uncertainty in making predictions, and interpreting subtle aspects of language.
5. <i>Critical</i>	Critical, questioning engagement in familiar and unfamiliar contexts that do not involve proportional reasoning, but which do involve appropriate use of terminology, qualitative interpretation of chance, and appreciation of variation.
4. <i>Consistent Non-critical</i>	Appropriate but non-critical engagement with context, multiple aspects of terminology usage, appreciation of variation in chance settings only, and statistical skills associated with the mean, simple probabilities, and graph characteristics.
3. <i>Inconsistent</i>	Selective engagement with context, often in supportive formats, appropriate recognition of conclusions but without justification, and qualitative rather than quantitative use of statistical ideas.
2. <i>Informal</i>	Only colloquial or informal engagement with context often reflecting intuitive non-statistical beliefs, single elements of complex terminology and settings, and basic one-step straightforward table, graph, and chance calculations.
1. <i>Idiosyncratic</i>	Idiosyncratic engagement with context, tautological use of terminology, and basic mathematical skills associated with one-to-one counting and reading cell values in tables.

Looking in detail at the content of items and how they were distributed across the variable construct suggested the increasing sophistication of terminology across all levels from Level 1, the increasing engagement with context from Level 3, and the ability increasingly to criticise claims in context from Level 5. This is shown in Figure 1. These results support the interactive view noted above on how critical thinking develops and suggest that the model should be useful in developing classroom activities and assisting students to move to higher levels in relation to the statistical literacy construct. Examples from each of the three tiers are presented here to illustrate the development taking place.

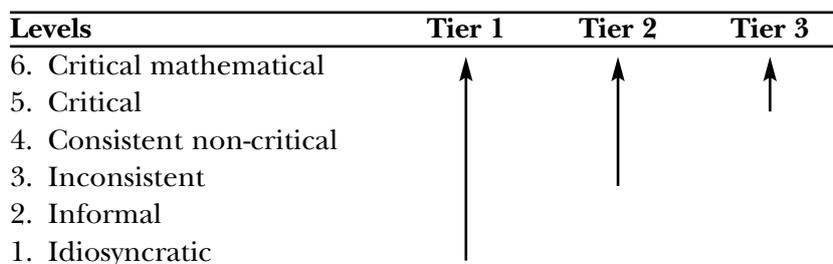


Figure 1. Tiers and levels of statistical literacy.

Terminology

Two of the words explored in relation to the development of statistical literacy were ‘average’ and ‘random’. For the first, students were asked, ‘If someone said you were “average”, what would it mean?’ For the second, the question was, ‘What things happen in a “random” way?’ It might be expected that the question on average would be easier than the one on random and this was indeed the case. The increasing codes for responses for ‘random’ are given in Table 2, as well as the levels at which these codes appeared in the statistical literacy construct. There are two aspects of increasing understanding being observed: one relative to responses to the specific task (code) and one relative to the many questions included in the survey (level).

Table 2. The development of understanding of the term ‘random’ (Watson et al., 2003).

Code	Description	Examples	Level
3	Definition + Example; To pick without any pattern	‘Random means something that does not happen in a pattern. In a Tatts lotto draw...’	6
2	Definition – No order, choose any, unpredictable; Multiple Examples from different aspects below	‘It means in any order. The songs on the CD came out randomly.’	4
1	Example – Natural (Weather), Human design (Breath testing), Game/selection (Tattslotto)	‘Choosing something. Random breath test.’	3
0	Inappropriate (ransom, fighting, everything); Chosen (weak), in order, random numbers/alphabet	‘Very quickly’	–

Although students were likely to be able to give an example of an average (‘I’m normal height’) at Level 2, the description of something that happens in a random way (lottery) was likely at Level 3. Giving a consistent mathematical description of an average (one

equivalent to mean, median or mode) was likely at Level 4, whereas the description of random with examples was not likely until Level 6, where mathematically-based critical skills also appeared. These changes in performance were likely, but not necessarily, seen with increasing grade.

Terminology in context

The contexts within which terminology were set appeared to have an influence on the difficulty level of interpretation by students. Reading and/or summing values from tables that only involved locating information cued by context (such as, ‘How many children were swimmers?’) were the easiest questions. Contexts familiar to students, such as conducting a survey in a school appeared easier than contexts found in media reports based on national or international affairs. Consider the question in Figure 2 based on surveying a school.

MOVIEWORLD
 A class wanted to raise money for their school trip to Movieworld on the Gold Coast. They could raise money by selling raffle tickets for a Nintendo Game system. But before they decided to have a raffle they wanted to estimate how many students in their whole school would buy a ticket. So they decided to do a survey to find out first. The school has 600 students in grades 1-6 with 100 students in each grade.
 How many students would you survey and how would you choose them? Why?

Figure 2. Survey question based on Jacobs (1999).

The question in Figure 2 was intended to explore students’ understanding of sampling in a survey context. Students made many suggestions, which were coded in four categories. These are summarised in Table 3, with examples. Informal suggestions were made at Level 2, whereas students made reasonable suggestions at Level 4 and were likely to combine more than one statistically appropriate idea at Level 3.

Table 3. Development of understanding of sampling in context (Watson et al., 2003).

Code	Description	Examples	Level
3	Representative & random; Random only	‘10 from each grade, 5 boys and 5 girls picked at random.’ Random only: ‘Put all 600 student names in a hat and draw out 65.’	5
2	Based on one or more factors	‘You would survey 60 children, 10 from each grade so you could see an average for each grade.’	4
1	Just the students I meet; take them all	‘50 students that I meet.’ Entire population: ‘You would survey them all.’	2
0	Misinterpretation	‘Choose them all because the more raffle tickets they sell the more money they get.’	–

Critical questioning

Although questioning of statistical claims can take place in any context the most sophisticated level of questioning was likely to occur in less familiar social contexts. Three extensions of the previous question, however, show how critical questioning can be developed out of a straightforward context. Figure 3 contains a series of questions following the question in Figure 2.

Other students in the school conducted surveys.

MVE2.
Shannon got the names of all 600 children in the school and put them in a hat, and then pulled out 60 of them.
What do you think of Shannon's survey?
 GOOD BAD NOT SURE — Why?

MVE5.
Raffi surveyed 60 of his friends.
What do you think of Raffi's survey?
 GOOD BAD NOT SURE — Why?

MVE6.
Claire set up a booth outside of the tuck shop. Anyone who wanted to stop and fill out a survey could. She stopped collecting surveys when she got 60 kids to complete them.
What do you think of Claire's survey?
 GOOD BAD NOT SURE — Why?

Figure 3. Extension survey questions based on Jacobs (1999).

It would be expected that appropriate appraisals of these items would occur at Level 5 and this happened for these three items. A summary of the descriptions of the codes, along with examples and levels are given in Table 4. What is important about the Movieworld items is that they allow students to be given credit for partially appropriate answers and development can be monitored, for example from Level 3 inconsistent responses. In terms of a question that requires critical thinking, it is interesting to note that some of the lower level responses reflect an appreciation of what is happening in the context (Tier 2), without that added critical element.

The final critical questioning example was based on a media report that was considered to present an unfamiliar context (Watson & Chick, 2004). The three graphs in Figure 4 were accompanied by the newspaper headline and the instruction to 'Comment on any unusual features of the graphs.' The general statement was meant to avoid suggesting students look for specific errors. An examination of the graphs shows that the column labelled 95 in the first graph has the number 6 at the top but its height on the scale appears to be 2. The sum of numbers within the graph is not 46, as indicated at the top. Also the numbers in the last graph do not add up to 46.

Table 4. Development of critical questioning in a sampling context (Watson et al., 2003).

Code	Description	Examples	Level
MVE2 Shannon got the names of all 600 children in the school and put them in a hat, and then pulled out 60 of them.			
3	Random methods; range	'Good, because it's a good random way to survey.'	5
2	Fair chance; sample size; methodology (easy)	'Good, there's a lot of people.'	4
1	Method too random, inaccurate; inadequate sample size; unfair; time consumption	'Bad, he could pick the wrong people.'	3
0	Misinterpretation; no reason or logic	'Bad, too many people.'	-
MVE5 Raffi surveyed 60 of his friends.			
3	Lack of range &/or variation	'Bad, they would probably say the same thing.'	5
2	Unfair; vague friendship factor; uncertainty; adequate sample size	'Good, you get a lot of answers.' 'Not sure, it depends how many of his friends have different opinions.'	3
1	Inadequate sample size; 'easy'; good to use friends	'Good, because they are his friends.'	3
0	Misinterpretation; no reason or logic	'Good, more money for them.'	-
MVE6 Claire set up a booth outside of the tuck shop. Anyone who wanted to stop and fill out a survey could. She stopped collecting surveys when she got 60 kids to complete them.			
3	Non-representative	'Bad, some kids might go twice.'	5
2	Uncertainty; adequate sample size	'Good, you just have enough.' 'Not sure, because people who thought it was a bad idea wouldn't bother.'	5
1	Inadequate sample size; fairness; free choice; assuming range and variation; 'easy'	'Good, it is their own choice.'	3
0	Misinterpretation; no reason or logic	'Good, first in best served.'	-

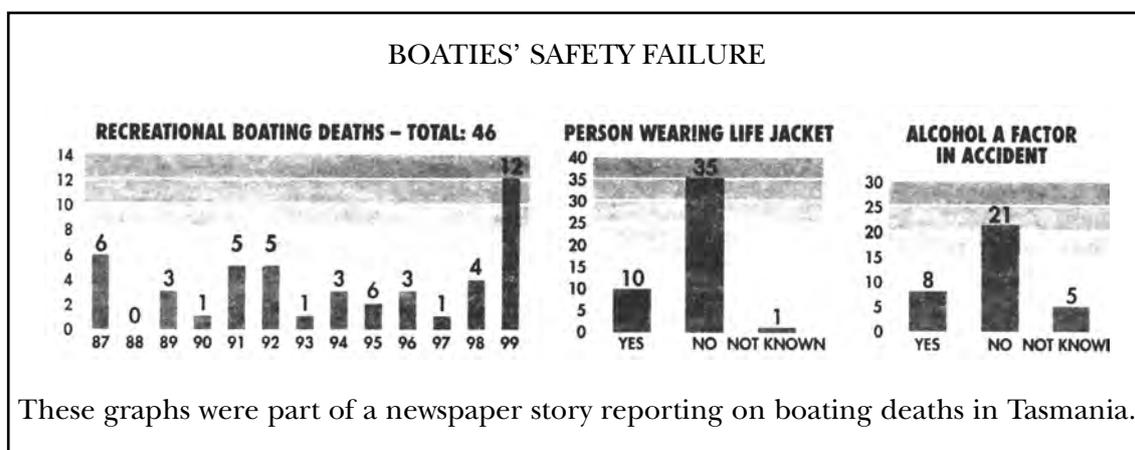


Figure 4. Set of three graphs used in the critical questioning task (Haley, 2000).

In the case of this item there was a large variety of responses, as might be expected from the general nature of the question. These are summarised within four codes in Table 5. Whereas the combined codes 1 and 2 showed that students at Level 3 were likely to be engaged in the task with responses that study the graphs carefully it was not until Level 6 that students were likely to find an error in the graphs.

Table 5. Development of critical graph questioning (adapted from Watson & Chick, 2004).

Code	Description	Examples	Level
3	In-depth graph analysis that recognises mistakes.	'The first graph has a mistake, the 6 is on 2.' 'Well on graph 1 it says there is a total of 46 but I counted and it has only got 38.'	6
2	Correct graph interpretation or comment but not the errors.	'The number of deaths has risen over the years.' 'The way they're set out. They don't have anything telling you what the Y and X axes are.'	3
1	Partially correct interpretation.	'They're all different graphs. They're [sic] all got different meanings.' 'Most people drowned in 1999. A lot of people were tanked [drunk].'	3
0	Statistically inappropriate response.	'They all look okay to me.' 'People should wear life jackets.' 'The graphs show us that boats are just as dangerous as cars are.' 'Hardly anyone wore life jackets in 99.' 'Less people died by not wearing life jackets.'	–

Of students in Grades 7 and 9, only about 3% found an error in the graphs, whereas about 40% considered other unusual features (Code 2). Mathematics teachers are natural sleuths when reading the newspaper but it would appear that students are not. There is a need to make students aware of the need to check and perhaps question every claim made in the media, even if it appears in an authentic-looking graph.

Summary

This paper presented examples of items used in research in order to document student understanding of statistical literacy as tasks become more sophisticated. Whereas a statistician might only be interested in the time when students are likely to achieve the most statistically appropriate level of response, the coding schemes adopted here give indications of progress along the way. Using such schemes in classroom assessment would give teachers a feel for how students are progressing and what understandings might be needed to progress to a higher level response.

It is also important for teachers to be aware of the three goals of statistical literacy: understanding language, understanding language in context, and being able to criticise claims made without justification in any context. It is not sufficient to stop when students can demonstrate the meaning of a term, even if that meaning is quite sophisticated. It is

also necessary to place the term in context and question understanding of the statistics and the context. The fact that partially appropriate responses to all items appear across the construct gives clues to the overall level of student thinking and where help might be given to increase performance. Particularly striking is the presence of increasingly appropriate responses to definitions across the levels of statistical literacy. It would appear that whereas informal understanding of definitions will allow students to begin to engage with other tasks, the ability to provide sophisticated descriptions develops alongside contextual and critical understandings. This may reflect the overall development of literacy skills more generally, including students' ability to express themselves in writing, and items such as these may provide links to research more broadly based across the school curriculum.

It is hoped that these items and their rubrics may provide starting points for teachers to develop their own items to assess the growing understanding of statistical literacy concepts by students in their classrooms.

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Quality pedagogy and teaching in K–12 mathematics*

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Aligning elements of the current New South Wales pedagogy model for quality teaching with a number of community perceptions of school mathematics suggests some areas for concern. By taking a deeper look at the nature of mathematics, this paper argues that mathematics teaching of both early ‘empirical’ and later ‘invented’ mathematics too often has tenuous links to previous knowledge and at best provides superficial applications to life. It is argued that quality teaching at both levels, while having different emphases, should employ a similar approach to ‘engagement’, ‘problematic knowledge’, background knowledge’ and ‘connectedness’, namely teaching through generalisation from known contexts. However, what constitutes meaningful learning varies with the individual, and invented mathematics may be inappropriate for a large number of students

Constructivists’ views of knowledge which emphasise the active construction of knowledge rather than receiving transmitted knowledge have provided the impetus for learning and teaching mathematics now for a number of years. The strong focus on students’ individual construction of their own knowledge in social settings resulted in the teacher’s role being seen primarily as one of facilitator and nurturer. The active role of teaching was in effect de-emphasised. A swing back to acknowledging the importance of teacher actions has been prompted by research which showed that a prime factor behind successful student learning is the quality of teaching (Hill & Rowe, 1998). In New South Wales (NSW), the Department of Education and Training (2003c) argues that recent developments in educational research have shed light on what constitutes quality teaching, and that quality learning comes from quality teaching and those key teacher actions which promote such teaching and learning can be identified and communicated to teachers. The basis then of this focus on quality teaching is effective pedagogy and consequently a new model for pedagogy in New South Wales schools has been established (NSW Department of Education & Training, 2003b).

Pedagogy and quality teaching

The NSW model has as its cornerstone the characteristics of *authentic pedagogy* (Newmann and Associates, 1996). Two key factors in authentic pedagogy are higher-order thinking

* This paper has been accepted by peer review.

and connectedness to the real world. The latter is seen as making learning relevant to students and thus providing them with a curriculum which is authentic for them. The Queensland School Reform Longitudinal Study (2001) expanded the authentic pedagogy model to what it called *productive pedagogy*, adding areas relating to language and problematic knowledge. The Queensland longitudinal study has in turn provided the basis for the New South Wales model, which comprises three dimensions. *Intellectual quality* is the central dimension, with *quality learning environment* and *significance* making up the other two dimensions:

- *intellectual quality* focusses on producing deep understanding of substantive concepts, skills and ideas;
- *quality learning environment* focuses on classrooms where students and teachers work productively in an environment clearly focused on learning;
- *significance* refers to learning which is meaningful to students.

Each dimension has six elements. There is no suggestion that all the elements in the model should be instantaneously embraced. First, the list is extensive and trying to incorporate them all at once would be a daunting task. Second, individual teachers may recognise attributes that they currently use with success. Thirdly, the model is generic and it may be that some aspects are not appropriate to mathematics. So, which elements are not seen as strong aspects of mathematics teaching?

What is not done well?

A first negative community perception of mathematics in school is that students get it wrong, do not like it and so 'hide' in class. Relevant elements in the model are 'problematic knowledge' (intellectual quality) and 'engagement' (quality learning environment). The NSW Department of Education and Training (2003b) defines these two elements as follows:

Problematic knowledge: Students are encouraged to address multiple perspectives and/or solutions and to recognise that knowledge has been constructed and therefore is open to question. Tasks require students to present or analyse alternative perspectives and/or solutions.

Engagement: Most students, most of the time, are seriously engaged in the lesson or assessment activity, rather than going through the motions. Students display sustained interest and attention.

The interpretative nature of the first element promotes mathematical activity as more than finding a mystical answer. The relationship between the second element and avoidance of participation is self evident.

A second community perception relates to the common cry in mathematics' classrooms of 'When are we ever going to use this?'. Relevant elements in the model are 'background knowledge' and 'connectedness' from the significance dimension. The NSW Department of Education and Training (2003b) defines these two elements as follows.

Background knowledge: Lessons regularly and explicitly build from students' background knowledge, in terms of prior school knowledge, as well as other aspects of their personal lives.

Connectedness: Lesson activities rely on the application of school knowledge in real-life contexts or problems, and provide opportunities for students to share their work with audiences beyond the classroom and school.

Both elements involve significance because they can connect to real-life situations. To begin, a closer look at the nature of mathematical ideas is considered.

The nature of mathematics

Booth (1990) describes a paradigm shift in school mathematics, from ‘empirical mathematics’ to ‘invented mathematics’. Empirical mathematics arises from real-life situations and can be explored by contextual investigation. Invented mathematics is developed and extended solely in terms of earlier mathematics. The shift means that the mathematics curriculum cannot be regarded as a single entity, but needs to be examined at the two stages.

Empirical mathematics

Empirical mathematics is the main focus of the early school years, where the relation to concrete experience is clear. However, empirical mathematics does not end there. Topics such as financial mathematics, chance and data, and graphs continue until the end of secondary school.

The study of empirical mathematics consists essentially of two components:

- recognising where a particular mathematical concept arises; and
- learning how to use that mathematics more effectively.

For example, in primary school, students learn that equivalent grouping situations lead to multiplication, and learning ‘times tables’ and other techniques enables them to solve problems more effectively. The same applies to more sophisticated but still empirical ideas such as ratio, angle and rates of change.

- *Ratio*

Partitioning, fair sharing, odds in betting, proportions in making cakes involve a similar type of comparison between like quantities called a ‘ratio’. Learning how to manipulate ratios abstractly enables one to understand a wide variety of discourse in the press and to carry out one’s own calculations.

- *Angle*

Corners, slopes, and turns can be identified in the environment. All involve the inclination between two lines through a point – an ‘angle’. Learning about angles in abstract diagrams enables one to make more accurate constructions and use trigonometry.

- *Rates of change*

Rate of change is significant in motion, population growth, etc. Representing such situations graphically leads to the ideas of gradient, average rate of change, and instantaneous rate of change, all of which assist in the interpretation of real-life situations. Learning about differentiation provides a means to investigate change situations more precisely.

These examples show the power of mathematics for empirical situations: instead of having to investigate a problem situation concretely, a solution can be predicted using standard manipulations of symbols which represent that situation.

Quality teaching of empirical mathematics

The early years

Several recent, successful numeracy initiatives (e.g., Count Me In Too in New South Wales, and the Early Numeracy Research Project in Victoria) have focussed on building concepts on children's own ideas and strategies. Because counting, shape, and length (for example) are so close to the world, teaching is also linked to children's real-life experiences through the use of teddies, fingers, and counters. In general, then, the recommended approach to teaching empirical mathematics in the early years incorporates a strong emphasis on engagement and background knowledge. The fact that children and adults alike see the skills and concepts taught as important supports them as being connected. The interpretation of different empirical situations also supports problematic knowledge.

The later years

Unfortunately, with empirical mathematics at the secondary level, there is strong evidence that many students do not make any connection with the mathematics they are learning and real situations. This disconnection explains many of the difficulties they experience in learning and applying the mathematics. Consider again the three examples.

- **Ratio:** Hart (1982) indicates a weak link between the concept of ratio and real-life multiplicative contexts.
- **Angle:** Many student difficulties arise because the angle diagram does not seem to be easily linked with any real angles. In one study, one third of Year 8 students could not identify angles in slopes and turns (Mitchelmore & White, 2000).
- **Rate of change:** White and Mitchelmore (1996) found that many first-year university students do not see the symbols in calculus as representing anything, so they cannot use the manipulative techniques they have learned to solve contextual problems.

Teaching empirical mathematics without linking it to experience seems to be very common. For example, recently, most Diploma of Education students observed by the author chose to start their lesson with an abstract definition. When asked why, the students referred to how they had been taught themselves, to the resources available to them and to their desire 'not to confuse the students'.

How can the quality elements be used in teaching higher level empirical mathematics? Consider the three examples for a third and final time:

- ratio can be taught by exploring a variety of multiplicative situations, identifying their common features, practising ratio manipulations, and then applying the skills learnt;
- angle teaching can be based on identifying the angular similarities between varieties of real contexts;
- an understanding of rates of change via graphs of real-life situations is now seen by many as fundamental. Materials by Barnes (1992) have been published along these lines, but the approach has not been adopted by most mainstream texts.

The approach suggested here is that students begin with appropriate examples and finish with the abstract definition, not the other way around. Hence, they see how the appropriate mathematical idea is common to all the contexts encompassed by the concept. This approach addresses the four key pedagogy elements because there is a natural 'connection' to situations outside the classroom.

Invented mathematics

A common description of invented mathematics is that it is very abstract, meaning that it is totally removed from reality. The essence of this claim is that mathematics is self-contained.

- Mathematics uses everyday words, but their meaning is defined precisely in relation to other mathematical terms and not by their everyday meaning. The syntax of mathematical argument is also precise and concise, with none of the redundancy common in everyday language.
- Mathematics contains objects that are unique to itself. For example, although everyday language occasionally uses symbols like x and P , objects like x^0 and $\sqrt{-1}$ are unknown outside mathematics.

Self-containment is a crucial feature of invented mathematics. It is the lack of reference to any specific context that makes the mathematics applicable to many different contexts and therefore contributes to its usefulness and power.

Historically, mathematics has become increasingly independent of experience as more systems and structures have been invented. Mathematicians look for completion — ways to apply current ideas and results to higher degrees of generality — by extending them to larger domains. For example, expressions like 15^2 and 2^3 arise in real-world situations involving area and volume. The empirical concept of a power is then applied to expressions for very large and small numbers and to compound interest calculations. To incorporate these ideas into a complete consistent system, however, requires the invention of concepts like zero, negative, rational and irrational powers. At each point in this extension/completion process, it is crucial that the new objects be related to each other and the previous objects in such a way that they can be operated on without any appeal to any external meaning they might have.

If invented mathematics is self-contained and removed from reality, how can it have anything to do with problematic knowledge, background knowledge and connectedness?

Quality teaching of invented mathematics

A large part of invented mathematics consists of rules for operating on mathematical objects and relationships. Some students can learn these ‘rules of the game’ well and gain high marks in examinations. It is, therefore, not surprising that there is often no attempt to apply invented mathematics to anything other than symbolic contexts. Other examples include symbolic algebraic manipulation in calculus; graphing of polynomial, rational and trigonometric functions; and proving geometric theorems and trigonometric identities.

It would be unfair to suggest that teaching in the higher grades never attempts to apply new mathematical knowledge in some way. Such attempts usually fall into one of two categories:

- *artificial exercises*
White and Mitchelmore (1996) report on students’ responses to an exercise involving a cube of volume 64 cm^3 shrinking at a rate of 96 cm^3 per minute. As one student doing this example (who was obviously well connected) observed, the cube was in its last moments of existence. This type of example only requires students to strip away a façade of context and uncover the mathematical exercise underneath.
- *applications in finance, statistics and graphs*
For example, Quality Teaching Program Local Interest Group (2001–03) produced assessment modules to support the implementation of the Stage 6 General Mathematics Syllabus in NSW. In 2001, the modules were line of best fit for age and

height graph, financial mathematics, and data analysis. These topics are essentially empirical mathematics, which is why realistic applications can be easily found.

It would appear that large sections of the senior mathematics syllabus deal with abstract ideas where it is too difficult to find related realistic contexts. Does this mean that invented mathematics should be treated as a special case where many of the key pedagogy elements are not relevant?

Background and problematic knowledge

It is true that self-contained mathematics can only be related to other mathematics, but it is still possible to link learning to something meaningful. Otherwise, there is no case for teaching invented mathematics at all! There are at least three ways to make invented mathematics meaningful.

- An example-based approach similar to that for empirical mathematics. For example, consider the index laws. The sequence 64, 16, 4 can also be written 4^3 , 4^2 , 4^1 , which to be consistent should extend to 4^0 , 4^{-1} ... The similarity between the two sequences provides the basis for zero and negative powers. In this approach, the rationale for completing a mathematical system can also be expounded.
- Mathematicians do not invent mathematics out of thin air — they build on previous work. So, any mathematics has at least a tenuous link back to reality. For example, algebraic manipulation can still be regarded as making generalisations about numbers.
- Linking in the world of mathematics need not be a purely academic affair — it can have a human face. The history of mathematical thought is a rich source of real life stories which give mathematical results a human perspective and provide opportunities to engage in narrative (another pedagogical element).

Connectedness

Topics like graphing complex trigonometric functions or solving n th degree equations never have any direct real-life applications for a student in school. Connectedness, therefore, appears unattainable. However, the definition given by the NSW Department of Education and Training (2003a) suggests a broader interpretation:

Connectedness: Students recognise and explore connections between classroom knowledge and situations outside the classroom in ways that create personal meaning and highlight the significance of the knowledge. This meaning and significance is strong enough to lead students to become involved in an effort to influence an audience beyond the classroom (p. 59).

The key phrase here is ‘situations outside the classroom... that create personal meaning’. There are a number of ways to provide personal meaning beyond the classroom. For example:

- students may see success with algebraic high level mathematics as contributing to their overall competence, their self-esteem as well as their future career prospects;
- challenge can provide personal meaning through involvement in the processes of generalisation and explanation — a dozen undergraduate teacher education students of the author all agreed that the challenge of a problem was a great source of engagement and connectedness;
- the creation of self-contained, consistent and complete systems within the world of mathematics can provide aesthetic satisfaction to some students.

Therefore, that engagement and connectedness can be achieved in invented mathe-

mathematics through qualities unique to the subject.

To summarise, in the ‘invented’ paradigm, quality teaching can foster engagement through the use of background and problematic knowledge, narrative, and connectedness, even if these elements need to be interpreted in a wider sense.

Conclusion

Approaches to teaching both empirical and invented mathematics have been presented which could help engage students by making the content more meaningful to the students. For higher level mathematics using challenge, purpose and narrative may, however, only provide a meaning beyond the classroom for the ‘true believers’. Others may find these suggestions just ‘maths for maths sake’ (as one student put it) and so another source of disconnection. There is also the challenge (for the teacher) of how to explain the purpose of many topics in higher level mathematics. Our conclusion is to advocate empirical mathematics for all, but not necessarily invented mathematics for all.

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The year in which my love of maths changed: Pre-service primary teachers' self-image as mathematicians*

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In this study pre-service teachers reflected on their own school experiences and their views of themselves as learners of mathematics while studying school students' experiences of mathematics. This paper discusses pre-service teachers' written responses to readings on students' learning problems, such as maths anxiety, and shows how their reflections brought about changes in their self-image of themselves as students, and their assessment of their capacity to learn and teach mathematics. They emphasised the potential of individual teachers to have a lasting influence, and showed increasing awareness and exploration of alternatives to the approaches that they experienced.

Introduction

This research investigated the effect of an explicit study of mathematical anxiety and problems faced by school students on pre-service teachers' self image as learners of mathematics. It is hypothesised that reflecting on their own experiences during a study of mathematical learning difficulties may enable pre-service teachers to better identify and deal with them. This reflection may enable them to develop a more positive self-image as learners of mathematics, through enhanced self-awareness — a necessary corequisite to developing a deep and connected knowledge and consequently becoming a more effective teacher of primary-aged children. Participation in the study may also provide students with greater insight into how children's anxiety about mathematics can be minimised by teachers.

Background

The study arose from two parallel streams of research: how teachers' images of themselves as mathematicians impact upon their teaching practices and the factors contributing to anxiety about mathematics among both school aged students and pre-service primary teachers. Research into the effectiveness of primary teachers of mathematics has consistently pointed out the need for effective teachers to have a deep and connected knowledge of mathematics and to have a positive view of themselves as learners of mathematics. Askew et al. (1997), studying effective teachers of numeracy in primary schools,

* This paper has been accepted by peer review.

identified three types of teachers: transmissionist, discovery and connectionist. The study found clear evidence that teachers' own perceptions of mathematics and how it is learned were more important in promoting positive outcomes for students than particular teaching methods or classroom organisational practices. Ma (1999) in a landmark study of the mathematical knowledge of US and Chinese elementary school teachers, identified that Chinese teachers possessed a deep knowledge of simple mathematics that enabled them to effectively address issues that were likely to impact on students' mathematical understanding in the elementary classroom. This deep knowledge is a function not only of mathematical content but also of teachers' views of themselves as learners and doers of mathematics. Hence there is a strong argument that mathematical anxiety among pre-service teachers must be overcome if they are to develop the knowledge necessary to become effective teachers of mathematics.

Mathematics anxiety has been identified as a specific learning difficulty in mathematics for many children (Dossel, 1993). It is characterised by a feeling that mathematics cannot make sense, a feeling of helplessness in the face of mathematics, and an inability to take control of one's own learning. It is suggested that, in many cases, this anxiety can be traced to inappropriate teaching practices, and to a belief in the wider society that 'some people can do maths and some cannot'. There is strong evidence that many pre-service primary or early childhood teachers have a fear of mathematics, and see themselves as unable to learn effectively (Haylock, 2001). Previous research such as that conducted by Trujillo (1999), who carried out in-depth interviews with pre-service primary teachers in the United States, has attempted to trace the roots of mathematics anxiety. However, relatively little has been done to investigate how studying subjects at university might impact upon this anxiety. Where such research has been conducted it has often focused on how subjects that teach mathematical knowledge have assisted students to develop deeper knowledge (Chick, 2002), or on how subjects that deal with mathematics teaching strategies impact on pre-service teachers' beliefs and attitudes (Frid, 2000).

Rationale for the study

This study adopted a different approach. While this study did not examine teacher effectiveness in a school situation, or pre-service teachers' own knowledge of mathematics, it attempted to look at pre-service teachers' images of themselves as learners and doers of mathematics. Enhancing this self-image may contribute to pre-service teachers developing the capacity to see mathematics as making connections, to see learning as developing deep knowledge, to see their role as teachers as being to provide opportunities for school students to solve rich and complex problems, and to adopt a view that all students can learn mathematics (AAMT, 2002). It investigated the effect of an explicit study of mathematical anxiety and problems faced by school students on pre-service teachers' self image in mathematics. It was felt that to focus on the factors impacting on school students' mathematical difficulties, pre-service teachers may better understand how their own school experiences affect their views of themselves as learners of mathematics and potentially bring about a more positive self-image of themselves as students and of their capacity to learn and teach mathematics. The reasoning is that pre-service teachers may be able to identify themselves through the case studies of children.

This process can be compared to the technique of bibliotherapy in which people are assisted in dealing with problems in their lives by reading about similar situations happening to a third person, or in some cases, to an animal. The technique is based on the active, dynamic process of reading, enabling the person to identify with the protagonist in the

story, followed by individual or group discussion in a non-threatening environment (Aiex, 1996).

The setting and methods

The setting for this study was the subject Mathematics and Learning Difficulties offered as part of an inclusive education major at the University of Canberra. The subject looked specifically at difficulties school-aged children experience in mathematics, both as a result of specific learning difficulties and as a result of cultural and attitudinal factors, and examined research conducted into how school children feel about mathematics and about themselves as they learn mathematics. Thirteen pre-service primary teachers — twelve females and one male — were members of the class. At the first session, students were asked to prepare a written description of a critical incident in their own school mathematics education that made a major impact on their image of themselves as learners of mathematics. This could have been a positive or negative experience during their mathematics learning that had meaning for them as an adult. This critical incident reflection was then sealed and stored until the subject had been completed. The responses were not read by the principal researcher until after the end of semester.

During semester as part of the assessment for the subject students were required to keep a log of reflections on readings, personal observations in schools and voluntary further reflections from their own schooling. They were asked to recall and write about incidents that they experienced as a student at school or university that had an impact on how they felt about mathematics. Prompts were provided for journal writing, including:

- ‘Something I learned’
- ‘Something I felt reassured by’
- ‘Something that surprised me’
- ‘Something I disagreed with’
- ‘Something I would like to know more about’.

This was presented as an open-ended task and students were not required to address every prompt. Students could voluntarily submit copies of the weekly journal entries for the research project, to be stored until formal assessment had been completed. In addition they had the option of submitting additional sealed personal reflections which would not form part of the assessment for the course and did not need to satisfy formal assessment requirements. A clear distinction between criteria used for formal assessment in the subject and the use of reflective writing as a research tool was made. After the completion of the unit, the critical incidents and journals were summarised, but not interpreted.

The critical incidents

In the accounts of the critical incidents, several themes predominated. These were the role of the teacher, the cycle of fear, failure and avoidance, the students’ perceptions of the nature of mathematics, their self-image as a learner of mathematics, and, less commonly, the influence of parents. These themes are consistent with five themes identified from ‘mathematics autobiographies’ of seventy-two pre-service teachers training to teach at primary, middle, and secondary level in the USA (Sliva & Roddick, 2001). They found that almost all students mentioned the role of the teacher in the development of their mathematics understanding, placing them on a continuum from ‘enabling’ (patient and understanding, giving full explanations and answers to student questions) to ‘disabling’

(intimidating students, not fully explaining concepts or not considering students' feelings); and many described a trend of fear, failure and then avoidance in their mathematics experiences.

Ellsworth and Buss (2000) collected autobiographies from sixty-one pre-service teachers studying elementary education methods classes in the USA. They defended the validity of using these recollections, even though they admitted they could be biased, on the grounds that it was the way that students recalled situations that influenced their current belief systems, even if their memories were not precise. They identified five themes which included the powerful effect of teachers and three facets of the ways mathematics was presented (relevance, comprehension, and emphasis on skills and memorisation).

The role of the teacher

There are parallels between these themes and those identified by the students in the current study. Students' written critical incidents tended to focus on descriptions of a teacher or a way in which they perceived mathematics was presented as a subject; and their image of themselves as learners of mathematics and the feelings that this invoked. Students retained intense memories of their experiences with 'disabling' teachers. Heather described her teacher writing continuously on the board and not explaining:

I can still see the teacher, I can picture her face as if it were yesterday, I can hear her accent and I can remember vividly the year in which my love of maths changed. The doubts that crept in back then creep in still now when I attempt to learn new mathematical strands.

Barbara described her discomfort during the line up for times tables questions before students were allowed to go out for lunch in Grade 4:

After a number of failures Miss A's facial expressions became unbearable to see. I remember 'putting up a wall' avoiding her looks but still trying to get the right answer in the right time. Miss A eventually took to getting cross and impatient, detected through facial expression, body language and her tone of voice. I shut down, I did not make eye contact. I did not react.

Odette recalled a similar experience at a new school in Grade 3 when children had to stand in front of the class and recite the week's tables:

This absolutely terrified me and I hated the entire experience. To this day I still don't like saying my times tables and I often need to double check that I have a correct answer.

It is worth noting, that even a good experience with an 'enabling' teacher was sometimes not enough to overcome pupils' perceptions about the attitudes of their peers. Patsy wrote:

I felt very slow but this teacher seemed to have endless patience and gave 1 on 1 attention and eventually I succeeded. While I was very happy with my personal success I still felt very slow. I felt all the students at the tables were watching me and thinking I was stupid.

The students showed an awareness of the potent effect that an individual teacher could have on a student, and even at the start of the subject, in the description of the critical incident, some were reflecting on the type of teacher that they aimed to be.

‘I don’t want to be a teacher that I had as a child. I believe I know a lot about the damage that can be done to avoid it.’ (Barbara)

‘I want to be good at teaching maths and ensure positive outcomes for my students.’ (Cathy)

‘I am aware of what happened to me in primary school and there is no way I want to repeat that so I’m [sic] determined to have great maths lessons even though I will have to study beforehand.’ (Felicity)

‘I believe there are more positive ways of encouraging one to learn them [times tables] rather than singling out individual children with such a daunting experience.’ (Odette)

Image of themselves as a learner — fear, failure, and avoidance

The cycle of fear, failure and avoidance described by some students has implications for their self-concept as learners of mathematics. This behaviour reflects the coping mechanisms that some students used in class in situations which they found extremely stressful. The vocabulary of the critical incident descriptions is replete with negative terms: ‘fear’, ‘failure’, ‘not fast enough’, ‘freeze’, ‘not good at rote learning’, ‘struggled with basic things’, ‘believed I could not see what everyone else could’, ‘believed I could not “do” maths’. Several students who classified themselves as ‘confident’, ‘relaxed’ or ‘enthusiastic’ in early primary school described changes that occurred in their beliefs about themselves as learners in the middle years. Even some students who appeared to have achieved some success showed an awareness that they were lacking in understanding: ‘Thankfully the answers were provided at the end of the book so I always looked as if I did well but really I had not an idea,’ (Felicity), and: ‘What amazes me is that I snuck through high school not truly understanding maths but yet passing’ (Noel). The potential of individual teachers to have a lasting influence is indicated by their comments: ‘Yeah. I wished I could say something to those teachers. They still impact on me,’ (Barbara, a mature-age student whose teachers said she did not have the ability to continue at school).

Journal reflections

The writings in the journals included a variety of responses. Some students used the opportunity to reflect on themselves as learners, others reflected on observations from their classroom experience, and did not explicitly connect their observations to their own experience. Almost all the journals included deliberations about how to apply the readings to their future role as teachers

Some reflections showed progress in students’ perceptions of themselves as learners of mathematics: ‘The article helped answer many questions I had about my own experiences of learning maths,’ (Barbara). In some cases, these included development of a deeper understanding of what it means to learn mathematics such as the student who now perceives her grades from school ‘as a reflection of my ability to observe and imitate’, not of her mathematics understanding (Odette), and their awareness that there are alternatives to the approaches that they experienced: ‘Basically I feel a bit cheated — like I got a second rate education,’ (Jenny).

The results of using these readings to assist pre-service teachers’ understanding of their own learning parallels one of the outcomes of the bibliotherapy technique, namely

that the person realises that they are not the only one who has the problem (Aix, 1996). Felicity wrote: 'The biggest thing I have learned this week is that I am really not alone in this anxiety.'

Extensive sections of some journals were devoted to a consideration of the effects of the readings on their intended teaching practices. The comments about teaching fell into two categories. One group of comments talked about the reassurance that pre-service teachers felt when faced with research that concluded that the best teachers were not always those who had performed best in mathematics at school: 'It gives me great comfort to know that although I may not graduate at the top of the mathematics class, this will have no lasting bearing on my ability to teach it,' (Jenny). The second group of comments reflected a determination that negative learning experiences would not be transferred to their students and continue a cycle of negative attitudes beliefs and feelings about mathematics:

This also leads me to my second thought that, for those teachers, who like me, have never believed maths to be their 'thing', there is the distinct possibility that our desire not to let students suffer our fate and to improve on our own childhood experiences in classrooms could well be the factor that makes us the more effective teachers. We are more open to the need for reflective teaching and professional development, and more willing to look for alternate explanations and examples. (Jenny)

The pre-service teachers started to discuss the implications of the readings for themselves as teachers and identify specific strategies that they might adopt in their teaching. Some of the comments detail specific issues such as the need to ensure that students see purpose and make connections in their mathematics learning. Others show specific detailed analysis of applications taken from particular readings, and descriptions of learning tools that they intend to incorporate into their classrooms. In several cases during the eight weeks of the journal reflections the focus of the comments moved from discussions of the self as a learner to later comments which focussed almost exclusively on teaching considerations.

Although comments about the influence of parents and families tended to be less common, a graphic description of 'arguments and intense yelling sessions' when parents helped with homework (Odette) shows the powerful effect of such incidents. Mature age students among the class included some comments about their role as parents of mathematics learners, and the strategies that they observe their children using, adding another dimension to their discussion.

Conclusion

A focus on the factors impacting on school students' mathematical difficulties developed pre-service teachers' understanding of their own school experiences and their views of themselves as learners of mathematics, and brought about changes in their self-image of themselves as students and assessment of their capacity to learn and teach mathematics. The potential of individual teachers to have a lasting influence, and the students' awareness that there are alternatives to the approaches that they experienced, were emphasised in their comments.

Clearly such a study, conducted over a short time in a University environment, cannot reliably predict how these pre-service teachers might convey their feelings about mathematics to students. Thus follow-up longitudinal workplace-based studies investigating the

link between mathematical anxiety of pre-service teachers and their effectiveness as teachers of mathematics to young children will be an important element of further research in the area.

Future research could also investigate the application of the techniques used in the study, such as critical incident analysis and journal writing, possibly in combination with bibliotherapy, to investigate their potential to combat maths anxiety in primary and high school students.

Teaching mathematics well, in an engaging way, to pre-service primary teachers is clearly a key aspect of their education. However an explicit focus on learning difficulties may well be a powerful additional element in addressing some of the well-documented anxiety felt by many pre-service primary teachers.

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Workshop papers and notes

Geometer's Sketchpad: Some basics

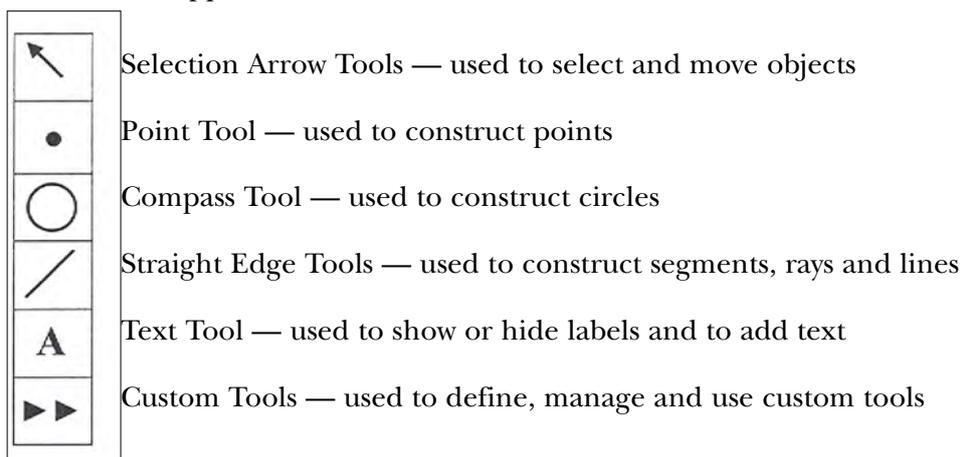
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Features of Geometer's Sketchpad

The Toolbox

The toolbox appears on the left of the screen. It includes six tools.



The Menus

File — allows you to create, save, and print a file

Edit and Display — contain commands that alter the appearance or format of objects

Construct — provides commands for geometric constructions e.g.; construct segment, midpoint, parallel line, or angle bisector

Transform — allows you to apply geometric transform-ations e.g.: translations, rotations, dilations and reflections

Measure — allows measurement of numeric properties of objects e.g.: distance, slope and area. It has a calculator which enables relationships between measurements to be determined

Graph — lets you create coordinate systems and draw graphs

Window — enables you to manage open document windows

Help — provides information on how to use the program

Context — appears when you right-click in a sketch. It presents the options relevant to selected objects.

Using the Construct Menu

Geometrical constructions may be created using the Construct Menu.

To enable any of the commands in this menu you first need to select one or more objects. These are called the selection prerequisites. The commands will only be enabled after the appropriate prerequisites are selected.

If the command you wish to use is unavailable it is likely that you have too few or too many objects selected. Deselect the unwanted objects and try again.

Summary of the selection prerequisites for commands in the Construct Menu

Command:	To use this Command select:
Point on object	One or more path objects
Midpoint	One or more segments
Intersection	Two intersecting objects such as a straight line, circle, or arc
Segment, Ray, Line	Two or more points
Parallel Line	A straight object and one or more points; or a point and one or more straight objects
Perpendicular Line	A straight object and one or more points; or a point and one or more straight objects
Angle Bisector	Three points. The vertex needs to be the second point selected.
Circle by Centre and Point	Two points
Circle by Centre and Radius	A point and a segment or distance measurement
Arc on circle	A circle and two points on that circle; or a centre point and two other points equally distant from the centre point
Arc through 3 points	Three points that do not lie on the same line
Interior	Circle Interior — One or more circles
	Polygon Interior — Three or more points
	Arc Sector Interior — One or more arcs
	Arc Segment Interior — One or more arcs

Using the Measure Menu

To measure an object's properties you need to first select the object and then choose from the available commands in the Measure Menu. If the command you want to use is unavailable it is likely that you have not selected the correct prerequisites. You may have too few or too many objects selected. Deselect the objects and try again.

Summary of the selection prerequisites for commands in the Measure Menu

Command:	To use this Command select:
Length	One or more segments
Distance	Two points, or one point and one straight object
Perimeter	One or more polygon interiors, arc sector interiors, or arc segment interiors
Circumference	One or more circles or circle interiors
Angle	Three points. The vertex needs to be the second point selected.
Area	One or more interiors or circles
Arc Angle	One or more arcs, or a circle and two or three points on the circle
Arc Length	One or more arcs, or a circle and two or three points on the circle
Radius	One or more circles, circle interiors, arcs, or arc interiors
Ratio	Two segments or three collinear points
Calculate	Always enabled
Coordinates	One or more points
Abscissa (x)	One or more points
Ordinate (y)	One or more points
Coordinate Distance	Two points
Slope	One or more straight objects
Equation	One or more lines or circles

Some basic application

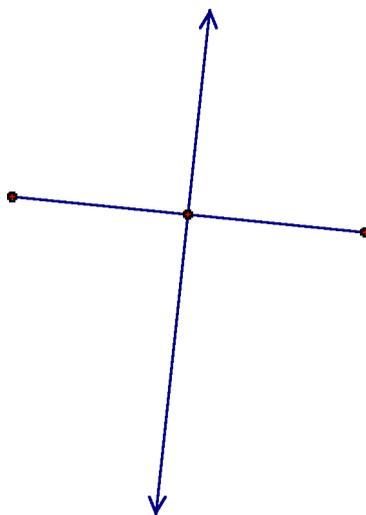
Constructing a Line Segment and its Midpoint

With the Point Tool construct two points in a new sketch. Keep the shift key held down when constructing the points and they will both be selected simultaneously. Go to the Construct Menu and select Segment. With the segment selected, go to the Construct Menu and select Midpoint. Your sketch will appear as follows:



Constructing a Perpendicular Bisector

Select the Arrow Tool and use this to select (highlight) the line segment and the midpoint. Go to the Construct Menu and select Perpendicular Line. The following should appear:



Constructing a Triangle

Select the Point Tool and use this to construct three points in a new sketch. Use the Arrow Tool to click on these points to select them simultaneously. Go to the Construct Menu and select Segments. This will give you a triangle. Now select the Text Tool and label the points with letters by touching each of the vertices.

Measuring an Angle

To measure an angle you need to select the vertices in correct order. The second point selected will be the vertex of the angle you wish to measure. For example, to measure $\angle BAC$, select points in the order B, A, C. The Arrow Tool is used to select the points. Go to the Measure Menu and select Angle. Sketchpad will measure the angle and write it onto the screen. Now measure the other angles in the triangle. Move the vertices of the triangle with the Arrow Tool to see how the angle measurements change.

Using the Calculator

With the Arrow Tool go to the Measure Menu and select Calculate. Click on the angle measurement written at the top left of the screen; e.g., $\angle BAC$. This will write the information to the calculator. Click + on the calculator and select the next angle. Add the third angle in the same way and press OK. Sketchpad should write something like this on the screen: $\mu\angle BAC + \mu\angle ACB + \mu\angle CAB = 180^\circ$.

Move the vertices around to see the angles change. The angle sum remains invariant.

Measuring Distance and Length

Open a new sketch.

Distance between two points

With the Point Tool, construct two points between which you want to measure. Highlight these points simultaneously with the Arrow Tool. Go to the Measure Menu and select Distance. Sketchpad will name the points and write the distance between them on the screen; e.g., $AB = 3.49$ cm.

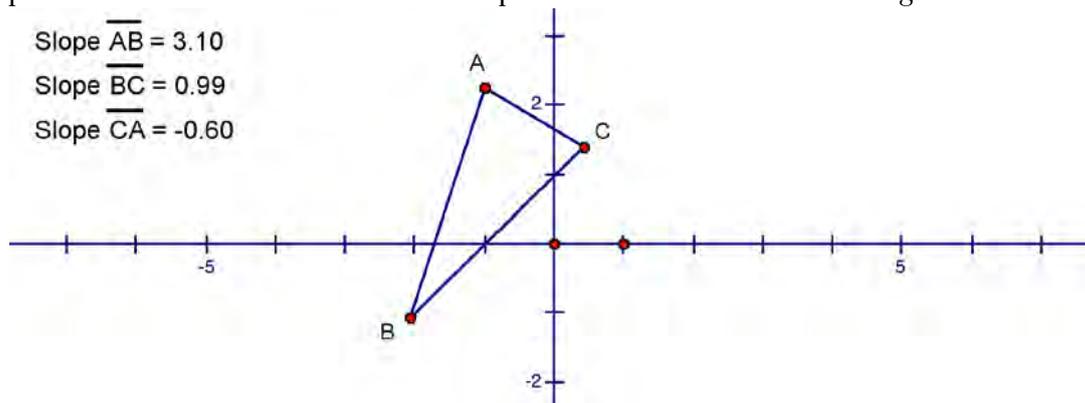
Length of a line segment

Construct a line segment between the two points. Go to the Measure Menu and select Length. Sketchpad will write the length of the line segment on the screen; e.g., $\overline{AB} = 3.49$ cm. Lengths can be measured simultaneously by selecting more than one line segment.

Measuring Slope

To measure the slope or gradient of a line or line segment, you first need to select it. From the Measure Menu, select Slope. Sketchpad will superimpose a coordinate system on the screen and calculate the slope. A number of slopes can be measured simultaneously by selecting them at the same time.

Open a new sketch and measure the slopes of the three sides of a triangle.



Measuring Area and Perimeter of Polygons

Open a new sketch.

Each part describes a different construction.

1. Construct 3 points and keep these highlighted. Go to the Construct Menu and select Triangle Interior. Sketchpad will display the interior. Use the Measure Menu to measure the Area and Perimeter. These quantities will be written on the screen.
2. Moving in a clockwise direction select 4 points with the Point Tool. Go to the Construct Menu and select Quadrilateral Interior. Use the Measure Menu to measure the Area and Perimeter of the quadrilateral.
3. Select 6 points in a haphazard way. Go to the Construct Menu and select Hexagon Interior. What happens? Measure the area and perimeter. With the Arrow Tool move the points around. Observe what happens.

Note that the points do not have to be connected for Sketchpad to construct a polygon interior.

Measuring Area and Circumference of a Circle

Open a new sketch.

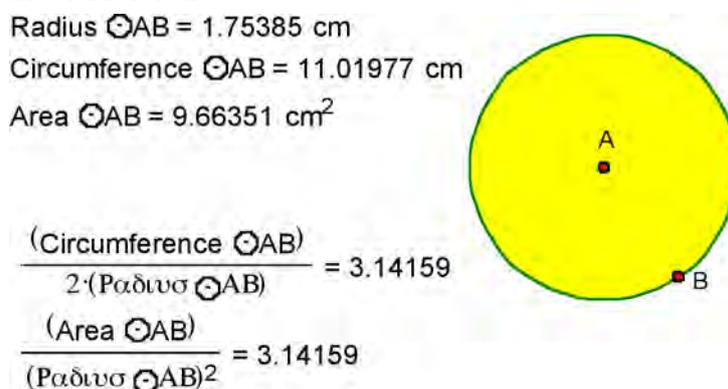
Select the Compass Tool and draw a circle. Go to the Construct Menu and select Circle Interior. Using the Measure Menu, measure its circumference, area and radius.

Use the calculator to determine the following ratios:

1. $\frac{C}{2r}$
2. $\frac{A}{r^2}$

Note that you need to use the brackets on the calculator pad for $C/(2r)$.

The sketch shown below uses 5 decimal place accuracy. The precision can be adjusted in the Edit Menu under Preferences.

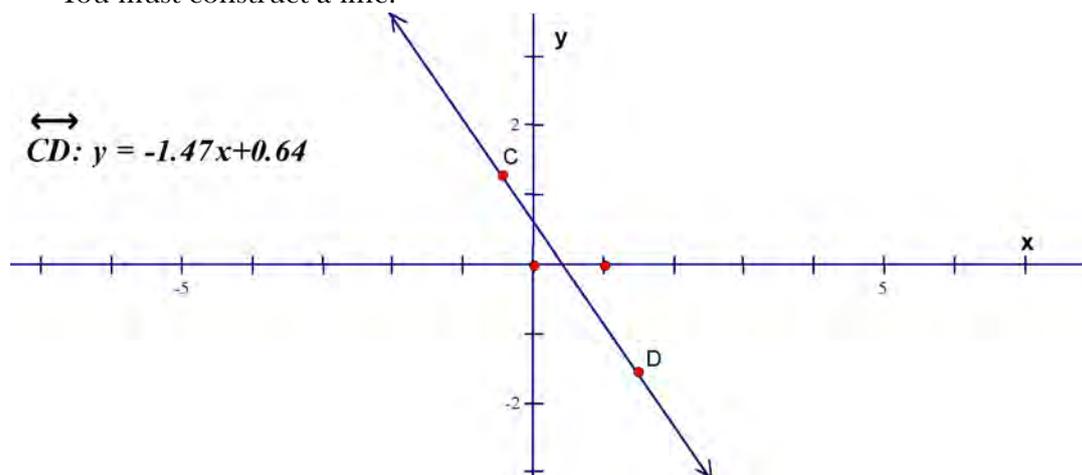


Alter the radius to see that the ratios $\frac{C}{2r}$ and $\frac{A}{r^2}$ are invariant.

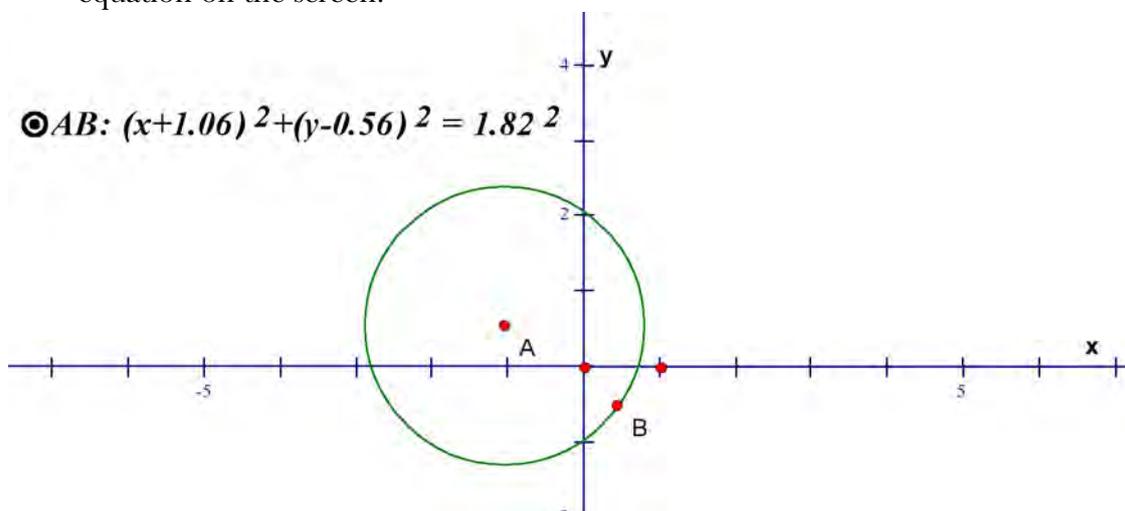
Determining Equations of Lines and Circles

1. Open a new sketch. With the Point Tool construct two points. Using the Construct Menu draw a line between these points. With the line selected, go to the Measure Menu and select Equation. This will superimpose a coordinate system on the screen and write the equation of the line. Move the points to see how the equation changes.

Note that Sketchpad will not be able to determine the equation of a line segment. You must construct a line.



2. In a new sketch, construct a circle with the Compass Tool. Go to the Measure Menu and select Equation. Sketchpad will superimpose a coordinate system and write the equation on the screen.



Using the Graph Menu

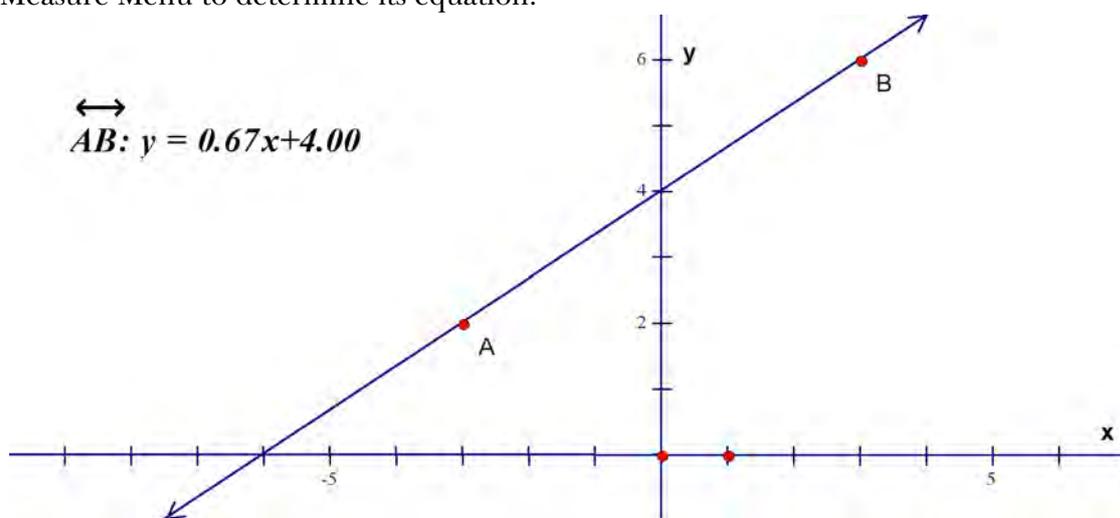
Open a new sketch

Plotting Points

Select Plot Points from the Graph Menu.

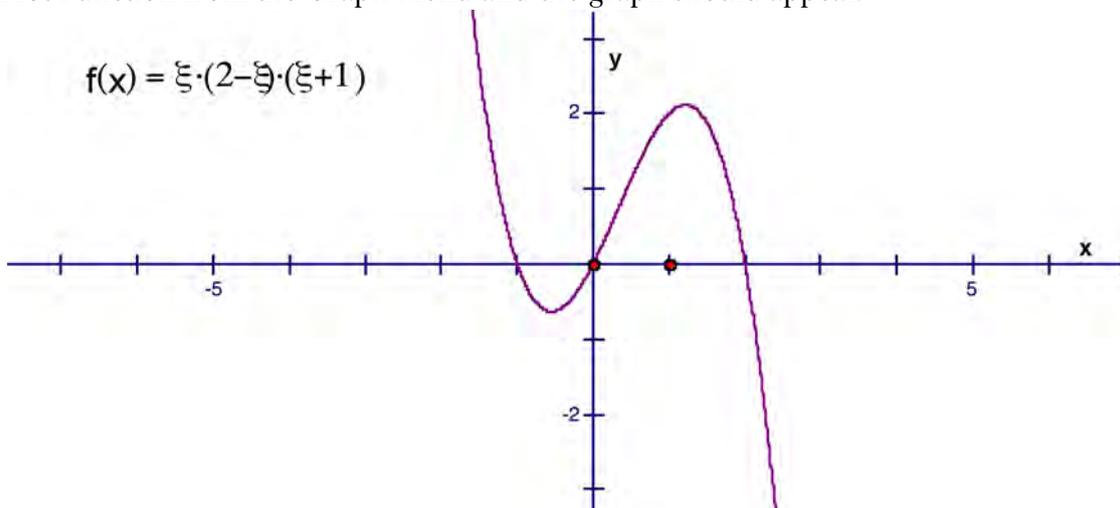
Select Rectangular and enter points (-3, 2) and (3, 6).

Construct a line between these points using the Construct Menu and then use the Measure Menu to determine its equation.



Sketching a Function

From the Graph Menu select New Function. Using the keypad, enter in the box. Press OK. Sketchpad will write the equation onto the screen. Whilst this is highlighted, select Plot Function from the Graph Menu and the graph should appear.



Some geometric constructions

The traditional tools for geometric construction are a point, an unmarked straight edge and compass. In order to construct a figure you need to know its properties and work out a way of creating it so that its properties remain invariant.

Constructing an Isosceles Triangle

Properties:	Two sides are equal length. The angles opposite the equal sides are equal
Method of construction:	Construct a circle. On the circumference of the circle construct two points. Select the centre and the two points and use these to construct a triangle. An isosceles triangle will appear inside the circle.
Hide tools of construction:	As we only want to see the isosceles triangle, we need to hide the circle. This can be done by selecting the circle and using the Hide Circle command in the Display Menu. Alternatively, right click on the circle and select Hide from the Context menu. When the circle disappears, only the isosceles triangle is visible.
Verify properties:	The side lengths and angles of the triangle can be measured. The measurements show that two sides are equal and two angles are equal. These properties remain invariant no matter how the vertices are moved. The triangle is Isosceles.

Complete the information for the following shapes and then construct the figures using Geometer's Sketchpad. After each construction you should verify that the properties of the figure are invariant. When the figure is completed you should hide the construction tools leaving only the required shape visible.

- Equilateral Triangle
- Square
- Rectangle
- Parallelogram
- Rhombus
- Kite

Constructing better mathematics classroom tests

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This paper helps participants walk through the guidelines in crafting the different test formats: short answer and completion, true or false, multiple choice and matching type. It also introduces the participants to the constructed response item format as well as the scoring guide or rubrics for a more subjective appreciation of the students responses. The practice exercises provide the hands-on learning experience intended to enhance their test construction skills.

Although authentic assessment or alternative assessment has caught increasing attention, most educators agree that the traditional paper-and-pencil tests should not be discarded. We need to combine all these tools to provide a more accurate picture of the learners. Writing effective tests is an indispensable task for all classroom teachers. Sound decisions about student performance as well as teaching effectiveness can only be arrived at through sound information gathered from sound assessment procedures such as a well-written test. Despite the fact that teachers undertake test construction year in and year out, research revealed that they need training on how to do the following: plan and write longer tests; write unambiguous test items; and measure skills beyond recall of facts.

Planning your next test

The following suggestions are intended to improve the quality of your next classroom test.

1. *Prepare a test blueprint or table of specifications.*

Invest adequate time for planning so that your test matches the content you are teaching. The blueprint serves as a tool for improving the validity of the interpretation of the results of the test as it describes the content the test should cover and the performance that is expected of the students in relation to the objectives of that content. It also describes the type of thinking skills to be assessed. This will ensure that one writes items that test skills other than recall. As mentioned earlier, research revealed that teacher-made tests emphasise recall of information. It is important for tests to measure higher order thinking skills as well. It also serves as a basis for setting the number of items and for assuring that the test should have the desired emphasis and balance. Figure 1 shows the suggested format of the test blueprint.

2. *Match question type to the level of learning targets desired.*

Well-crafted selected — response and brief constructed — response items do a

Contents/ Objectives	Levels of behavior / No. of Placement of Items			No. of hours taught	Total no. of items	% of items
	KF & P	CU	R			
1. Define and give examples of an arithmetic sequence	1	1		2 $2/54 = .04$	2	4%
2. Tell whether the finite sequence is or is not an arithmetic sequence		1		1 $1/54 = .02$	1	2%
3. Given two terms of an arithmetic sequence, find: the first term, the common difference, or a specified nth term			3	3 $3/54 = .05$	3	6%
...						
Total					50	100%

Legend:

KF & P: Knowledge of Facts & Procedures

CU: Conceptual Understanding

R: Reasoning

Figure 1. Suggested format of test blueprint.

good job of assessing knowledge and simple understanding particularly when students must recognise or remember isolated facts, definitions, concepts and principles. These include multiple-choice questions (MCQ), true/false, matching type, and fill-in-the-blank questions.

Deep understanding and reasoning skills are demonstrated most efficiently in constructed response items; however, MCQ can also be used to assess reasoning depending on the manner by which the item is formulated.

3. *Construct questions carefully.*

The most important process in test-making is selecting the wording of each question. The items should be carefully worded using straightforward language. Careful attention should also be given to writing the distracters or options.

Crafting selected response test items

1. Crafting short answer and completion items

- Word items specifically and clearly so that only one answer is correct.
- Put the blank toward the end of the sentence.
- Avoid copying statements verbatim.
- Omit the important words. A completion item should require a student to respond to important aspects of knowledge and not to trivial words.
- Use one or two blanks only.
- Attend to length and arrangements of blanks.
- Specify the precision you expect in the answer.
- Avoid irrelevant clues.

2. Suggestions for improving true-false items
 - Assess important ideas, rather than trivia, general knowledge, or common sense.
 - Make sure the item is either definitely true or definitely false.
 - Use short statements when possible.
 - Use positive statements and avoid double negatives, which many students find especially confusing. If you must use a negative function word, be sure to underline it or use all capital letters so it is NOT overlooked.
 - Avoid copying sentences verbatim.
 - True and false statements should have approximately the same number of words.
 - Do not present items in a repetitive or easily learned pattern (e.g., TFTF..., TTFFTT..., TFFTF...).
 - Do not use verbal clues (specific determiners) that give away the answer (e.g., always, all, never, absolutely and every tend to make propositions false. Often, usually and frequently tend to make propositions true).
 - Attribute the opinion in a statement to an appropriate source.
 - Focus on one idea.
 - Avoid long sentences.
 - Avoid using vague adjectives and adverbs such as frequent, sometimes, occasionally, typically and usually

3. (a) Crafting basic multiple-choice items
 - Focus item to assess specific learning targets
 - Prepare the stem as a question or problem to be solved.
 - Write a concise correct alternative.
 - Write destructors that are plausible
 - Edit the item to remove irrelevant clues to the correct answer.
- (b) Crafting the stem of the item
 - Write as a direct question.
 - Put alternatives at the end.
 - Control vocabulary and sentence structure.
 - Avoid negatively worded stems.
 - Avoid textbook wording.
 - Create independent items.
 - Definitions go in the alternatives.
- (c) Crafting alternatives or foils
 - Strive for creating 3–5 functional alternatives.
 - All alternatives should be homogenous
 - Put into the stem words or phrases that are repeated in each alternative.
 - Use consistent and correct punctuation in relation to the stem.
 - Arrange alternatives in a list format rather than in tandem.
 - Arrange alternatives in a logical or meaningful order (magnitude or size, degree to which they reflect a given quality, chronologically, or alphabetically).
 - Avoid a collection of true-false alternatives.
 - Avoid using ‘none of the above’.
 - Avoid using ‘all of the above’.
 - Avoid verbal clues.
 - Avoid technical and unfamiliar wording.

4. Crafting basic matching exercises
- Assess only important performance and content.
 - Match tasks to your learning targets and the test blueprint.
 - Make a matching exercise homogeneous.
 - Explain completely the intended basis for matching. Avoid long written directions which place unnecessary premium on reading skill.
 - All responses should serve as a possible option for each premise.
 - Use short list of responses and premises. Put no more than 5 to 15 elements in the response list.
 - Avoid perfect matching.
 - Longer phrases appear in the premise list, shorter phrases in the response list.
 - Arrange the response list in a logical order.
 - Identify premises with numbers and responses with letters.

Evaluating test items

Exercise: Put a G in the space before the items you feel are good, and put a P before the items you feel are poor. Explain your answer. If the item is poor, improve the item.

Completion items

Identify the word(s) being referred to by the following statements.

1. The smaller angle formed by the hands of the clock at 2:30 is _____.
2. _____ is the term given to the abscissa and ordinates of a point.
3. The graph of _____ is a _____ that opens _____.
4. The Indian king who thought that his servant was asking for a modest reward when he requested for one grain of wheat for the first square of the chess board, two grains for the second, four for the third, and so on was _____.
5. The technique of adding a number to both members of an equation to make one side a perfect square is called _____.
6. The circumference of a circle with radius equal to 26.5 cm is _____.
7. At exactly six o'clock, the angle formed by the hands of the clock is a _____.

True-false items

Write T if the statement is true and F if otherwise.

- ___ 1. When rounding-off we always reduce a number to a given place value.
- ___ 2. 7 is greater than 4.
- ___ 3. We usually round off numbers to the nearest whole number.
- ___ 4. $4 + 5 \times 2 = 18$
- ___ 5. Numbers that are not prime do not have more than two factors.
- ___ 6. The value of 6 in the number 2 680 234 is six hundred thousands.
- ___ 7. Lines that never intersect are parallel and those that intersect are perpendicular.

Multiple-choice

Encircle the letter of the correct answer.

1. The diameter of a circle is approximately _____ of its circumference.
A. $\frac{1}{2}$ B. $\frac{1}{4}$ C. $\frac{1}{3}$ D. $\frac{2}{3}$
2. Which of the following is not equal to 1?
A. $7 \times \frac{1}{7}$ B. $\frac{4}{5} \times \frac{5}{4}$ C. $2\frac{1}{2} \times \frac{2}{5}$ D. None of the above
3. Which of the following is not a non-terminating decimal?
A. .12
B. 3.67
C. 0.574
D. all of the above
4. Which is in order from least to greatest?
A. 12497; 12507; 12521
B. 12497; 12521; 12507
C. 12507; 12497; 12521
D. 12521; 12507; 12497
5. A _____ has a whole number and a fractional part.
A. fraction
B. mixed number
C. least common multiple
D. simplest form
6. In the metric system, temperature is measured in degrees _____.
A. Celsius
B. Fahrenheit
C. all of the above
D. none of the above
7. The most reasonable unit for measuring the liquid in a fishbowl is _____.
A. mm
B. L
C. mL
D. kg
8. Mr. Lopez solved the division problem below on the chalkboard.
 $252 \div 7 = 36$
Which of the following could Mr. Lopez use to check his answer?
A. 7×42
B. 36×7
C. 36×252
D. 252×7
9. Of the following, which is not true for all rectangles?
A. The opposite sides are parallel
B. The opposite sides are equal
C. All angles are right angles
D. The diagonals are equal
E. The diagonals are perpendicular

10. Sound travels at approximately 330 metres per second. The sound of an explosion took 28 seconds to reach a person. Which of these is the closest estimate of how far away the person was from the explosion?
- A. 12 000 m
 - B. 9000 m
 - C. 8000 m
 - D. 6000 m
11. Sheila is solving this problem.
- $$(3^2 - 4^2)^2 - ?$$
- Which step is correct in the process of solving the problem?
- A. $(3^2 - 4^4)$
 - B. $(9^2 - 16^2)$
 - C. $(7^2)^2$
 - D. $(9 - 16)^2$

Matching items

Exercise: Match the two columns

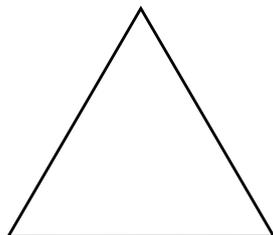
- | | |
|---------------------------------|--|
| ___ 1. composite | A. Numbers with only two different factors |
| ___ 2. equivalent fractions | B. Numbers that name the same amount |
| ___ 3. simplest form | C. The greatest common factor of the numerator and denominator is 1 |
| ___ 4. prime | D. Numbers with more than two different factors |
| ___ 5. least common multiple | E. The smallest number that is a common multiple of each of two numbers |
| ___ 6. greatest common factor | F. The greatest factor a pair of numbers have in common |
| ___ 7. least common denominator | G. Fractions with different denominators |
| ___ 8. unlike fractions | H. You can use the least common multiple to write fractions with this |
| ___ 9. multiples | I. When you find a common denominator for fractions, you change the fraction to this |
| ___ 10. like fractions | J. Denominators such as eighths are ___ of fourths |
| ___ 11. volume | K. The distance around a circle |
| ___ 12. circumference | L. The number of square units needed to cover a surface |
| ___ 13. area | M. The measure of the space inside a solid figure |

Crafting constructed response test items

Constructed-response items for mathematics provide students with an opportunity to

- solve problems and explain their methodology for solving problems;
- communicate mathematical ideas in a variety of ways;
- apply estimation strategies; and
- select and use appropriate technology to enhance mathematical understanding.

Constructed-response items for mathematics will generally allow students an opportunity to show what they know and are able to do in relationship to several different content standards.



1. Find the area of the triangle above.
2. Suppose the base of the triangle above is increased by 10% and the height is decreased by 10%. What is the ratio of the area of the new triangle to that of the original triangle? Show or explain how you found your answer.
3. Consider any triangle with a base, b , and a height, h . Suppose the base of the triangle is increased by 10% and the height is decreased by 10%. Will the ratio of the area of the new triangle to that of the original triangle remain the same for all values of b and h ? Justify your answer mathematically.

Constructed-response scoring guide

Score description

- 4 Student demonstrates a thorough understanding of the area formula for triangles by correctly determining ratios of areas in both a particular and a general case.
- 3 Student demonstrates a general understanding of the area formula for triangles by determining ratios of areas in both a particular and a general case, with only minor errors and/or omissions.
- 2 Student demonstrates a basic understanding of the area formula for triangles by correctly completing or using correct strategies to complete a significant portion of the required tasks.
- 1 Student demonstrates minimal understanding of the area formula for triangles.
- 0 Response is incorrect or contains some correct work that is irrelevant to the skill or concept being measured.

Activities to develop fraction concepts

Richard Evans

Introduction

Currently, fractions are introduced in the early primary grades, with successive grades spending more time on them each year. Depending on the textbook, addition and subtraction of fractions may begin around Grades 3 or 4, with multiplication and division commencing around Grades 5 or 6. Review of operations on fractions continues through Grades 6, 7, and 8. For many students, the heavy emphasis on procedural knowledge (symbolic rules and manipulation) is not built on a strong conceptual knowledge of fractions. Researchers have discovered some of the reasons why children have difficulties with fractions, and suggest that we might do things differently.

We will examine the following questions relating to fractions and decimals.

1. Should models be used when learning about fractions or decimals, and if so, which models should be used?
2. How do we develop a firm conceptual knowledge of fractions and decimals and when do we introduce symbols?
3. What kinds of activities should be done before computation with fractions and decimals in order to develop 'fraction and decimal sense?'

Modeling fractions

Researchers from the Rational Number Project (Cramer & Henry, 2002) devised a curriculum based upon four beliefs:

1. children learn best when actively involved with multiple concrete models;
2. in order for students to develop mental images needed to think about fractions conceptually, they need to use concrete models over extended periods of time;
3. students need opportunities to discuss fraction ideas as they construct their own understanding of fractions as numbers; and
4. students need to develop conceptual knowledge of fractions prior to formal work with symbols and algorithms.

Three commonly used models are the region or area model, the length or linear model, and set models.

Region (area)

In this model, a surface or region is subdivided into smaller congruent parts. Usually, the regions subdivided are ones students are familiar with. However, there are some very good reasons for using specific models, such as rectangles. Other models and aids which could be used are Fraction Bars, Cuisenaire Rods, Pattern Blocks, graph paper, Base Ten Blocks, and paper strips. These same activities can be used to develop an understanding of decimals and percentage.

Activity 1

Divide the rectangle and circle shown in Figure 1 into fifths and discuss with your partners the advantages and disadvantages of using with a rectangle versus a circle.

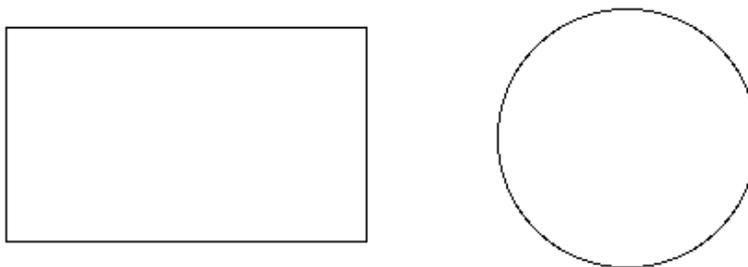


Figure 1

Length or measurement models

This is similar to the area model, except that the length of the object is compared instead of the area. Models for this include number lines, string, and rulers.

Activity 2

Take a piece of rope and three clothes pegs. Have two students hold the ends of the rope and have three volunteers place the clothes pins where they think $\frac{2}{5}$ or some other fraction would be located if the ends of the rope are zero and one. After placing the three clothes pins, have the class vote on which of the three they think is closest to the actual location of $\frac{2}{5}$. Now fold the rope over from one end to the first of the $\frac{2}{5}$ markers. The length of the fold is $\frac{1}{5}$ according to the first person. Keep folding counting the fifths until you get to five. Do it for the other two estimates as well. (Students can do this at their desks with strips of paper and they can change the target fraction to $\frac{5}{3}$ with the endpoints being zero and four or one and four, etc.)

Set models

In this model, the whole is actually a set of objects and subsets of the whole make up the fractional parts. Any set of counters can be used for set models. The set model may well be the most common representation of fractions in everyday life.

Activity 3

Draw a set model illustrating $\frac{3}{5}$.

Beginning activities for developing fractional concepts

Show students pictures of sets of fractions (Figure 2) and ask them which ones represent fourths.

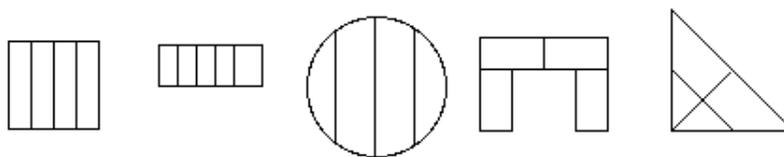


Figure 2

Which of the following models represents one-third (Figure 3)? If a model does not represent one-third, explain why it does not.

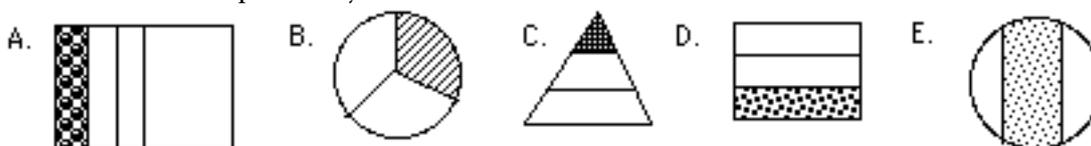


Figure 3

Activities where students count fractional parts (or decimals) are good exercises. Students need to realise that $\frac{3}{5}$ is a representation of having three of the pieces called fifths. Similarly, eight-tenths is a set of eight things called tenths. Students should be shown how four-fourths can be seen as one whole and that ten-fourths is two wholes and half of another. Counting fractional pieces and comparing them helps students understand why five-ninths is less than three-fourths. We want students to think of $\frac{3}{5}$ as three $\frac{1}{5}$ pieces.

Activity 5

Use a calculator to count fractional parts. What does the calculator read when adding one-fourth to three-fourths?

Activity 6

Guess my number: Each of you will be given a piece of paper to stick on someone's back. That piece of paper will have a fraction or a decimal written on it. Your job is to guess the fraction or decimal on your back by asking 'yes or no' questions. You may ask a person at most two questions, then you must go ask another person. This activity is used to develop fraction and decimal vocabulary.

Mixed numbers and improper fractions

Mixed numbers and improper fractions are topics that can be natural outcomes from counting fractions. They should be taught right along with other fractions, and not as something different. Rules for changing from mixed numbers to improper fractions should be delayed until students discover it. Students should be given experiences where they change things like $3 \frac{5}{8}$ to $2 \frac{13}{8}$, $1 \frac{21}{8}$, and $\frac{29}{8}$. This is the natural progression from a concrete approach. This is also very helpful in doing operations later on such as $4 \frac{1}{3} - 1 \frac{3}{5}$. We really change $4 \frac{1}{3}$ to $3 \frac{4}{3}$ and eventually to $\frac{30}{3}$.

Activity 7

$4 \frac{3}{5} = 3 \frac{8}{5} = 2 \frac{13}{5} = 1 \frac{18}{5} = \frac{23}{5}$ What happened as you decreased the unit by one each time?

Part/whole activities

As much as possible, these activities should always be done at first using physical models and not diagrams. Be sure your questions can be answered with the models given. Problems in which unit fractions (fractions whose numerators are one) are usually the easiest ones to work with. The mixed number questions or improper fractions are the most difficult for students to understand. Students should be given opportunities to discuss among themselves and try to convince one another they are correct.

These same kinds of activities can be done with decimals and percentages.

Whole-to-part activities

Give students the whole and ask them to divide it into some fractional part. This gets back to our original activity where a rectangle represents one whole and you are asked to find two-thirds or three-fourths. You could also use the other models to do the same kind of activity.

Activity 8

If the counters in Figure 4 represent one unit, what would represent two-thirds?



Figure 4

Activity 9

Solve the following problems using Cuisenaire Rods.

- If the brown Cuisenaire Rod represents one whole, what rod would equal three-fourths?
- If the blue rod is one, what rod would equal two-thirds?
- If the black rod is one, what rod would be one-half?

These same activities could be done using other models, such as Pattern Blocks.

Part-to-whole activities

Give the students some fractional part and ask them what the whole might look like. In essence, we are turning the previous problem around.

Activity 10

If the rectangle in Figure 5 represents $3/5$, what is the whole.



Figure 5

Activity 11

Solve the following problems using Cuisenaire Rods.

- If the purple rod is two-thirds, what rod represents the whole?
- If the dark green rod is three-fourths, what rod represents the whole?
- If the brown rod is $4/3$, what rod is one?

Part-to-part activities

These activities are similar to the part-whole activities above. The difference is that instead of arriving at one, we want to obtain another fraction.

Activity 12

Solve the following problems using Cuisenaire Rods.

- (a) If the purple rod is $\frac{1}{2}$, what is $\frac{3}{4}$?
- (b) If the blue rod is $1\frac{1}{2}$, what is $\frac{2}{3}$?
- (c) If the rectangle in Figure 6 is $\frac{2}{3}$, what is $\frac{1}{2}$?
- (d) If the rectangle in Figure 6 is $\frac{4}{3}$, what is $\frac{1}{2}$?

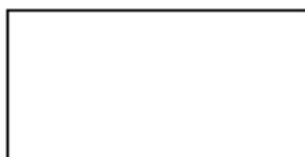


Figure 6

Activity 13

Solve the following problems.

- (a) If the diagram in Figure 7 represents 0.3, what represents one? One half?



Figure 7

- (b) If the brown rod represents 40% or 0.4, what could represent 10% or 0.1?

Ordering, comparing and equivalent fractions

Comparing unit fractions

Students often carry over their whole number concepts to fractions. Thus, they may believe that $\frac{1}{8}$ is greater than $\frac{1}{5}$, because $8 > 5$. Having students compare fractional pieces from the same sized whole is important. They need to come to the realisation that the larger the number on the bottom, the smaller the fractional part. Students will need experiences doing this.

Activity 14

Put '>', '<' or '=' in the blank space below and explain your thinking.

- (a) $\frac{1}{5}$ $\frac{1}{8}$
- (b) $\frac{1}{10}$ $\frac{1}{7}$
- (c) $\frac{2}{3}$ $\frac{2}{6}$
- (d) $\frac{3}{9}$ $\frac{3}{5}$

Close to zero, half, or one?

Students need experiences with fractions so that they can readily know that $1/25$ is close to zero, $4/9$ is close to one-half, and $9/10$ is close to one.

Activity 15

Tell whether the following fraction is closer to 0, $1/2$ or 1 and explain your reasoning.

- (a) $2/12$
- (b) $1/3$
- (c) $2/9$
- (d) $7/9$
- (e) $6/5$
- (f) $3/4$

Activity 16

Put '>', '<' or '=' in the blank space below and explain your thinking by relating the fractions to zero, half or one.

- (a) $4/5$ $7/8$
- (b) $5/11$ $4/7$
- (c) $6/11$ $5/9$
- (d) $2/9$ $1/5$

Estimating fractions

Students should be shown portions of figures which have been shaded and be asked to estimate a fraction that might be used to represent it. Look at the exercises below.

Activity 17

Name a fraction which would be close to the shaded portions in each diagram in Figure 9.



Figure 8

Activity 18

The chocolate bar activity. In this activity six chocolate bars are placed in three rows. One row has three bars, another row has two bars and the third row only has one bar. One-by-one students enter the room and stand next to the row of candy bars they think will give them the most to eat at that given instant. When they are done, questions are asked to help them develop their fractional understandings. The number of chocolate bars and the number of rows can vary.

Equivalent fractions

The procedure for identifying equivalent fractions, namely multiplying or dividing the numerator and denominator by the same number, leaves a lot to be desired. The process is very procedural and not very conceptual. Students need to discover those rules on their own.

The rectangular area model is a rich and powerful model to use when working with

equivalent fractions. Examine the rectangles in Figure 9. You could divide the first rectangle into sixths in two different ways. Students should recognise they need twice as many pieces to get sixths as thirds. You could divide each of the thirds in half to get sixths or you could divide the rectangle in half horizontally. Each yields that $\frac{2}{3}$ equals $\frac{4}{6}$. Students should be given lots of experiences in doing activities like this by either folding pieces of paper or by drawing rectangles and cutting them vertically and horizontally.

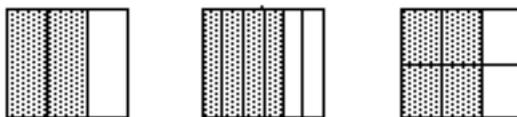


Figure 9

Students should discuss how we got six pieces by dividing each of the thirds in half obtaining twice as many shaded regions. Collecting data and organising it can help students conjecture that multiplying the numerator and denominator by the same number will result in equivalent fractions.

Activity 19

Paper folding is a good way to explore equivalent fractions. Pose some questions which will develop conceptual ideas about equivalent fractions.

- Take two strips of paper the same length and fold them to illustrate whether $\frac{2}{3}$ is greater, less than, or equal to $\frac{3}{4}$.
- Use paper folding to show how many eighths is $\frac{3}{4}$.
- Use paper folding to show how many ninths is $\frac{2}{3}$.

Simplifying fractions

When students change the fraction $\frac{6}{8}$ to $\frac{3}{4}$, it is often called ‘reducing the fraction.’ Unfortunately, the term ‘reducing fractions’ is a poor name for this process. Fractions are not reduced, they are merely simplified or renamed. We would highly recommend using the term ‘simplify’ the fraction or ‘rename’ the fraction when working with equivalent fractions. Using the idea of the multiplicative identity element is probably best left for the seventh and eighth grades, when students are beginning to explore algebraic concepts. Many students when asked what they did to change $\frac{3}{4}$ to $\frac{6}{8}$ reply by saying I multiplied by two. They do not recognise that they used the identity element for multiplication.

Comparing fractions

Usually students are taught rules for comparing or ordering fractions (e.g., cross multiply), and usually these rules are not based on any conceptual understanding. Instead, students should think about the fractions and try to justify their conclusions based on conceptual understanding.

- More of the same-size part: this is the case where students are comparing fractions such as $\frac{2}{6}$ and $\frac{4}{6}$. Since the numerator is counting the same size pieces, and $2 < 4$, then $\frac{2}{6} < \frac{4}{6}$.
- Same number of parts, but the parts are different size: this is the case when comparing $\frac{3}{4}$ and $\frac{3}{6}$. Since fourths are greater than sixths and we have the same number of each, $\frac{3}{4} > \frac{3}{6}$.
- Greater than or less than one half or one: this would be the case where students compare two fractions such as $\frac{3}{7}$ and $\frac{5}{9}$ or $\frac{7}{8}$ and $\frac{4}{3}$. Since $\frac{3}{7} < \frac{1}{2}$ and $\frac{5}{9} > \frac{1}{2}$, then $\frac{3}{7} < \frac{5}{9}$. Similarly, since $\frac{7}{8} < 1$ and $1 < \frac{4}{3}$, then $\frac{7}{8} < \frac{4}{3}$.

- (d) Closer to one-half or one whole: students should be able to compare $\frac{4}{5}$ and $\frac{9}{10}$. Since each is just one piece away from one and $\frac{1}{10}$ is less than $\frac{1}{5}$, therefore $\frac{9}{10}$ is closer to one and is the greater fraction.
- (e) Changing to an equivalent fraction to apply the above strategies: students could be asked which is greater $\frac{6}{9}$ or $\frac{3}{5}$. Since $\frac{3}{5}$ is equivalent to $\frac{6}{10}$ and ninths are greater than tenths, $\frac{6}{9} > \frac{3}{5}$. We use the idea of ‘common numerators’.

Same number of parts away from zero, half, or one?

For example $\frac{4}{5}$ is less than $\frac{7}{8}$, since they both are missing one part to equal one whole. Since fifths are greater than eighths, $\frac{7}{8} > \frac{4}{5}$.

Activity 20

Which is greater: $\frac{5}{9}$ or $\frac{7}{13}$? Explain.

Activity 21

Order the fractions below from least to greatest using the strategies above. Be able to explain your reasoning.

$\frac{2}{3}$, $\frac{4}{7}$, $\frac{3}{8}$, $\frac{4}{5}$, $\frac{2}{9}$, $\frac{5}{4}$

Reference

Cramer, K. & Henry, A. (2002). Using manipulative models to build number sense for addition of fractions. In B. Litwiller & G. Bright (Eds), *Making Sense of Fractions, Ratios, and Proportions* (pp. 41–48). Reston, VA.: NCTM.

Changing the focus of computation instruction in primary schools: Putting the research into practice

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Computation is recognised as an aspect of mathematics education in need of a change in focus. Shifting the emphasis from teaching the traditional written algorithms to mental computation based on strategies and number sense has been advocated by researchers across the world and has been adopted in syllabus documents across Australia. As yet there has been little widespread change evident at the classroom level. This paper outlines attempts by the author to enact change in the whole staff of two primary schools in Queensland.

Why change computational focus?

Recent curriculum documents in Australia and overseas the US *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000), the Dutch *Proeve van een Nationaal Programma* (Treffers, DeMoor & Feijs, 1989), and the Australian *National Statement on Mathematics for Australian Schools* (Australian Education Council, 1991) have indicated that mathematics education needs to change emphasis to match the developments in the world today. For school mathematics to be useful, it needs to reflect the computational techniques used in everyday life (AEC, 1991; Clarke, 2003; Irons, 2001; Willis, 1990). Where computation, as part of school mathematics, continue to be oriented towards paper and pencil techniques (McIntosh, 1990, 2002, Willis, 1990), those outside the classroom are predominantly mental (Carraher, Carraher & Schliemann 1987; Northcote & McIntosh, 1999). Kilpatrick, Swafford and Findell (2001) specified that the goal of mathematics instruction should be mathematical proficiency which has been defined by Schoenfeld (1992) as needing to involve conceptual understanding, computational fluency, strategic mathematical thinking and a productive disposition.

Teaching computation

The traditional didactical approach to teaching computation, where the teacher directly instructs the whole class in the procedures of the traditional written algorithms followed by pencil and paper practice exercises: all set out, usually in a text book and often expected to be completed in silence, is quite comfortable for many classroom teachers of mathematics. Generally, this is how they were taught themselves. They like the routine and it is quite a straightforward method of instruction. To change to a focus on mental

computation requires a change to having students discuss and share strategies and methods of solution, is unsettling to many teachers. Becker and Selter (1997) stated, 'teaching is no longer seen as treatment and learning as the effect. Learners are people who actively construct mathematics' (p. 511). Mental computation can play an important role in constructing understandings in mathematics, stimulating not only conceptual understanding and procedural proficiency but also number sense and the understanding of number relations (McIntosh, Reys, & Reys, 1992).

Brownell (1935) stated that mathematics instruction could not be significantly improved by turning away from more complex methods of instruction because teachers are not sufficiently prepared to implement them. This statement was made a significant time ago and unfortunately this factor is having an impact on the potential change in focus of computation in primary schools today. If there is to be a shift in computational instruction focus teachers will need to adopt teaching strategies more in line with the development of conceptual understanding than computational procedures.

Supporting the classroom teacher

All over the world calls for reforming school mathematics are being heard. These calls are urgent, but not new (Becker & Selter, 1997). Selter (1997) stated that desired reform 'cannot happen simply by setting the scene. Practising teachers need to know how to deal with the subject matter in a way differing from the so called traditional one' (p. 55). The teachers' knowledge base has to consist of domain specific and topic specific background knowledge, but also of specific knowledge about instructional activities and material to support learning.

Clarke and Peter (1993) stated that it has been clear for a number of years that, for many teachers, transforming their teaching entails changes in their beliefs, changes in their knowledge of mathematics and the processes by which children learn mathematics, and changes in their own instructional practice. They saw the process by which teachers change their practices and their knowledge and beliefs as fundamentally a learning process. Teachers need pedagogical support, mathematical content support, practised models and resources and methods for utilising informal written methods.

Pedagogical support

The draft Queensland Years 1–10 Mathematics syllabus (Queensland Studies Authority, 2003) includes a stronger support for mental computation in schools. Through access to professional development teachers believe that a shift in focus to mental computation is appropriate. However, they would like support to put this into practice. When working with teachers who agree with the need for the shift, questions arise that include:

- What teaching approaches work best for the development of mental computation strategies?
- Is there a list of mental computation strategies to teach or look for in student work?
- Should teachers teach the strategies to students or should students invent their own strategies for mental computation?
- Is there a sequence for the development of mental computation strategies?
- Is there a set of descriptions for mental computation strategies to facilitate focussed discussions?
- What models (resources) could teachers use to aid the students' understanding of

mental computation?

- Does the traditional written algorithm have a place in primary mathematics at all if the focus is on mental computation?

To change the focus on computation instruction in primary school teachers are going to need help to view the teaching of mathematics differently. The classroom activity needs to have a focus on discussion where the sharing of strategies is common and expected. Students should be encouraged to be flexible in their thinking, with spontaneous thinking being encouraged. Students should be encouraged to reflect on their strategies and methods used. The shift away from the traditional teaching methods for computation is going to be difficult for teachers as the new Queensland Years 1–10 mathematics syllabus (QSA, 2003) will likely be implemented with little or no professional development or support materials in terms of computational instruction.

Mathematical content support

Successful mental computation requires accuracy, flexible use of a range of strategies and affective traits like persistence and confidence. Teachers often have a very limited personal, conscious repertoire of mental computation strategies themselves and as such are hesitant to take on the content (or intent) of the new syllabus. Knowing what strategies exist and having time to use these and to rate the usefulness of them will influence teaching and learning in classrooms.

Categorisation of strategies

A variety of researchers (Beishuizen, 1993; Cooper, Heirdsfield & Irons, 1996; McIntosh, 1990; Reys, Reys, Nohda & Emori, 1995) have attempted to categorise mental computation strategies. These categorisations have been developed for the study of students' strategies when being interviewed by researchers, i.e., for analysis.

A proposed categorisation of mental computation strategies was trialled by the author with teachers and students to provide a common language (see Table 1). While it was found to be useful for the teachers a major stumbling block in seeing this categorisation utilised in the classroom was the lack of time available for the teachers to really understand the strategies and to recognise them. Some class groups developed their own names for some strategies which served to personalise them but this understanding remained localised in these classrooms.

Table 1. Proposed categorisation of computation strategies.

Strategy	Description
Counting on/back	Counting forwards or backwards in steps
Doubling/halving	Applying knowledge of multiples and factors and their relationship
Breaking up numbers	Splitting numbers into manageable parts
Adjusting and compensating	Changing one or more numbers and then compensating for the adjustment
Using place value	Applying knowledge of place value to simplify a computation
Using compatible numbers	Choosing and using combinations of numbers that 'go together'

Within each strategy there were many variations and teachers were presented with examples of these and references to resources to support each of the strategies. The intended aim was for the document to form a reference for teachers to support their teaching and fostering of mental computation strategies throughout classroom activities. Teachers found the resource useful and were amazed at the possible strategies. They did need time to familiarise themselves with the strategies before they felt comfortable to make major changes to their practice.

Models and resources

Currently the most common model to support the teaching and learning of computation in Queensland primary schools is multi attribute (MAB) base-ten blocks. Miura and Okamoto (2003) found that children can be taught to use the blocks to make canonical (tens and ones) constructions, but this does not necessarily indicate a change in their cognitive representation of number. Instead, it appeared that children might simply acquire proficiency in using blocks to construct numbers. Once the children received instruction on using the base ten blocks, they almost always made canonical construction to represent numbers. However, this did not automatically lead to an understanding of place value. For students to successfully develop flexible strategies for mental computation, multiple representations of number are useful.

Resnick (1982), criticised the use of base-ten blocks for the teaching of computation because the materials provided a strong conceptual but weak procedural representation of operations on numbers. The base-ten blocks not only provide only one (canonical) representation but also lead to very procedural methods of computation — usually traditional written algorithms.

Classroom models recommended throughout the professional development program used by the author to support mental computation include number boards and number lines. Number boards are generally in the form of hundred boards where the numbers 1-100 are represented on a ten by ten grid. The benefit of this model is that students can utilise the format to easily add and subtract tens by moving up or down rows and add or subtract ones by moving right or left in a row. This reinforces mental computation in that natural usage is to do the larger numbers first (the tens) as opposed to the traditional written algorithm which has students work from the ones to the tens.

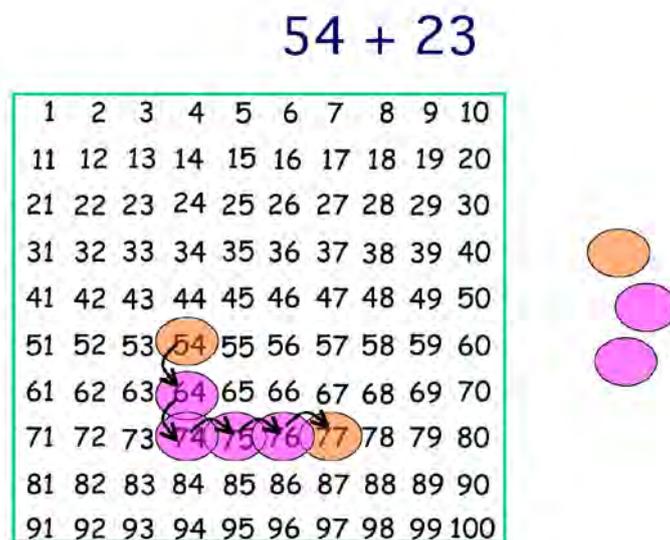
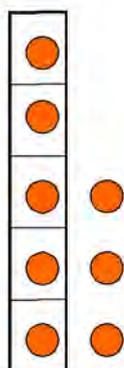


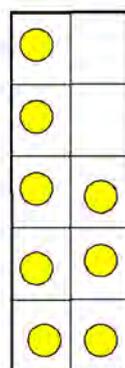
Figure 1

Number lines take many forms from number frames which have spaces for numbers or physical objects to go in representing the number as well as indicating addition and subtraction utilising benchmarks of 5 and 10 depending on the particular frame being used.

Fives frames : **Tens frames**



8 is 5 and
three more



8 is 2 less
than 10

Figure 2

Number lines have been included as part of the curriculum in the Netherlands (Beishuizen, 1997). A variety of forms of number lines has been reported in the United Kingdom where they referred to number tracks, numbered lines, unnumbered lines and then empty number lines. Empty number lines ‘serve both basic number operations and flexible mental strategies’ (Beishuizen, 1997, p. 18).

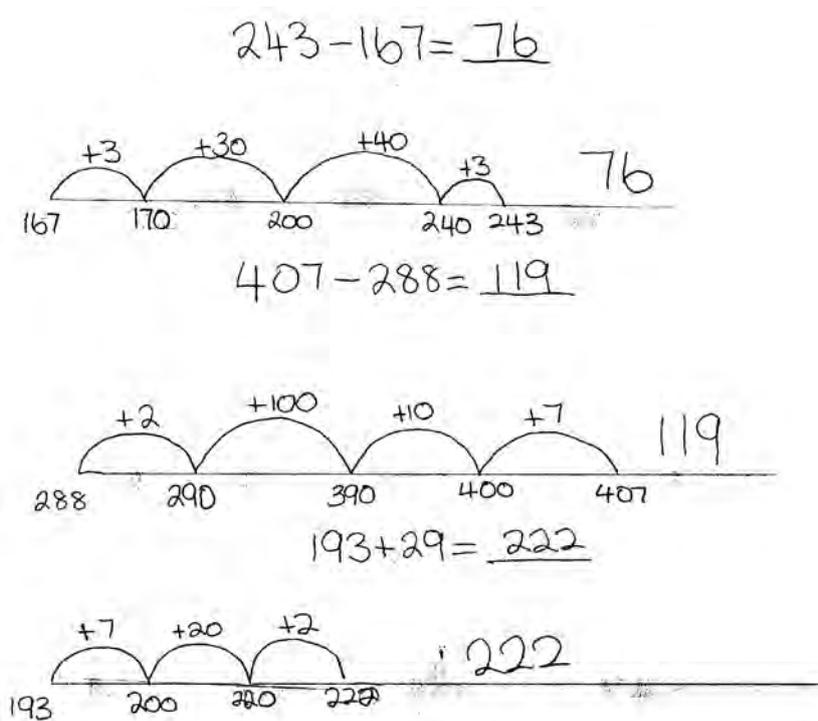


Figure 3

Klein, Beishuizen and Treffers (1998) discussed many reasons for the adoption of the empty number line. The empty number line is well suited to link up with informal solution procedures because of its linear character and lack of set representations of numbers. It also provides the opportunity to raise the level of students' activity to give students freedom to develop more sophisticated strategies. The empty format stimulates a mental representation of numbers and number operations and seems well suited to the representation and solution of non-standard context or word problems. Students using the empty number line were cognitively involved in their actions. In contrast students who used materials such as base-ten blocks or the hundred square sometimes tend to depend primarily on visualisation, which results in a passive 'reading off' behaviour rather than cognitive involvement in the actions. Students using empty number lines also keep track of what they are doing, leading to a reduction of the memory load while solving a problem.

Teachers in Queensland are very heavy users of classroom textbooks for supporting the teaching of mathematics. There are very limited commercial resources readily available to support a strategy based mental computation focus. Most available resources have a focus on the instant recall of answers to computations and are marketed as maths mental books. These books are often used by teachers to set homework for the students. By introducing a focus on mental computation and the use of models like number boards and number lines teachers have begun to see other options – however these have been seen as extra activities rather than as replacements for the focus on traditional algorithms and teaching models to replace current classroom resources like MAB base ten blocks.

Informal written methods

What is the teacher to do when children are faced with [mental computations] where the digits are difficult to hold in the head while calculating? Should the teacher ignore practices and understandings related to mental computation of smaller numbers which the child has acquired and teach the standard formal written algorithm? Or should the teacher build on practices and understandings related to the mental computation of smaller numbers and help the child develop extensions of these practices? (McIntosh, 2002, p. 377)

A shift to mental computation is often seen as requiring the removal of students using any pencil and paper to compute. Thompson (1999) discussed partial written methods used in the United Kingdom, referred to in the Numeracy Project framework as *jottings*. He saw a sensible extension of these jottings to develop informal written methods, and then move to more formal strategies, culminating in the adoption of standard algorithms by those who understood them and/or wish to make use of them. McIntosh (2002) reported teachers using this approach by following a six-stage developmental sequence as a basis for developing informal written computation (see Table 2).

Table 2. Development of informal written computation from mental computation (McIntosh 2002).

Stage	Process
1	Strengthen children's mental computation with two digit numbers
2	Encourage children to explain their mental methods using paper and pencil
3	If method is 'sound', conference and refine recorded explanation
4	Strengthen this method on examples of a similar difficulty
5	Extend its use to more difficult calculations
6	Consolidate it as an understood, secure written method

Research is continuing on non-standard recordings, particularly in the Netherlands. The important aspect for teachers to see is that a focus on mental computation strategies is possible, and is in fact preferable, while utilising written methods. The focus is not to remove pencil and paper from the classroom but to change the focus of computation away from learned procedures including the traditional written algorithms which have for so long taken up time in Queensland schools (and beyond).

Putting the research into practice

As discussed above, when teachers have been introduced to the possibilities of a higher inclusion of mental computation into maths programs they see the value in the shift of focus. What is lacking is the teachers' own pedagogical and mathematical knowledge and knowledge of strategies to enable them to lead the students through activities to support mental computation skill development. Having support in the form of a consultant/researcher in the classroom taking the lead and modelling lessons using strategies and also by having a reference document with examples of possible strategies has shown the beginnings of change as discussed in this paper. The teachers so far, have not felt confident enough to make a big change in their general practice when the consultant/researcher is not present.

It is a belief of the author that mental computation can replace the traditional written algorithms as the main computational instruction focus in primary schools if the teachers are supported in terms of knowledge and skills for implementation. This idea will be trialled during 2005 in a teaching experiment with the proposed outcomes being the students' increased accuracy, flexibility of strategy choice, and attitudes; and the teachers' confidence, increased personal repertoire of strategies and positive beliefs and attitudes about mental computation. The teachers will be supported in terms of pedagogy and mathematical content and strategies by a researcher in the classroom leading lessons and activities. The students will be exposed to a variety of computational strategies, models and methods of recording. The researcher will document the program of learning throughout the study. It is proposed that it will be possible to teach students alternative computation strategies and to see flexibility in strategy choice and recording methods in the students' standard repertoire by the end of a year. This study is planned to guide a wider shift in focus for schools with support to implement the new Qld Maths syllabus.

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Algebra revisited*

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Poor concepts of the symbols used in algebra contribute to students' difficulties. Some concerns include the understanding of the addition sign, the equals sign and the variety of meanings for the pronumeral x . Following a discussion of student understanding of algebraic concepts, some activities are suggested which foster discussion around some of the 'big ideas' of algebra and have the potential to make the concepts of algebra explicit.

Introduction

The mention of the word algebra often brings a negative reaction from the listener. Many adults comment that mathematics was 'okay' until they started algebra. It then became hard and sometimes they add that they subsequently failed mathematics. I have heard teachers comment that when the word algebra was mentioned it was like a chilly wind blew through the classroom. The perception seems to be that algebra is difficult.

Why is it that algebra causes so many difficulties for children learning mathematics? Many children seem to 'hit a brick wall' in their mathematics learning early in Years 7 and 8 and this is usually attributed to algebra. Recently there has been a lot of international attention on early algebra being introduced in the first few years of school. Changes in curriculum in many places have included algebraic development right from the start of schooling and this should make a difference but these changes will take some years to filter through and affect students at Years 7 and 8.

Understanding of the operation symbols

So what are some of the causes of these difficulties with learning algebra? One of the difficulties is that although one aspect of algebra is *generalised arithmetic*, the signs and symbols in algebra are not exactly the same as they are understood by many students in arithmetic. For example, when some students see $5 + 7$ they immediately recognise the $+$ as a sign to combine the two numbers and give the response 12. Once the number is seen as 12 the original components such as the $+$ sign are no longer visible and a single number, 12, replaces the expression $5 + 7$. In algebra however, $a + 7$ is different: the plus sign does not mean 'combine the two parts to make a single number in the same way' as

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it did for the arithmetic expression (Chalouh & Herscovics, 1988). The expression $a + 7$ can be considered as a single object made by combining the two components a and 7 but these components maintain their identity within the object.

Many children try a variety of ways to combine the separate components. Most teachers of Years 7 and 8 have seen expressions such as $4x + 3$ simplified by students to $7x$. Some students will have learned the procedure for simplifying expressions and can use it to simplify quite complex expressions but then add an extra step to write the final expression as a single term, thus eliminating the plus sign. Students who leave the expression as $4x + 3$ without trying to combine the parts are said to have ‘acceptance of lack of closure’ (Collis, 1975). This understanding is a critical part of algebraic development.

Understanding of ‘=’

Another aspect of differences between symbol use in arithmetic and algebra is the equals (=) sign. Freudenthal (1983) claimed four different categories of meaning for the equals sign:

- the result of sum;
- quantitative sameness;
- a statement that something is true for all values of the variable (identity); and
- a statement that assigns a value to a new variable.

A full understanding of the equals sign as it is used in algebra requires all of these meanings. However, for many students = is the sign that indicates the need to do something — an operator sign — or to move to the next step; or even as an indicator of where to write the answer — a syntactical indicator (Carpenter & Levi, 2000) — so they will record incorrectly using equals signs. I saw this demonstrated in a classroom recently where students were solving a problem. The question concerned how many legs there were with two lions, four cubs and four storks. After the sharing time at the end of the lesson, the display shown below was on the blackboard.

$$\begin{aligned} 2 \times 4 &= 8 + 4 \times 4 \\ &= 8 + 16 = 24 + 4 \times 2 \\ &= 24 + 8 = 32 \end{aligned}$$

This misuse of the equals sign during the solution to a problem is also common among secondary school students who use = as a sign to do the next step in solving an equation; for example, $\cos A = 0.5 = 60^\circ$. Other students will use the equals sign at the start of a row, so it becomes the sign for the next line of a solution even if the task is solving an equation as in the example below.

$$\begin{aligned} 3x - 4 &= 2x + 5 \\ &= 3x - 4 - 2x = 5 \\ x - 4 &= 5 \\ &= x = 9 \end{aligned}$$

The meanings of these symbol components of arithmetic and algebra need to be made explicit for the students. The new curriculum in Queensland addresses this by including the algebraic structure as part of early understanding of mathematics and recognising that this structure underlies both arithmetic and algebra. In other places also, the primary school curriculum has recognised the need for improved understanding of the equals

sign. However, for Year 7 and 8, students who have not had these experiences and who are still operating with the idea that the equals sign is an indication of where to write the answer, there needs to be discussion about these issues so that the reasons for using the symbols in a particular way are based on understanding rather than ‘because the teacher tells me I have to set it out that way.’

Understanding of the pronumeral

Yet another group of errors arise because of lack of explicit explanation for the different uses of the pronumeral. Here is a list of equations where pronumerals can be considered to have different meanings.

1. $x + 2 = 5$
2. $3x + 4 = 15$
3. $x(x - 2) - 15 = 0$
4. $a(x + b) = ax + ab$
5. $2x + 3x = 5x$
6. $y = 2x - 4$
7. $4x + 3y = 12$
8. $y = mx + c$
9. $A = l \times w$

Sometimes the x represents a number which is known, and sometimes it represents an unknown number. Sometimes it represents one number and sometimes many. On some occasions the pronumeral is a variable, and on others, a constant. A more complete list of possible meanings for the ‘ x ’ is given here.

- a specific known number
- a specific unknown number
- more than one specific number
- any number
- (any object)
- a variable which may be dependent or independent
- a constant
- a quantity that can be measured
- a quantity that can be calculated

In the first of the equations above, most students look at it and know immediately that $x = 3$. In this situation x is not an unknown. The equation is transparent. Many students thus find it difficult to understand why the textbooks use complicated algorithms to ‘solve’ such equations. The methods of solution given make much more sense when applied to the second of the equations as nearly all students would need a formal method of solution. The third equation not only requires a method of solution but yields more than one value for the pronumeral.

The distributive law, which is the fourth equation, is an identity which is always true for all possible values of the pronumerals involved and relates to Freundenthal’s third category of meaning for the equals sign. In the fifth equation, the x is not restricted to pronumerals or algebraic objects made up of pronumerals, but indeed could be any object. This is the root of what has become to be known as ‘fruit salad algebra,’ based on using the letter to represent an object often starting with that letter so $3a + 4a = 7a$ is read with the a being ‘apples’ rather than the desired understanding at this level of a representing a number.

The understanding of x which relates to functions and relations is as a variable, and is represented in equations six to eight. It also relates to Freudenthal's fourth category for the understanding of the equals sign. As an independent variable, x does not just represent any number but rather all numbers in the possible domain. For students there is a difference between an expression written in the assignation form such as equation 6 and the linear relation represented in equation 7. Equation 8 also raises the idea of constants and variables. I remember being puzzled over this distinction for years as a student in high school and at university.

In the final equation, the l and the w represent the length and width of a rectangle and as such in the student's eyes are known quantities because they are easily measured. The A is different because it is not measured directly but is rather calculated. This difference in understanding explains why students who otherwise can solve an equation like $3x = 21$, have difficulty finding the length when the area is 21 cm^2 and the width is 3 cm (Usiskin, 1988).

Students often adopt one meaning for the pronumerals and do not attend to others. A classic situation arose when a teacher was returning a test to a Year 8 student. The student complained that he had been unfairly treated as the teacher had marked the question wrong when it was correct. The teacher looked at the linear equation, for which the student had the answer 39 and explained to the student that 14 was the correct value as it made the equation correct. The student responded in frustration, 'But all last year you told us that x could be any number and so what is wrong with 39?' The symbol x has many different meanings which are rarely if ever made explicit and this can contribute to students' misunderstandings. These multiple meanings need to become part of the classroom conversation.

One key aspect of algebra is its use in generalisation. The student above has over generalised the meaning of the x but on other occasions we want students to generalise. Algebra has often been described as generalised arithmetic, and part of it is the abstraction from specifics in arithmetic to general underlying structures.

Difficulties in this abstraction process often occur because students may focus on inappropriate generalisations and interpretations, as well as obstructions caused by semantics and alternative approaches to semantics deduced from the 'concrete' situation. For example, given a simple one- or two-step linear equation in early algebra, students will often solve it by a guess and check method in spite of the teacher presenting a different approach. This is reinforced by success in the problems in the Year 7/8 textbooks, and becomes entrenched but does not lead to further understanding and allow transfer to more difficult situations. Similarly in arithmetic, the equality symbol is often seen as a signal to perform operations, but this is a limited conception and causes an obstacle in algebra. Left to their own devices without direction, students are unlikely to develop the semantics of algebra as we know them because the types of experiences they have are limited and often lead to alternative representations which do not then relate to other situations. Another example of students developing entrenched but non-productive understanding is with students using a backtracking method. They might record it happily as $5x + 3 = 8 = 5 = 1$ and all students involved at the time understand what this means, but it is a misuse or different meaning of the symbols and will limit future development. This means clear guidance is needed to assist the students to construct knowledge and use mathematical language and sign systems that are compatible with the language and sign systems of others.

Backtracking causes a further obstacle. Students are often shown how they can solve fairly complex equations with one occurrence of the variable on the left hand side of an equation and a single number on the right hand side. They practise this skill and become

adept at using it. This often leads to a strong reluctance to relinquish it when in the following years they meet equations for which backtracking cannot be used, and thus handicaps their further development in algebra.

Algebra sense

Students need to develop a sense of algebra. What do I mean by algebra sense? Algebra sense is an understanding of the objects of algebra and the different representations as well as the ability to sense the form of the result of a particular process (Horne & Maurer, 1998). It is the ability to visualise the nature and form of the solution and to move readily between the representations or mathematical sign systems rather than the ability to work with the objects to produce the required solutions, although of course producing solutions is also necessary in developing algebra. In many ways, algebra sense corresponds to number sense, though algebraic experiences are not as much a part of the students' world as numerical experiences.

A critical part of developing algebra sense is encouraging discussion where the use of language and student explanation can assist them in their developing understanding. The few activities below are designed to allow all students to participate in developing mental algebraic skills and more particularly to make sense of algebra. The key part of these activities is the ensuing discussion in which the issues can be made explicit and the big ideas of algebra be raised. Part of the focus is on some of these key principles of algebra. For example, the first activity focusses on the approaches students use in solving equations. The idea is to enable the students to share their ideas about how to solve equations. The ensuing discussion should also raise the issue of when different methods are useful and efficient and the difference between 'arithmetic' linear equations which can be solved using backtracking methods and 'algebraic' equations which have more than one occurrence of the x . Filloy and Sutherland (1996) call this separation between what they see as arithmetic and algebraic, the didactic cut.

Activity 1

Which of these equations

- A. are easy to solve in your head?
- B. could you solve in your head but it requires extra thinking?
- C. would you prefer to use a pen and paper to solve?

1. $2x + 5 = 9$
2. $3x - 4 = x + 2$
3. $4x + 3 = 12$
4. $5 = 2x + 1$
5. $3x - 8 = 5x + 2$
6. $6x - 5 = 3x + 2$
7. $3(x - 4) = x + 2$
8. $2(x + 5) = 9$
9. $5x - 2 = 9$
10. $(11x + 5)/3 - 4 + 2 \times 3 = 11$

Another question to ask then is what different methods could be used to solve these equations and which is the most efficient method for each question? We know many students use guess and test even though teachers have often tried to insist on the students setting out their solutions to equations by using a balance method. For this activity the key focus for the students could be which methods are most suitable for which equations. Instead of the question above about doing the problems in their heads, the question might be:

For which of these equations could you use

- A. guess and test;
- B. backtracking;
- C. the balance method; or
- D. other?

The activity can also involve group discussion before the whole class has a sharing time. Allow the students some time of individual work to decide on their answers then have them share their strategies in groups of 3 or 4. The question did not actually ask for the solutions to the problems but in the discussion about the strategies the solutions will arise. Following a time of group discussion, the key approaches can be discussed with the whole class with the students also suggesting how to decide on the best method to use each time. The other key point that will arise is that there is not one best method. While the balance method always works and is often the taught algorithm, it is not the most efficient method for an equation like $20/(2x + 3) = 4$ or $32/(3x + 1) = 4$. The focus of this question was solution of equations. Activity 2 is also focusing on solving equations.

Activity 2

Write down five different equations that have a solution of $x = 3.5$.

The approaches that students use to do this task can be shared with the class. In order to elicit a variety of answers from the students, criteria can be added such as at least one of the equations has to have an x on each side of the equals sign.

The activities used can be from any aspect of algebra. The critical aspect is that they are fairly open and encourage the students to share and discuss meaning. Activity 3 is an open task that focusses on equivalent expressions and raises the whole issue of simplification.

Activity 3

Ask the students to write down three different expressions equivalent to $2x + 3$. Collect verbal answers from all students (the teacher acting just as scribe), arranging them in up to five different groups on the board as students give their answers to you. The answers should be recorded on the board with no corrections. It is up to the students to discuss any discrepancies. The groups might be

- those which change the order of terms or insert symbols: e.g., $3 + 2x$, $x \times 2 + 3$;
- those in which the number term is changed: e.g., $2x + 6 - 3$, $2x + 1 + 10/5$;
- those in which the coefficient of x is altered or a series of x terms are added or subtracted: e.g., $8x/4 + 3$; $2x + 3 + x$;
- those which are a combination of the last two groups;
- a miscellaneous group which may include changes to the x : e.g., $x^2 + x + 3$.

If too many answers are coming in for any of the first three groups, ask them to try to change some other aspect of the expression. The students also will need to check that they agree with each recorded expression. When answers have been collected from the whole class, the students can explain why you have grouped them in the way you have by explaining the common aspects of each group and the differences between them. Of course with older students expressions can be with different powers. There should be class discussion about how we know the expressions are equivalent and students should try to explain how they arrived at their answers. Another way to do this is to put up the expression and focus the nature of the student answers by specific questions while still leaving them partly open. For example:

- Write down an expression with no 3 in it.
- Write down an expression with no 2 in it.
- Write down an expression with a $-$ sign.
- Write down an expression that begins with a negative number.
- Write down an expression with a fraction in it.
- Write down an expression with a b in it.

One of the early rules students suggest is often to change the order so the negative raises that question. Students often think $a - b$ is the same as $b - a$. Rather than immediately correcting the students who suggest that the order does not matter, follow up by using the same task but with the starting point $2x - 6$, or some other similar expression. As part of the discussion one of the questions becomes, 'How do you know when two expressions are equivalent?' Another key issue to raise in the discussion is which of the expressions is simplest. For many students $x + x + 1 + 1 + 1$ is the simplest as it shows the basic meaning.

Activity 4

Write down an ordered pair which satisfies the equation $2x + 3y = 6$.

An important part of all these activities is the discussion which ensues. Students should explain how they arrived at their answers and discuss the relative ease of using different types of numbers and approaches.

Try it again with $y = x^2 + 3$. Did strategies change for this problem and if so why?

Concluding comments

These activities and the associated discussions are an attempt to engender in students a sense of algebra. Estimation and number sense are acknowledged as critical to our teaching. An important part of the introduction of ordinary calculators in schools is the corresponding emphasis on estimation skills as students develop the number sense necessary in tandem with calculator skills. Symbolic manipulators (computer/calculator algebra systems) are to algebra as ordinary calculators are to number, although there is one important difference. Students are continually meeting number and measurement in a variety of ways in the world around them and in their out-of-school experiences. A corresponding algebraic world experience is not as accessible. Algebra provides a language, notation and procedures that enable problems from the world to be more easily and efficiently solved. The rarity of this experience in everyday life means we must be extra careful to include experiences that can support the development of algebraic estimation skills and assist in the development of algebra sense. Our approach to teaching algebra has to allow for a variety of approaches. Efficient mental methods are not always the same

as written algorithms and change more with the components of the question rather than with the nature of the question. Number sense plays an important part in this. How will the corresponding algebra sense be developed? We will need to change our teaching programs to include approaches which will build algebra sense.

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Numeracy development: What it looks like in the classroom

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The interpretation of the terms 'numeracy', 'real-world problems' and 'real-life problems' has varied considerably. Consequently, the ways in which numeracy is developed in the classroom has also varied. In this paper, various views of numeracy are examined and a different model for numeracy development is put forward. The introduction of the model is supported by examples from classroom based action research.

What is numeracy?

Seeking a definition

The theme for this conference is 'making mathematics vital'. Considering what this paper purports, it is worth examining the *Macquarie Dictionary's* definition of the word 'vital':

of or pertaining to life; having remarkable energy, enthusiasm, vivacity; necessary to life; necessary to the existence, continuance, or well-being of something; indispensable; essential, and, of critical importance.

(Macquarie Dictionary, 1990)

The stance taken here is that the development of numeracy ranks equally in importance with the development of literacy, and that the above definition of 'vital' encompasses the significance of numeracy development in our schools. During the last thirty years, the term 'numeracy' has come to mean many different things to different people. The Australian Association of Mathematics Teachers (AAMT), in its report of the 1997 Numeracy Education Strategy Development Conference, put forward the following definition of numeracy:

To be numerate is to use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life.

In school education, numeracy is a fundamental component of learning, performance, discourse and critique across all areas of the curriculum. It involves the disposition to use, in context, a combination of:

- underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic);

- mathematical thinking and strategies;
- general thinking skills; and
- grounded appreciation of context. (p. 15)

This view contrasts with the ‘traditional’ view of numeracy, which equates it with efficiency in computation. Fuson (2003), Devlin (2000) and Steen (2001) also took issue with the traditional and comparatively narrow view, noting that numeracy, or ‘quantitative literacy’, is concerned with promoting greater understanding and application of mathematical ideas. Steen further clarified the distinction between mathematics and numeracy; the former being abstract, organised in categories mostly inherited from the past, and mostly applicable to academia; the latter being practical, concrete and contextual, focusing on the way knowledge is used. ‘Whereas mathematics asks students to rise above context, quantitative literacy is anchored in the messy contexts of real life. Truly, today’s students need both mathematics and numeracy’ (Steen, 2001, p. 1).

Some models of numeracy development

The particular model of numeracy teaching that underpins the exploration of issues in this paper draws heavily on models put forward by Watson, Willis and Van den Heuvel Panhuizen. Watson’s views on ‘statistical literacy’ have been well documented (Watson, 1995, 1997, 2004) and underline the importance of maintaining a strong focus on problem solving as part of numeracy development. Watson (2004) noted recent attention being given to the term ‘quantitative literacy’ and compared Steen’s (2001) definition of quantitative literacy to the definition of numeracy generated by the Australian Association of Mathematics Teachers (1997) which adopted many of the ideas put forward by Willis (1992), who stated:

Being numerate, at the very least, is having the competence and disposition to use mathematics to meet the general demands of life at home, in paid work, and for participation in civic life.

Steen’s (2001) defines of quantitative literacy as: ‘An aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem-solving skills that people need in order to engage effectively in quantitative situations arising in life and work’ (p. 7) and notes that it is very ‘contextual’ in nature. This aligns well with the final point of the AAMT (1997, p. 15) definition that numeracy involves ‘grounded appreciation of context’. Watson inferred that there was great similarity between the two definitions and hence quantitative literacy and numeracy could be considered one and the same.

It is useful to compare the ‘numeracy framework’ of Hogan and Willis (Hogan, 2000) with Watson’s (1997, 2004) three-tiered hierarchy for the related field of statistical literacy. Tier 1 involves understanding the terminology involved; this relates well to the ‘mathematical knowledge’ and ‘being a fluent operator’ in Hogan and Willis’ framework. Tier 2 involves understanding the terminology within the context in which it is used; this relates well to the ‘contextual knowledge’ and ‘being a mathematical learner’ in Hogan and Willis’ framework. Tier 3 involves a critical awareness to question contextual claims made without proper justification; this relates well to the ‘strategic knowledge’ and ‘being a critical user of mathematics’ in Hogan and Willis’ framework.

Van den Heuvel-Panhuizen’s (2001) description of ‘Realistic Mathematics Education’ (RME) in The Netherlands provides a similar perspective on numeracy. Though the term,

‘numeracy’ is not mentioned, many aspects of what is described as being standard practice come into the realm of numeracy, quantitative literacy, or statistical literacy, as illuminated by a variety of other sources. For example, ‘mathematics must be connected to reality, stay close to children and be relevant to society’ with the underlying ideal of ‘mathematics as a human activity’ where students are given ‘the guided opportunity to ‘reinvent’ mathematics by doing it within a process of progressive mathematization’ (2001, p. 50). Each of these points sits well with the aforementioned views of numeracy.

Van den Heuvel-Panhuizen (2001) pointed out that the term ‘realistic’ could be misunderstood. Essentially, in the context of RME, it denotes any situation or context that is real in the mind of the student. This can be a ‘real-world’ problem, or it can be from the fantasy world of fairy tales, as long as it is real for the students involved. The difference between ‘real world’ and ‘real life’ situations is highlighted in the hierarchical model for numeracy teaching that is developed later in this paper.

RME is based on six principles (Van den Heuvel-Panhuizen, 2001), which can be summarised as follows:

- activity principle — mathematics is best learned when students are active participants in the process;
- reality principle — mathematics is best learned within rich contexts, rather than in isolated situations divorced from reality;
- level principle — the condition for arriving at the next level of understanding is to reflect on activities and tasks conducted;
- intertwinement principle — components of mathematics cannot be separated;
- interaction principle — mathematical learning is best achieved through social interaction in a whole-class setting;
- guidance principle — students need the opportunity to construct and ‘reinvent’ mathematical tools and insights.

Numeracy and ‘real’ contexts

Despite the best efforts of teachers to make mathematical learning as ‘real’ as possible, Kemp and Hogan (2000, p. 13) made the point that, ‘many of these “real world problems” appeared contrived rather than real’ and, ‘left out factors relevant to the real situation’. The main reason for this, according to Kemp and Hogan, seemed to be that the central purpose was to teach the mathematics associated with the problem, rather than for developing student numeracy. What needed to occur, and probably still needs to occur, is a genuine attempt to give students the opportunity to regularly experience using mathematics to solve problems, rather than using the problem to teach classroom mathematics.

At the very heart of the numeracy debate are the associated notions of transferability of mathematical learning, and numeracy across the curriculum. Boaler (1993, p. 12) noted that traditional approaches to developing student numeracy were based on the assumption that ‘mathematics can be learned in school, embedded within any particular learning structures, and then lifted out of school to be applied to any situation in the real world’. However, as Kemp and Hogan (2000, p. 13) pointed out, ‘evidence suggests that students do not automatically use their mathematical knowledge in other areas’. Indeed, if learning was freely transferred from the mathematics classroom to any of a number of outside situations, it is unlikely that the numeracy debate would have begun, or at least, reached the proportions it has. It is this suggestion of a gap between the learning of mathematical concepts and the application of them to wider contexts that has prompted the development of the following model.

A model for numeracy teaching

The hierarchical model for numeracy teaching.

The ‘hierarchical model for numeracy teaching’ (Figure 1) is based on the belief that there has been a missing link in the development of student numeracy. In the traditional mathematics classroom, there was a focus on the teaching of mathematical content,

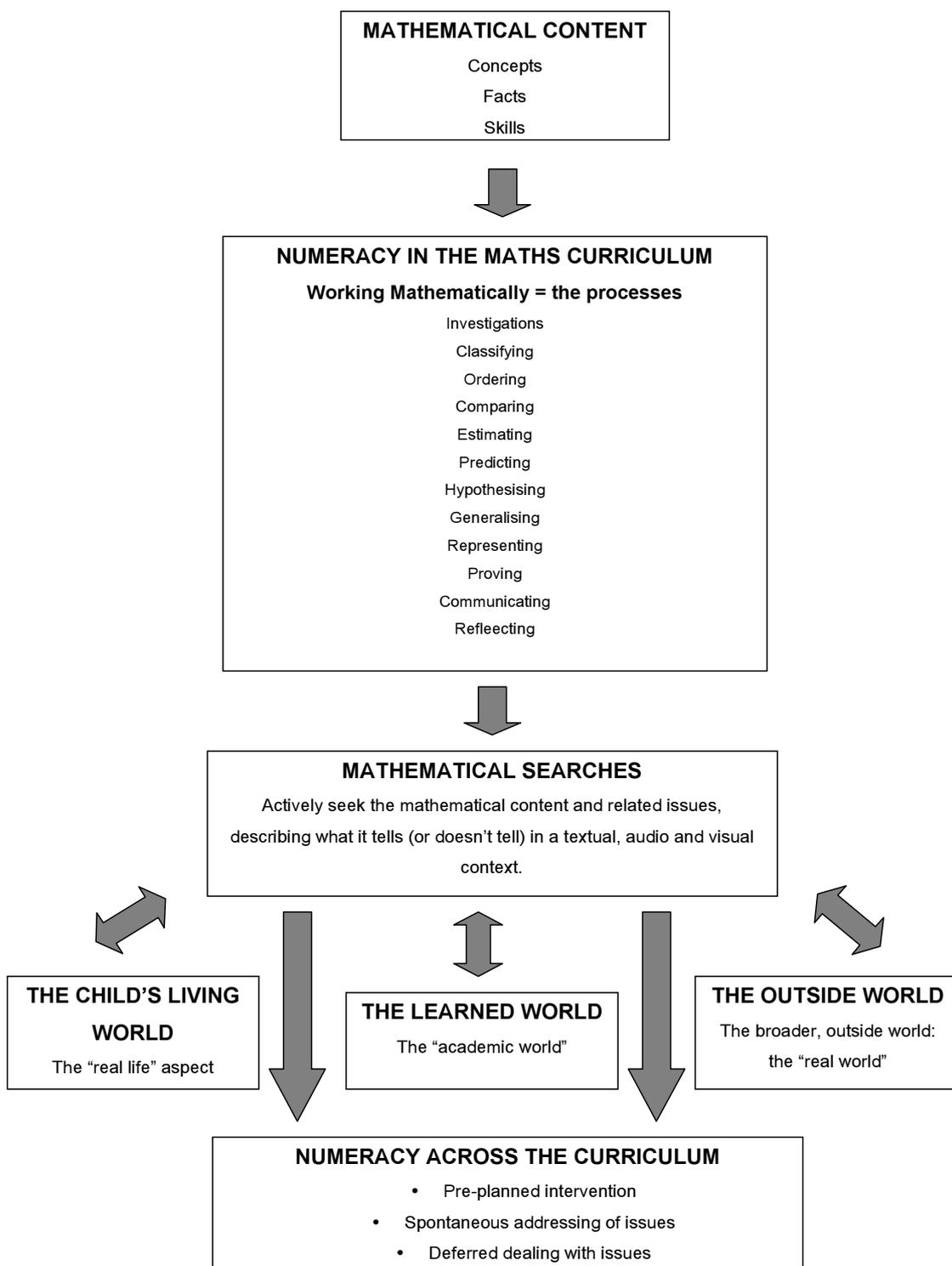


Figure 1. The hierarchical model for numeracy teaching.

and, during the past ten years or so, ideas generally described as ‘Working Mathematically’ in most Australian mathematics curricula, have been more or less adopted. Currently, there is a significant call to develop ‘numeracy across the curriculum’. However, it is suggested here that there needs to be a more conscious effort to link the mathematical content and the mathematical processes learned in the maths classroom with other learning areas, and with situations that reflect ‘real life’ and ‘real world’ issues.

This assertion is supported by Peter-Koop (2004, p. 454), who noted the following with respect to children’s ability to solve simple real-world related problems.

Due to difficulties with the comprehension of the text and the identification of the ‘mathematical core’ of the problem, primary school children frequently engage in a rather arbitrary and random operational combination of the numbers given in the text. In doing so, they fail to acknowledge the relationship between the given data and the real-world context.

With this in mind, it is further suggested that the ‘missing link’ referred to above could be the notion of *mathematical searches*.

The mathematical search

The purpose of a mathematical search is to encourage students to actively seek mathematical concepts and facts embodied in a contextual situation. The context can be any one of a variety of situations: students could be given some text to read, pictorial information to observe, or an audio-visual or oral presentation to regard. The basic task given to them is embodied in a proforma titled ‘Looking for mathematics’ which asks the following questions:

- What mathematical ideas are there in the text you have been using?
- How did you recognise that there were mathematical ideas?
- Describe what the mathematical ideas tell you about what is in the text. Show working out if this helps you.
- What did you know about these mathematical ideas before you read the text?

It is significant that this involves several levels of understanding, from recognition of mathematical ideas and facts, to interpretation and inferring, and involves many of the key ideas contained in the second part of the model (see Figure 1).

Action research

Classroom tasks

In order to gain an appreciation of the effectiveness of mathematical searches, and also of the usefulness of intervention to teach concepts prior to a search, a simple qualitative study was conducted. It was intended that this would be a guide as to whether or not a more detailed study should be conducted in the future. Several activities were presented to a group of seventeen Year 6 students. In the first instance, the students, who had done a brief study of endangered species, were given several pages of text about the numbat, an endangered Western Australian marsupial. The questions contained in the ‘Looking for mathematics’ proforma were posed. No prior teaching of any concepts contained in the text was carried out. A debriefing session to discuss their findings was conducted.

A week later, a second mathematical search was presented, based on a selection of text excerpts on the theme of 'Australia's Gold Rushes', a topic which students had also studied briefly. Once again, the proforma was used. In this instance, specific teaching about bar and column graphs and line graphs was undertaken, prior to the mathematics search. These two tasks could be described as meeting the criteria in the model as a part of the 'outside world' (the numbat task) and the 'learned world' (the gold rushes task).

Approximately six weeks later, a third task was completed by the students. This was different in that no set text was provided. The students were given the brief to bring some information sources of any type that gave specific information about something in which they were vitally interested. They were then asked to apply the proforma questions to their own choice of information. Issues and concerns related to specific aspects of mathematical content that surfaced during this task were deferred for future teaching. The third task is best described as being applicable to the 'child's living world', or a 'real-life' task.

Analysis of the student work samples

Question 1 and 2: Recognising mathematical ideas

The work produced by the students was extremely varied, especially for Task 1, in which some students recognised many mathematical ideas while others recognised only one or two. Responses for Task 2 were more evenly spread across the group with more students providing multiple examples of mathematical ideas in context. Task 3 responses were similar in number to those in Task 2, but were significantly more varied in nature. In Task 3, students recognised a greater variety of aspects of mathematics pertinent to the topic they had chosen.

Question 3: What the mathematics is telling us

There was a significant increase in the number of responses from Task 2 compared to Task 1. Also, more students contributed ideas in Task 2 than for Task 1. In Task 3, the level of synthesis of the mathematics was much greater than for either Task 1 or 2. Every student contributed at least two examples of what the mathematics told them and connected the mathematics to other aspects of their chosen topic.

Question 4: Prior knowledge of mathematical ideas before reading the text

The responses for Tasks 2 and 3 were far greater (three times as many) than for Task 1.

Other tasks to develop numeracy

In conjunction with the mathematical searches, a variety of other activities were completed by the students, in an effort to use the three sources of rich tasks: the child's 'living world', the 'learned world', and the 'outside world'. For example:

Bruce's Café

Students were asked to plan a floor and seating plan for a new restaurant, given an area of fifteen metres by fifteen metres. Students needed to consider measurement issues such as chair and table sizes and groupings of tables, as well as other logistical issues.

Car park activity

Students were asked the question, 'How many cars can we fit on the school oval to accommodate parking for the school speech night?'. Similar logistical issues to the above task needed to be considered.

Student interest survey

Students selected an area of their own interest and constructed a survey to administer to other students/classes in the school. Topics such as favourite fast foods, electronic games and sporting teams were chosen. Results were documented using Microsoft *Excel*.

Traffic counts

Following class discussions about popularity of car makes and types, a traffic survey was conducted outside the school. Associated issues such as difficulty of counting on a major road, location of road and suburb, and time of day were considered. A second survey was conducted in a railway station car park and similar issues were raised, such as the effect that counting in a particular suburb might have on vehicle makes and types.

Capturing the ‘numeracy moment’

If cross-curricular numeracy is to be successfully developed, it is important that teachers take advantage of ‘numeracy moments’ that arise in the context of class activities. (Morony, Hogan & Thornton, 2004, p. 7). A numeracy moment could be described as the use of a mathematical idea in context, and could be seen as a demand on which understanding depends, or an opportunity to enrich a learning experience. Whether or not such opportunities are pre-planned, dealt with at the time of encounter, or deferred, is a matter for the teacher’s professional judgement. This is outlined in the final section of the model in Figure 1.

Conclusion

In comparing the responses from Tasks 1, 2 and 3, it is clear that there was a much greater level of engagement with Tasks 2 and 3, than with Task 1. The number of responses from every student was greater for Tasks 2 and 3 than for Task 1. This may be due to a greater degree of familiarity with the task type, or because of the intervention lessons that were conducted prior to Task 2. In Task 3, responses to Question 3, indicated a greater level of synthesis of the mathematics than for either Task 1 or 2. This seems to indicate that, if working on a topic of interest to them (real-life context), students will display a greater level of understanding of how the mathematics actually ‘works’.

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Exploring space and measurement with the ClassPad 300*

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Hand-held technologies have offered less support for learning about space and measurement than they have for other strands of mathematics. This paper describes a recent development which seems promising in this respect, Casio's ClassPad 300. The mechanisms of providing interactive opportunities for students are described and illustrated, and some possible ways in which this sort of device might be used productively in mathematics learning are offered. A range of mathematical ideas are used, including geometric properties and relationships, coordinate geometry, transformations in the plane, mensuration and conics.

Introduction

More than a decade ago, *A National Statement on Mathematics for Australian Schools* (Australian Education Council, 1990) identified five content strands for school mathematics, which continue to be used in curriculum frameworks around Australia. At around the same time, personal technologies in the form of graphics calculators began to be used in many schools. Since that time, the major uses of hand-held technologies in secondary schools have centred on the Number, Algebra and Chance & Data strands with many fewer opportunities concerned with the other two strands of Space and Measurement. Although there have been exceptions to this generalisation (such as the development by Texas Instruments of a version of *Cabri Geometry* for the TI-92 graphics calculator), the constraints of screen size have generally discouraged the development of suitable technology for dealing with spatial objects satisfactorily. This paper offers a brief description of the way in which the Casio *ClassPad 300* supports mathematics in these two strands, and considers some implications of this sort of device for school mathematics.

ClassPad 300

The *ClassPad 300* is a relatively new example of personal technology, about the same size as a modern graphics calculator, but with a larger graphics display screen and a smaller keyboard. The photograph in Figure 1 illustrates these two characteristics and also shows the main menu, from which various applications can be accessed.

* This paper has been accepted by peer review.

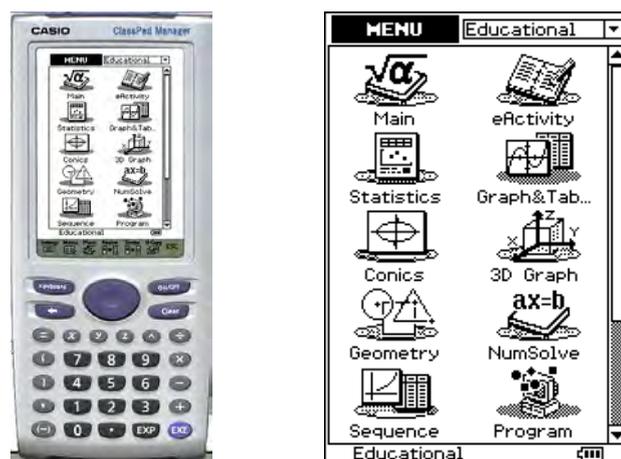


Figure 1. Photograph and opening menu of the ClassPad 300.

As well as the keyboard shown in the photograph, the device has a touch-sensitive screen, which can be operated by a stylus, and a variety of soft keyboards. A PC-based emulator, called *ClassPad Manager*, is also available. This paper focusses upon the *geometric* resources provided by such an environment, the two main features of which are the personal nature of the technology (more portable and hence more accessible than, say, a conventional computer) and the use of a stylus for interaction between machine and person. Readers interested in details of the various other *ClassPad 300* capabilities can access them from the manufacturers (e.g., Casio Computer Corporation, 2004), and may well find similar kinds of issues arising for other aspects of mathematics.

Interactive geometry

A major feature of the *ClassPad 300* concerns interactions between the user and the machine, made possible by the stylus. In the case of geometric work, the use of the stylus gives the experience a feel not unlike the use of physical manipulative materials or drawings, since there is a direct link between the screen elements involved (such as geometric objects) and the person using the stylus. Interactive experience of this kind is potentially very fruitful for learning, as argued in more detail in Kissane (2004a). Similarly, Mason (1995, p. 10) observed that, ‘Screen-objects present a new form of apparatus or manipulable.’

Constraint-based geometry

Geometric objects are available on the *ClassPad 300* through the use of the Geometry application, visible as one of the icons in Figure 1. While so-called ‘dynamic geometry’ software such as *Cabri Geometry* and *Geometer’s SketchPad* has been used for some time in secondary schools, the *ClassPad 300* uses ‘constraint-based geometry’, described briefly by the software company responsible for it, Saltire Software (2004a). An earlier version of this sort of software seems also to have been used on a hand-held device by Hewlett Packard (Saltire Software, 2004b).

To illustrate the nature of ‘constraint’ in this context, the first screen in Figure 2 shows a parallelogram ABCD drawn on the graphics screen, using one of the pull-down icons on the Geometry toolbar. The vertices and sides of the parallelogram can be moved using the stylus, some possible results of which are shown in the second and third screens. Despite

these movements, the figure ABCD remains *constrained* to be a parallelogram by the software. The defining constraints for a parallelogram used by the software ensure that opposite sides of ABCD remain parallel and congruent, regardless of how the vertices are moved or the dimensions and shape changed through the use of the stylus.

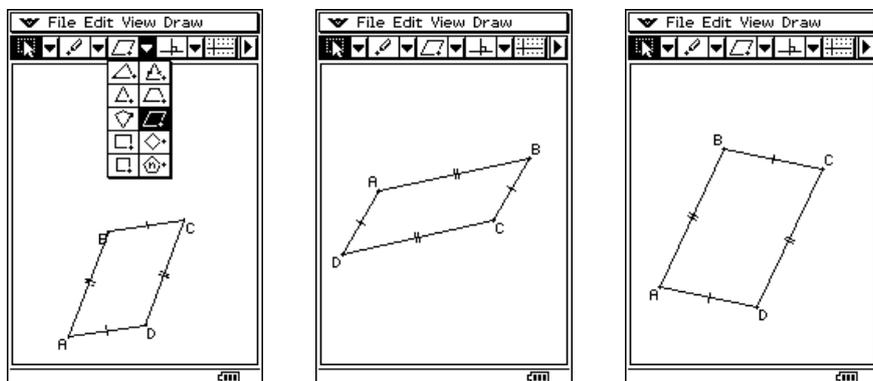


Figure 2. ABCD is constrained to be a parallelogram.

Interacting with such a figure seems likely to help students get a good feel for the nature of a parallelogram and some of its essential properties, such as relationships between opposite sides, opposite angles, adjacent angles and so on.

Making measurements

In addition to the manipulation of screen objects, a second important feature of the Geometry application concerns the making of measurements of geometric objects, which might help students get a more quantitative sense of the properties of objects. In Euclidean geometry, we are interested in a range of measurements, such as those of length, angle, area, direction and so on. On the ClassPad 300, measurements are made via a ‘measurement box’, illustrated directly under the menu items in Figure 3. The first screen shows that the length of the selected segment AB is about 8.05 units. The second screen identifies the angle at vertex B by highlighting the adjacent sides AB and BC, and the measurement box shows the angle size of 103.0418° . Other possible measurements might be chosen from a pair of adjacent sides, as shown in the third screen; the measurement box indicates ‘No’ to the ‘congruence checker’, since the two highlighted line segments are not congruent.

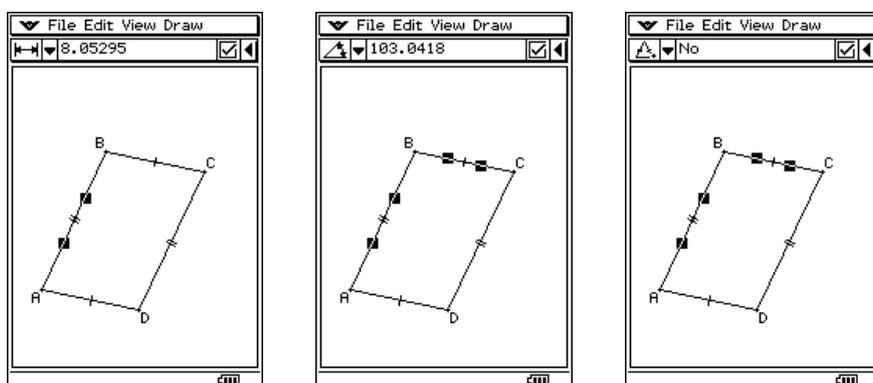


Figure 3. Some measurements of ABCD.

In a similar way, different selections of parts of an object offer different opportunities for measurement and the resulting explorations. The first screen in Figure 4 shows that selecting all four vertices of parallelogram ABCD allows for either the area or the perimeter to be measured. The measurement box shows that the area is 41.03547 square units. The selection of only three vertices (which define a triangle, of course) allows for the areas of triangles DAB and ABC to be determined, as shown in the final two screens of Figure 4. In this case, these measurements help to see that each triangle has half the area of the parallelogram, or 20.51774 square units (within a rounding error), allowing students to verify that the diagonals of the parallelogram divide it into two parts of equal area.

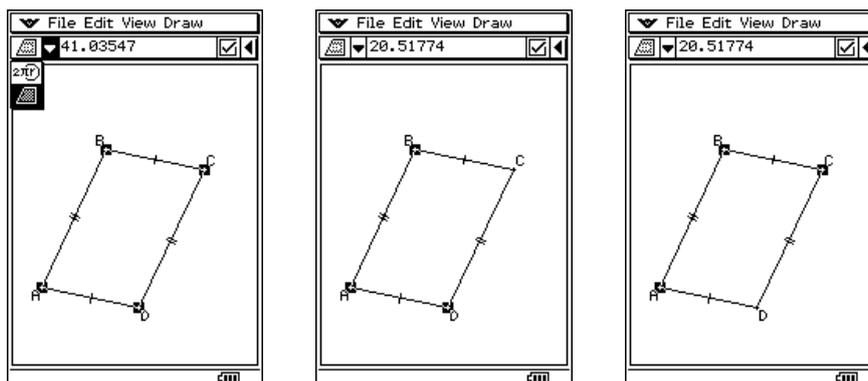


Figure 4. Measuring areas associated with parallelogram ABCD.

Coordinate geometry

The measurements made in Figures 3 and 4 are relative to a particular scale of course, and the *ClassPad 300* allows for the rich connections between algebra and geometry to be explored by students, through the medium of coordinate axes. The first screen in Figure 5 shows the same parallelogram ABCD as in Figure 4, but with the coordinate axes revealed, as well as a grid marked to assist with locating objects in two-dimensional space. Although the measurement box is not showing, the first screen indicates that the coordinates of point C are (3.6,2.4), relative to the axes shown.

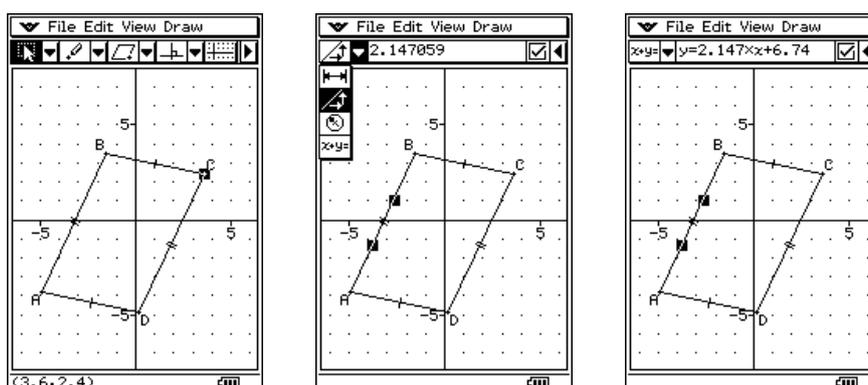


Figure 5. Revealing the coordinate system and making measurements.

In keeping with Rene Descartes' wonderful invention, students now have access to algebraic ways of describing various geometric objects, as shown in the rest of Figure 5. The middle screen shows that one of the measurements of side AB is the slope of the corresponding line through A and B, which in this case is 2.147059. The drop down tools indicate that students might have just as easily chosen to measure the length of AB, the

angle of inclination of AB with the horizontal axis or the equation determined by AB. The latter is illustrated by the final screen in Figure 5, with a clear link between the gradient of AB and the equation for AB.

There are many opportunities for students with these tools at their disposal to explore the properties of objects and to make the links between geometry and algebra that characterize coordinate geometry.

Geometric constructions

A common activity for students with dynamic geometry software involves making geometric constructions and then exploring them to look for patterns, such as invariant properties. Many others have indicated the possibilities here for students to make discoveries for themselves, and to be thus encouraged to look for ways of proving that their observations are universally true (rather than relying on observations alone). Such activity is supported on *ClassPad 300*, since, as Saltire Software (2004a) indicate, construction-based geometries are a subset of constraint-based geometries.

To illustrate this kind of learning opportunity, Figure 6 shows the use of a perpendicular bisector construction tool to show the three perpendicular bisectors of the sides of triangle ABC. Once the construction has been completed, the stylus can be used to move the vertices of the triangle to different locations, while the software ensures that the constructed perpendicular bisectors are moved accordingly. The middle screen suggests that these three are concurrent, while the third screen verifies that this seems to be the case, even when the three lines meet outside the triangle itself.

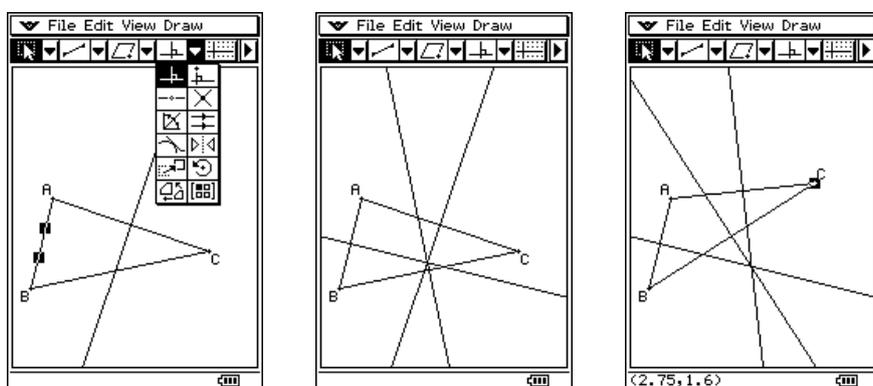


Figure 6. The perpendicular bisectors of the sides of a triangle are concurrent.

Armed with such a tool, students can be given opportunities to notice and to explore spatial relationships in ways that have not been available prior to the development of dynamic geometry software. In this regard, Mason (1989, p.46) conjectured that 'what is important about geometry is being aware of the fact that there are facts, rather than mastery of some particular few facts'. While such activity does not provide the mathematical reassurance required for a proof, it hopefully provides a stimulus to look for reasons for observed regularities and to thus provide a need for proof.

Spatial connections

A distinctive feature of the *ClassPad 300* environment is that different applications can be connected together in educationally powerful ways. In this section, some promising examples of this facility are briefly described and illustrated.

Exploring functions

A powerful and popular use of graphics calculators has been to represent graphs of functions and to quickly explore relationships between changes in the parameters of functions and their graphs; indeed this idea is probably the first one used by mathematics teachers and continues to be a reason for describing the devices as ‘graphing’ calculators. Since it includes all of the functionality of a graphics calculator, a *ClassPad 300* can be used in the same way. To date, it has not been possible to do the reverse on graphics calculators: to see how changes in graphs are related to changes in the parameters of the functions concerned.

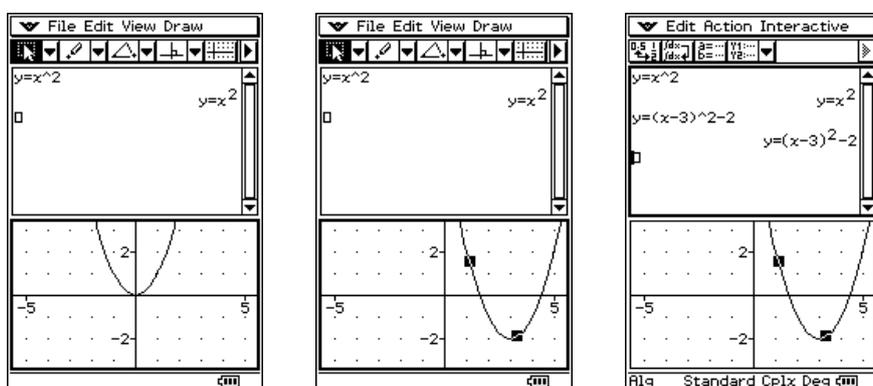


Figure 7. Dragging and dropping between algebra and geometry windows.

Figure 7 shows a sequence of three dual screens with algebraic expressions in the top and a geometry screen at the bottom. In the first screen, the stylus has been used to physically drag the expression for quadratic function to the geometry screen, thus producing the expected graph. (An alternative involves cutting and pasting via the Edit menu, in much the same way as a word processor operates.) The second screen shows that the stylus has again been used to drag the parabola to a new position, three units to the right and two units down. Finally, the third screen shows the effect of dragging the (re-located) parabola back to the algebra screen, which results in the algebraic representation of the associated function. (Of course, it is a great deal easier to physically do these things than it is to describe them on paper.)

This sort of facility seems to offer much promise for students making sense of functions and their graphical and symbolic representations, as it complements the graphics calculator capability of moving from symbols to graphs very nicely. Armed with such a tool, students can experience at first hand the lovely relationships involved, and can see that they apply to various families of functions, not only quadratic functions. A limitation is that it is restricted to dragging, so that only translations (both horizontal and vertical, as illustrated in Figure 7) are involved; stretches and the corresponding scale changes do not seem to be able to be produced in this way, so that more conventional methods of exploring these remain important.

Exploring transformations

A second example of connections between geometric and algebraic ways of representing things concerns transformations in the plane. The *ClassPad 300*, like recent versions of geometry software, includes a facility for the isometries of reflection, translation and rotation, allowing these to be constructed and studied. The series of three screens in Figure 8 shows an example of a reflection about a line.

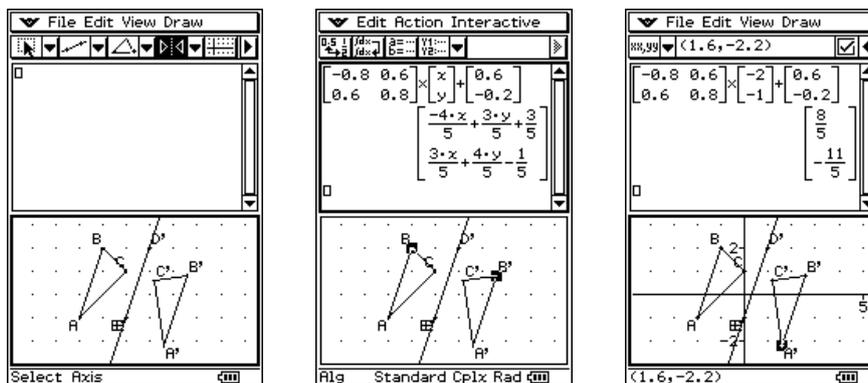


Figure 8. Exploring reflections about a line geometrically and algebraically.

A reflection of the triangle ABC about the line DE has been constructed in the geometry window at the bottom of the first screen. The middle screen shows the results of selecting a point (B) and its image (B') and then dragging these together to the top window: the general linear transformation involved is then shown algebraically, using a matrix representation. The third screen includes the axes (which of course are needed to make sense of the matrix formulation) and also shows how the matrix representation can be used to find images of a particular point. In this case, the image of (-2,-1) after reflection in line DE is (8/5,-11/5). Again, it seems reasonable to expect that providing students with the facility to move smoothly in these ways between geometric and algebraic representations will offer learning opportunities that were previously unavailable.

Mensuration

Interactions between geometric measurements and the main screen allow for verification of mensuration formulas, as illustrated in Figure 9. In this case, the area of triangle ABC has been determined in four different ways, once directly in the measurement box and the other three using half the product of the lengths of a pair of sides and the sine of their included angle.

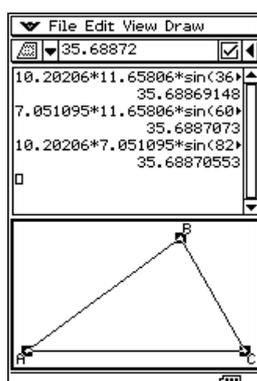


Figure 9. Finding the area of a triangle using two sides and the included angle.

All necessary lengths and angles have been copied and pasted from the measurement box in the geometry window into the calculation window. (Measurements are of course rounded, resulting in slight inaccuracies in final decimal places.) What the *ClassPad 300* provides here is an opportunity to verify that the formula produces equivalent results, even when different pairs of sides are chosen.

Dynamic linking

As a final example, complementary to the mechanism of dragging and dropping or cutting and pasting described above in the context of exploring functions, the *ClassPad 300* offers a means of dynamically linking algebraic and geometric objects, so that a change in a geometric object gives rise to a corresponding change in its algebraic representation and vice versa. This idea occurs in the context of ‘eActivities’, which are small user-created applications that are designed for students to experience various aspects of mathematics; an extensive discussion of these is provided in Kissane (2004b) and there are many examples illustrated on the Internet (e.g., at Saltire Software (2004c)).

Figure 10 shows an example of an eActivity. Each of the three screens shows a circle in the bottom geometry window and an associated equation in the top window. The first screen shows the initial situation. In the second screen, the equation parameters have been changed, resulting in the circle ‘moving’ and ‘shrinking’ accordingly. Similarly, the third screen shows that the circle itself has been dragged with the stylus to a new position, which has resulted in an updated version of the equation. In this particular case, as the centre of the circle has moved off the horizontal axis, the equation includes a linear y term, not previously evident. As for some earlier manipulations, it takes a good deal more words to describe this than to actually do it; again, it seems reasonable to expect that students given the opportunity to manipulate both expressions and geometric objects linked together dynamically in these sorts of ways will develop a strong sense of the connections involved.

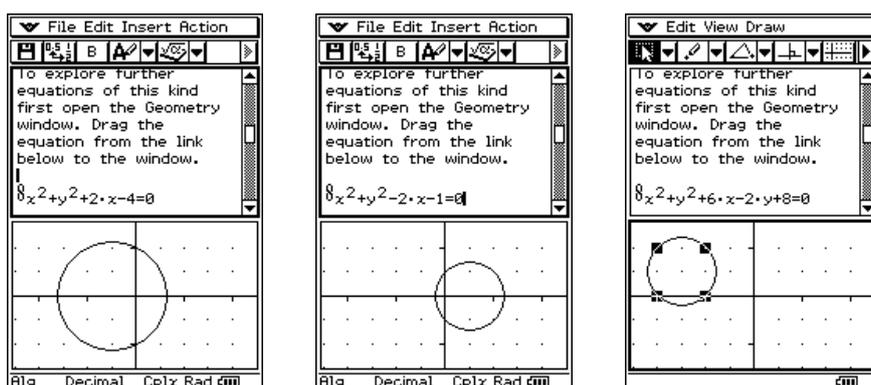


Figure 10. A dynamic link between circles and their defining equations.

Conclusion

An important role of technology is to provide students with experiences that are not otherwise available to them. (Kissane, 2002). To date, the experiences offered by hand-held technologies seem most likely to be accessible to a wide group of students, but these have been relatively scarce in areas related to the Space strand and its connections elsewhere, such as the Measurement strand. Mason (1989, p. 46) argued that, ‘geometry takes place

in a world of forms and images, entry to which is gained through the power of mental imagery, augmented and extended by dynamic images, drawings on paper and discussion with colleagues'. Technologies of the kind described here seem to offer new opportunities for enhancing mental images in students, through the design of productive classroom activities of new kinds, based on personal interaction between mathematical ideas and their representations. While taking advantage of these new opportunities is unlikely to be an easy matter, and it is too early to tell whether the educational effects are as promising as they at first seem, the design of new technologies like this offers important opportunities to explore these important issues in school classrooms.

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The role of manipulatives in developing mathematical thinking*

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The use of manipulatives as part of mathematics lessons has long been advocated as part of a comprehensive mathematics program. Recent developments such as virtual manipulatives, along with research have caused some to question the role that manipulatives play in learning mathematics. In this paper the authors re-examine the use of manipulatives within the constructivist paradigm. Attribute Blocks are used to illustrate how mathematical thinking may be developed with the aid of manipulatives.

Introduction

For many years teachers of mathematics, particularly at the primary school level, have espoused the virtues of ‘hands on’ learning. While children may have appeared engaged in the task of ‘manipulating blocks’ it is not clear what they were learning. In some cases rather than construct appropriate knowledge children were in fact mis-learning or mis-constructing knowledge.

Ball (1992, p. 17, as cited in Perry & Howard, 1997, p. 26) noted:

One of the reasons that we as adults may overstate the power of concrete representations to deliver accurate mathematical messages is that we are ‘seeing’ concepts that we already understand. That is, we who already have the conventional mathematics understandings can ‘see’ correct ideas in the material representations. But for children who do not have the same mathematical understandings that we have, other things can reasonably be ‘seen’.

As Clements (1999, p. 2) so eloquently sums up: ‘it cannot be assumed that concepts can be “read off” manipulatives’. It cannot be assumed that simply using manipulatives means that children will learn. If children are to construct meaning from the manipulative being used teachers need to be explicit about the mathematics to be developed from the manipulative. Children need to be given time to gain familiarity with the manipulative so that rather than focus on the manipulative itself they will be focussed on the mathematics to be developed. Stein and Bovalino (2001) noted that: ‘If not used with careful thought, manipulatives can become little more than window dressing, they are nice to look at and play with but superfluous to overall learning’ (p. 357).

* This paper has been accepted by peer review.

A brief overview of manipulatives research

The use of manipulatives has been studied over a number of years. The advent of virtual manipulatives has rekindled interest in the role that manipulatives have to play in the learning of mathematics. The following is a brief outline key research findings.

- Students who use manipulatives outperform those who do not (Clements, 1999). Kennedy (1986) noted that: ‘Although no single study validates the claim that children should use manipulative materials as they learn mathematics, the collective message garnered from many studies is that materials are worthwhile’ (p. 7).
- Attitudes toward mathematics improve when concrete materials are used (although there is a caveat: ‘provided... teachers are knowledgeable about their use’ (Clements, 1999, p. 1).
- Use of manipulatives declines in later years of the primary school (Gilbert & Bush, 1988; Perry & Howard, 1997).

A change in approach

Clements (1999) questioned the whole notion of moving from the ‘concrete to the abstract’. He revisited this commonly accepted approach, especially in the light of computer or virtual manipulatives. He stated:

...common perspectives on using manipulatives should be reconsidered. Teachers and students should avoid using manipulatives as an end — without careful thought — rather than as a means to that end. A manipulative’s physical nature does not carry the meaning of a mathematical idea. Manipulatives alone are not sufficient — they must be used in context of educational tasks to actively engage children’s thinking with teacher guidance. (Clements, p. 9)

The ability to engage children’s thinking should not be taken for granted. In the following section, *attribute blocks*, a common manipulative, are used to illustrate how children may be aided to construct knowledge.

Working mathematically with attribute blocks

The key mission of teachers of mathematics at all levels is to promote in their students the ability to ‘work mathematically’ and this involves being able to think sometimes creatively and sometimes logically (Southwell, 2003, p. 19). Southwell went on to explain two types of reasoning: inductive and deductive. It is the second type of reasoning that is used in logic. Attribute blocks, or logic blocks as they are sometimes called, may be used to help children ‘work mathematically’. A series of activities involving the use of attribute blocks is presented below. Consider how each activity contributes to the development of thinking skills.

Attribute block sample activities

A set of attribute blocks (usually) consists of sixty blocks. The different attributes of the blocks are: shape (circle, rectangle, triangle, hexagon and square); colour (red, blue and

yellow); thickness (thick and thin); and size (large and small). Each block is unique.

Activities with attribute blocks require a high level of abstraction when we speak of the ‘attributes’ of the blocks. Some children may find this difficult at first. Attributes are not part of the object. They are concepts that the mind attributes to the objects for the purpose of classification. Intellectual development consists partly of the ability to invent relevant attributes or categories and to deal with relationships between those attributes. Many attributes other than colour, shape, size and thickness could be invented for the blocks, but are in fact ignored.

When introducing the blocks to children, plenty of time needs to be allowed for them to explore properties such as size and shape. This may start off in the form of free play, but the play would soon become more directed, where the teacher directs the children in such ways as, ‘Can you tell me what different types of blocks are in your set?’. The instructions and activities should be enjoyable, but challenging.

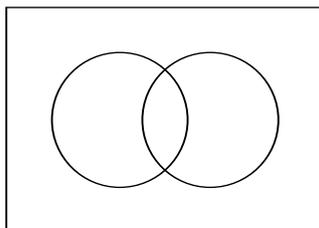
The following is a selection of activities of varying degrees of abstraction.

1. Examine the set of blocks. Is your set complete? If not, what pieces are missing? Are there any extra pieces? How many pieces should there be in a complete set?
2. Group the blocks by colour. Let one person build a tower or make a pattern using one colour. Try to copy your partner’s arrangement with a different colour. Vary the game by making the initial grouping by size, shape or thickness.
3. Use only the thick blocks. Choose two colours and two shapes (e.g. blue and red, square and triangle). How many pieces do you have? Can you describe them fully? Close your eyes while your partner removes a block. Can you say what has been removed?
4. Choose a colour and collect all the blocks of this colour. Close your eyes while your partner removes a block. Can you tell what was removed? Vary the game by choosing all the blocks of a particular shape.
5. Get all the yellow and blue circles and squares. Arrange the 16 pieces in some systematic way. One of these pieces is NOT yellow, NOT square and NOT small. Which piece is it?
6. Use all the large blocks. Arrange them systematically in colour and shape rows; i.e., in a matrix.
7. Use the small, thin blocks. Put four pieces out as ‘clues’ in a 3×5 grid and let your partner complete the grid.

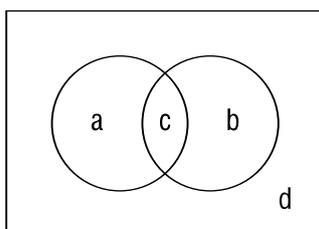
red triangle		
	yellow circle	
	yellow square	
red triangle		

8. Close your eyes and have your partner put any block in your hands. Keeping your eyes closed, how well can you describe the block?
9. What distinctive features of the blocks could be used to group them into:
 - (a) 2 equivalent sets?
 - (b) 4 equivalent sets?
 - (c) 6 equivalent sets?
 - (d) 12 equivalent sets?
10. Build a *one-difference train* using all the blocks. Each block must have *exactly one* difference from the preceding block.

11. Build a *two-difference* train. An extension: build a *three-difference* or *one-same-ness* train.
12. Use all the blocks. Place a hoop on the floor. Put all the yellow triangles in the hoop. What blocks will be inside the hoop? What blocks will be outside the hoop? How many blocks are inside the hoop?
13. Place two hoops on the floor. In one hoop place all the red shapes. In the other place all the blue shapes. How do you have to place the hoops? Repeat for red shapes and thick shapes.
14. Place two hoops on the floor so that they cross each other. In the left hoop put all the blue blocks, in the right hoop put all the rectangular blocks. Describe the set of blocks found in each of the regions. What blocks are left outside the hoops?



15. Place two overlapping hoops on the floor.
Let Set A = the triangles and Set B = the red shapes.
Place the blocks where they belong.



Describe the blocks that are:

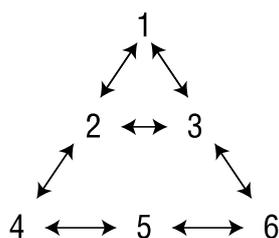
- (a) in region a
- (b) in region b
- (c) in region c
- (d) in region d
- (e) in regions c and d
- (f) in regions a and b

What is the intersection of a and b?

16. This is a game for 2 or 3 players. Use only these set descriptions with a single attribute: red, yellow, blue, thick, thin, small, large, hexagon, rectangle, square, circle, triangle.

Place 2 hoops on the floor so that they cross each other. Call one Hoop A and the other Hoop B. One player secretly names A and B from the set descriptions above. The other players try to deduce Sets A and B. They do this by selecting a block and asking the first player to place it for them. To win, sets A and B must be deduced within 8 moves.

17. Place six blocks to make a pattern like this.



The blocks joined by arrows must form one-difference trains.

Repeat, but make two-difference trains.

18. Sherlock's blocks logic problems.

- Block 1:
1. It is not red or it is not small
 2. It is a circle or a rectangle or a triangle
 3. It is not blue
 4. It is not large
 5. It is yellow or red
 6. It is not a circle
 7. It is not a triangle
 8. It is _____

- Block 2:
1. It is blue or large or square
 2. It is not yellow
 3. It is small or a triangle
 4. It is red or blue
 5. It is not a circle
 6. It is blue or large
 7. It is not blue
 8. It is _____

Conclusion

Manipulatives per se do not teach in and by themselves. They may be used to help children to construct knowledge, but only if there is a clear purpose for the activity in the teacher's mind. This is then translated into activities where this purpose is highlighted by the use of pertinent focus questions which allow children to build on their existing knowledge, not by accident but by design.

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Houdini, Fibonacci and Pythagoras: The link

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In this workshop a series of dissections will be presented in which a part of the material either disappears or gets bigger. The relationships between these dissections and the Fibonacci sequence are then explored including some manipulation of surds. Next, a standard dissection proof of Pythagoras' theorem is examined to determine that the sections do fit (unlike the first part of the presentation) to confirm that proof is an essential part of any mathematics course.

Students enjoy mathematics when they are presented with a situation that appears to be contradictory. Challenging them to explore possible explanations has the potential to enrich their experiences in classrooms. The following suggestions help to establish the importance of proof.

Take a 21 unit \times 21 unit square and cut it up as shown in Figure 1. These pieces can then be rearranged to form a rectangle as in Figure 2.

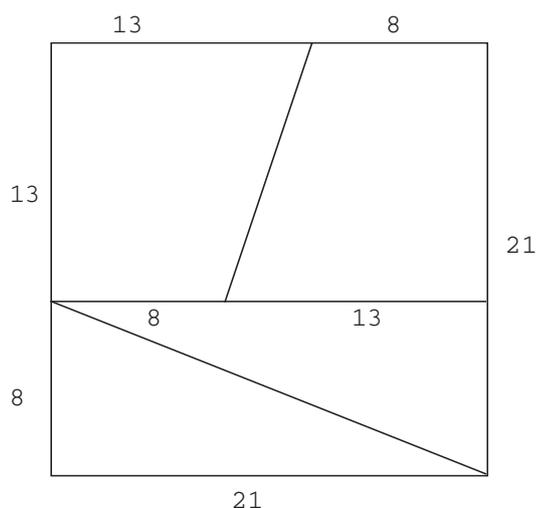


Figure 1

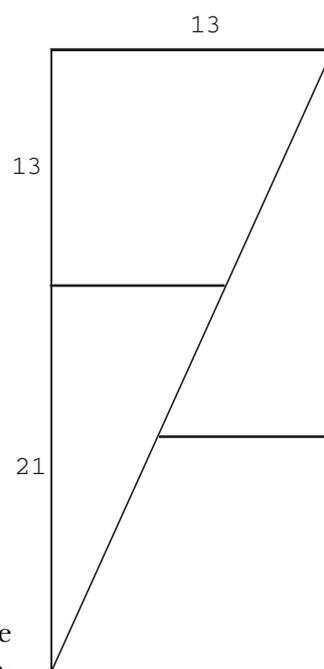
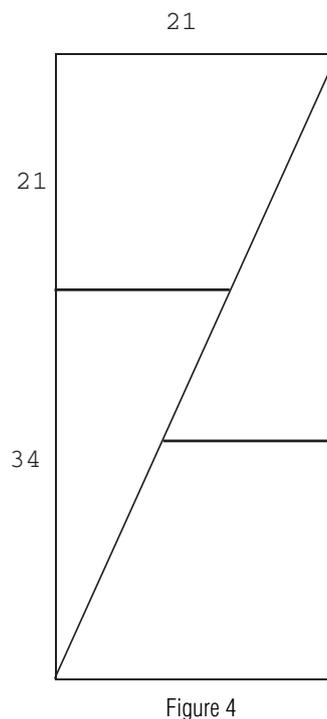
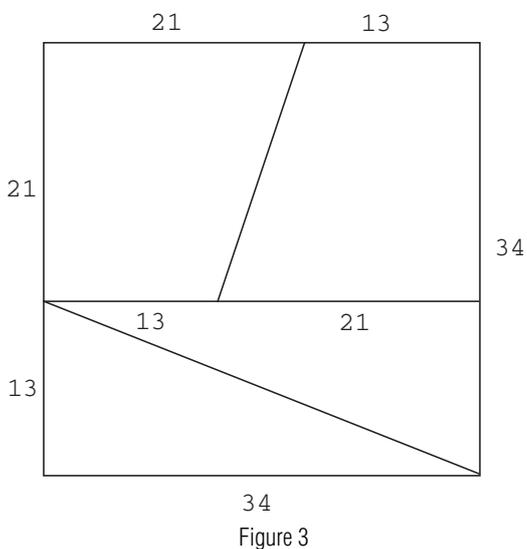


Figure 2

However, when we do this there is a slight problem in that the area of the square is 441 square units and the area of the rectangle is 442 square units. So how is it that we gained an extra unit?

Let us try the same thing with a different square, this time a 34 unit \times 34 unit square. (not to scale, but the dimensions are important).



A similar problem occurs in that 34×34 is 1156 and 21×55 is 1155. A square unit has been lost!

If we check the areas of the cut-up trapezia and triangles, we find that the total area is still that of the original square, which establishes that we have not contravened the Law of Conservation of Matter (and that a nuclear explosion did not take place — in deference to Einstein’s equation).

Let us look at the gradient of the diagonal of the rectangles.

For the smaller square (see Figures 1 and 2):

The gradient of the diagonal of the rectangle is $\frac{34}{13}$

the gradient of the sloping edge of the trapezium is $\frac{13}{5}$

and the edge of the triangle $\frac{21}{8}$

If we look at these as decimals, we have:

$$\frac{34}{13} = 2.615384\dots \quad \frac{13}{5} = 2.6 \quad \frac{21}{8} = 2.625$$

For the larger square (see Figures 3 and 4):

The gradient of the diagonal of the rectangle is $\frac{55}{21}$

the gradient of the sloping edge of the trapezium is

$$\frac{21}{8}$$

and the edge of the triangle

$$\frac{34}{13}$$

If we look at these as decimals, we have

$$\frac{34}{13} = 2.615384\dots \quad \frac{13}{5} = 2.619047\dots \quad \frac{21}{8} = 2.625$$

Not surprisingly, we could not detect the differences in the slopes. The numbers involved here have a certain familiarity about them.

Notice that 5, 8, 13, 21, 34, 55 are all part of the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55... and if we check, we can see that the square of each term, after the first, is one more or one less than the product of the terms on each side:

$$\begin{aligned} 1^2 &= 1 \times 2 - 1 \\ 2^2 &= 1 \times 3 + 1 \\ 3^2 &= 2 \times 5 - 1 \text{ etc.} \end{aligned}$$

or $F_n^2 = F_{n-1} \cdot F_{n+1} - (-1)^n$ for $n > 1$

We also know that the ratio of consecutive terms of the Fibonacci sequence approaches a limit

$$\lim_{n \rightarrow \infty} \frac{F_{n+1}}{F_n} = \Phi$$

where Φ is the ‘Golden Ratio’ 1.61803398... Table 1 shows the convergence of the ratios of terms of the Fibonacci sequence.

Table 1

n	$F(n)$	$F(n)/F(n-1)$	$F(n)/F(n-2)$
1	1		
2	1	1	
3	2	2	2
4	3	1.5	3
5	5	1.666666667	2.5
6	8	1.6	2.666666667
7	13	1.625	2.6
8	21	1.615384615	2.625
9	34	1.619047619	2.615384615
10	55	1.617647059	2.619047619
11	89	1.618181818	2.617647059
12	144	1.617977528	2.618181818
13	233	1.618055556	2.617977528
14	377	1.618025751	2.618055556
15	610	1.618037135	2.618025751
16	987	1.618032787	2.618037135
17	1597	1.618034448	2.618032787
18	2584	1.618033813	2.618034448
19	4181	1.618034056	2.618033813
20	6765	1.618033963	2.618034056
21	10946	1.618033999	2.618033963
22	17711	1.618033985	2.618033999
23	28657	1.61803399	2.618033985

These data are presented graphically in Figures 5 and 6.

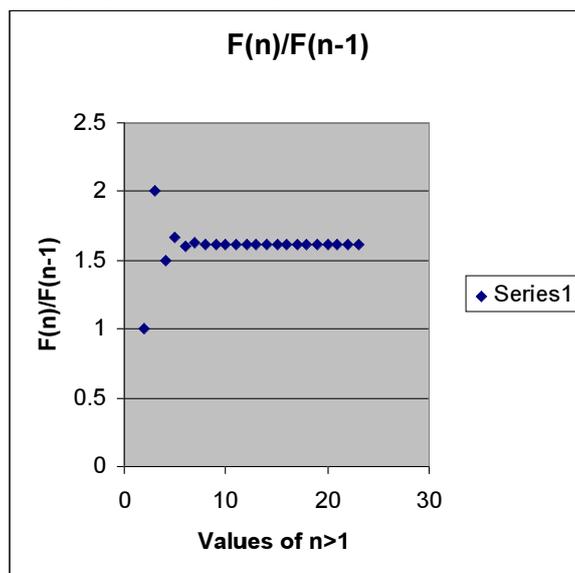


Figure 5

The gradients of those lines we looked at before seem to hover around 2.61803... or $\Phi + 1$ but the ratios here are of every second Fibonacci number so that

$$\frac{F_{n+1}}{F_{n-1}} = \phi^2$$

Figure 6 shows the convergence of the ratios of every second term of the sequence.

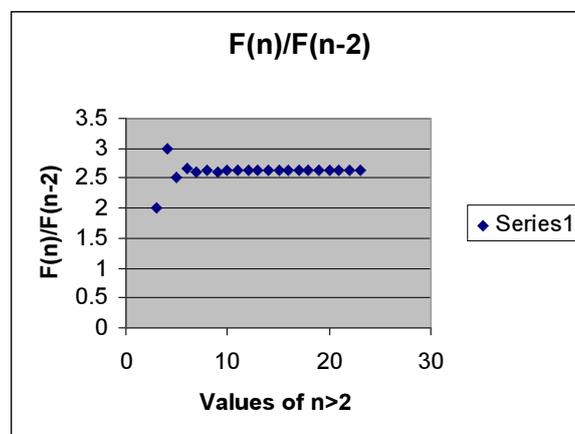


Figure 6

This gives us $\Phi^2 = \Phi + 1$. This leads to the quadratic $\Phi^2 - \Phi - 1 = 0$ which has the solutions that

$$\Phi = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \frac{1-\sqrt{5}}{2}$$

We should note that

$$\left(\frac{1+\sqrt{5}}{2} \right)^2 = \frac{6+2\sqrt{5}}{4}$$

which simplifies to

$$\frac{3+\sqrt{5}}{2} \text{ or } \frac{1+\sqrt{5}}{2}+1$$

Another fascinating thing about Φ is that

$$\frac{1}{\Phi} = \Phi - 1$$

and it can be readily seen that this yields the same quadratic that we had above. It is interesting to do the calculations on a calculator.

All of this means that we can make the puzzle with the missing square by making the dissections using the numbers in the Fibonacci sequence.

We can now turn our attention to Pythagoras' Theorem.

One favourite 'proof' of the theorem is the so-called 'Chinese Dissection'. In this proof, any right-angled triangle is taken and squares drawn on each of the sides. Then, the centre of the middle-sized square is found. Through this point, lines are drawn parallel to and perpendicular to the hypotenuse of the right-angled triangle (see Figure 7).

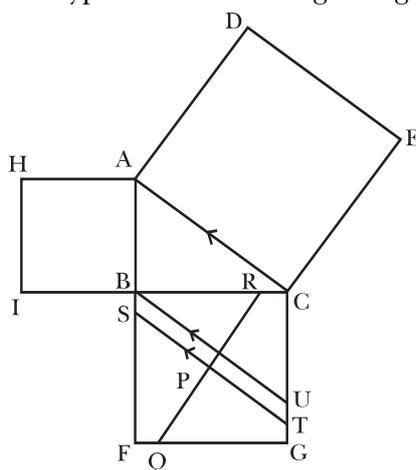


Figure 7

These pieces are then cut out and rearranged in the largest square, leaving a square space in the middle. The square on the smallest side will neatly fit into this space, establishing the veracity of the famous theorem (see Figure 8). The pieces marked 2 fit into the spaces marked 2 in the large square and the small square marked 1 fits into the space marked 1 in the large square.

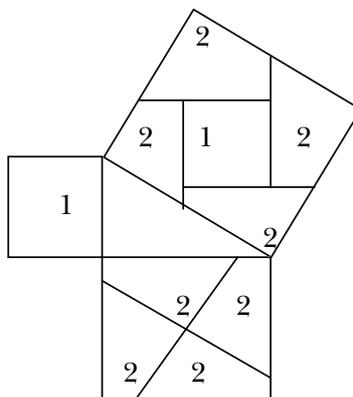


Figure 8

I started with a dissection in which pieces disappeared. We now need to ensure that the 'Chinese Dissection' is valid. When we look at the line drawn through the centre of the middle-sized square parallel to the hypotenuse, we can see that it forms a parallelogram with the hypotenuse and so is the same length as the hypotenuse. The dissection cuts the lines in half and rearranges the halves around the edges of the large square. Careful checking of the angles shows that the angles of the centre shape are right angles all that needs to be shown is that the length of the edge of the square is the same as the small side of the original triangle.

From arithmetic to algebra: Helping to give the 'letters' arithmetic meaning — A consideration of algebra and algebraic thinking within the realm of arithmetic

Ken Milton

University of Tasmania

A recognised path to algebra is through the generalisation of arithmetic. A significant number of writers and researchers have drawn attention to the situation that many students fail to see the 'letters of algebra' as having 'arithmetic meaning' and the rules which govern the fundamental procedures of algebra as being the generalised counterparts of arithmetic procedures and properties. This failure is widespread and often deeply entrenched. Little wonder that such students do not see algebra as having purpose, meaning, pattern and power!

This workshop seeks to make a contribution to alleviating the problem.

Introduction and background

There would be little argument with a claim that mathematics involves a study of pattern, structure, relationship and generalisation. Central to much of mathematical study is 'algebra'. This is so in the sense that algebra is a language for communicating, 'discovering' and proving many of the mathematical features and properties cited by Milton and Reeves (2002), on the one hand, and that 'algebraic thinking' is at the heart of mathematics generally, on the other.

For many students, algebra is about 'letters' and working 'algebraically' is concerned with learning 'what you do with the letters'! This can be baffling (to put it mildly) since, in such circumstances, the letters appear to have no meaning and the rules which govern the 'movement of the letters' may seem to come from nowhere. Little wonder that the mathematics education of such students is somewhat 'stunted' at least! If we think that such a situation is rare, then perusal of the work of Kuchemann (1981) and Booth (1981, 1982a, 1982b, 1984, 1986, 1988), among others, should inform us otherwise.

Most teachers would accept, from experience and study, that learning mathematics is not 'a single path activity'; nor is it linear. It is equally clear that the more mathematical content and 'relating' elements that are put in the way of a learner, the more likely it is that he or she can put enough of the pieces together and forge something of the connecting linkages, to produce at least a satisfactory and cohesive learning outcome.

A fundamental route to algebra is through the generalisation of arithmetic. In taking this pathway the student needs to realise that the 'letters of algebra' are the numbers of arithmetic, so to speak. The numbers of algebra are subject to the same operational behaviour 'rules' as the numbers of arithmetic. The rules have their genesis in the 'structural properties of arithmetic'. If students are unable to grasp the basic 'structural

congruences', as it were, between arithmetic and algebraic forms of the rules, difficulties arise. This is even before students come to terms with the very difficult and sophisticated notion of the 'letters' representing variables. The complexity of reaching an understanding of variable has been well documented by Quinlan (1992, 1996) who, additionally, suggests cohesive ways and means of developing such understanding (Quinlan, Low, Sawyer & White, 1993; Quinlan, Clark & Abrahams, 1997).

It must be made clear that this workshop considers the arithmetic to algebra link only, and the ideas seek to complement (not attempt to replace) the usual ways of approaching the teaching of algebra, particularly in the introductory stages.

The workshop has these primary intentions:

1. to indicate how teachers might help students feel at ease with arithmetic considerations, properties and relationships within which algebra can have a part to play;
2. to indicate something of the 'pattern and power' inherent in algebra, particularly with respect to articulating arithmetic relationships and generalisations.

The issue of developing and implementing an arithmetic background program to underpin and link to algebra has been considered thoroughly by Milton (2002). Broadly, the thrust of such an arithmetic program is to have students become familiar with 'numbers', number operations and behavioural properties of numbers 'under number operations'. This is not the principal focus of this workshop. The concern here is to be of help to students in coming to grips with pronumerals in contexts where arithmetic generalisations are made, and where students can be 'made aware' of the pattern and power of appropriate accompanying algebraic expressions or formula. Such a focus is particularly pertinent in the middle school or the early stages of secondary school when students are usually formally introduced to 'arithmetic with letters' although, as Wheeler (1989) demonstrates convincingly, even 'mature and seemingly capable' students need this form of reminder! Additionally, in the process, students are experiencing and exploring mathematically significant and often encountered arithmetic situations and becoming familiar with basic algebraic notation and conventions in settings where the links with arithmetic meanings can be made clear. As well, opportunities arise where mathematical proof can be discussed and considered.

It may be that the ideas presented in the workshop are considered novel: if this is so, then so much the better.

A starting consideration

Two aspects of arithmetic are needed to be accepted and 'internalised' by students when taking the arithmetic to algebra route, if a degree of relational and symbolic understanding is to prevail:

First, a recognition that 'equals' means 'names the same number as'. So, when it is stated that, for example, $2 + 3 = 1 + 4$ we know that '2 + 3' and '1 + 4' are alternative names for the same number. To highlight this meaning, as distinct from the often erroneously accepted notion that '=' is a 'do something symbol', it is helpful to record this as $(2 + 3) = (1 + 4)$. Without specific exposure to the contrary, the 'do something' signal of the '=' sign pervades student thinking well into the high school years (Herscovics & Kieran, 1980). In truth, equality (=) is an equivalence relation.

Second, a willingness to accept 'lack of closure', as Collis (1972) puts it; that is, accepting that, say, $(2 + 3)$ is a 'legitimate' name for a number in its own right, so to speak. There is no need to 'close' to call it 5. It is an acceptable name for the number that is '3 more than 2'. This acceptance, for example, enables us to understand the arithmetic statements

of equality:

$$4 \times (2 + 3) = (4 \times 2) + (4 \times 3)$$

$$(2 + 3) \times 4 = (2 \times 4) + (3 \times 4)$$

without a dependence on calculation.

Through establishing and accepting structural algebraic congruence with the stated arithmetic equivalences we are able to conclude that, say, $2x + 3x = (2 + 3)x$, where x is a natural number.

Ideas to be considered and activities to be undertaken in the workshop

- Whole numbers, consecutiveness and multiples (working with a hundred number chart and ‘tables’).
- Experiencing, discovering and expressing some generalisable arithmetic relationships (working with a hundred number chart, and arithmetic calculation).
- Applying algebraic statements (formula) to specific examples of the arithmetic relationships ‘captured’ by such statements. In this regard various ‘sums’ will be investigated.
- Throughout the session there will be an opportunity to discuss the ideas and illustrative examples presented.

Conclusion

Learning and understanding elementary algebra and algebraic thinking is a complex, multifaceted business which should begin in middle primary school, or even earlier. Students can begin to see arithmetic as having structural patterns applicable to ‘all number’. It is from this base that algebra can be developed as generalised arithmetic and can be seen by students to be a continuation of arithmetic. In this regard, the letters of algebra are the generalised numbers of arithmetic. Buxton (1984) put this nicely when he declares that there is no mystery in algebra for a learner at ease with arithmetic. This workshop attempts to assist the attainment of such ease.

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Technology enhanced mathematics education

Karim Noura

Introduction

The use of technology can enhance the teaching and learning of mathematics. This is a big statement and we should be aware of the effective use of technology in teaching mathematics. Contemporary mathematics text books are full of applications using technology and the Internet is full of mathematical games and activities; some of these applications are very interesting and reflect practical situations taken from real life while others are boring and poorly presented.

In the implementation of the *Curriculum and Standard Framework* (CSF) for Victorian schools, most of the mathematics books (for Years 7–10) have provided teachers and students with a number of computer and graphics calculator applications to develop problem-solving skills and to be part of CSF II Performance Assessment Tasks.

Also, as a part of VCE (2000–2003) mathematics, students have to successfully complete the outcome 3 for all units (1 to 4). ‘On the completion of this unit the student should be able to select and use technology appropriately to develop mathematical ideas, produce results and carry out analysis in situations requiring problem solving, modeling or investigative techniques or approaches in the area of study.’

We can notice clearly that technology is fully integrated (in line with VCE 2000 recommendations) into most of the mathematics books in Victoria, which refer to graphics calculators (which are now a feature of the course), spreadsheets, dynamic geometry software and several graphing packages.

Mathematics teachers should consider carefully before deciding to use any kind of information and communication technology in the class. It should be an integral part of the lesson plan and not just a prestigious feature to add on to the end of the class. Any mathematical application using technology should be well prepared and well designed to cater for the needs of the students and should be based on what the students already know in mathematics. Also, teachers must be well trained for the kind of technology that they are going to use in the class. They have to take into consideration the potential of the new technology and how we can use it to achieve the stated goals and outcomes of the class.

Students should be encouraged by teachers to learn more about this technology and to discover its features. They should be well motivated and encouraged to use the technology in a productive way and not to consider it as a game or an entertainment tool. They should be able to use it to solve hard problems and to handle more mathematical information. Also, they can use it to save and present their work in a very attractive way.

In this paper, I will look at a number of different strategies to solve a problem using different kinds of technologies.

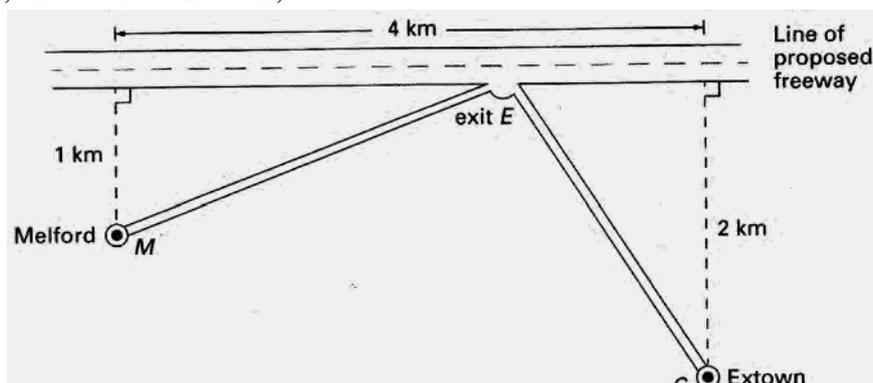
I have included with this report a copy of an application task that I have used in the classroom. This application contains different tasks to cater to the needs of students in different year levels and to respond to the curriculum assessment tasks that must be achieved in both middle and senior school levels.

The freeway exit

The problem

Building roads can be very expensive, so civil engineers try to make them as straight as possible. The line of a new freeway as it passes by Melford and Extown, two small towns, is to be as straight as a ruler. Only one exit is proposed for local residents, and it is to be joined to both towns by two straight roads.

Where should the exit be put to minimise the total length of road from Melford, M , to the exit, E , and back to Extown, C ?

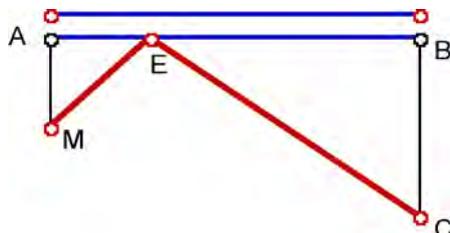


Hints

- Let x (km) the distance of the proposed exit to the point opposite to M .
- Let y (km) the length of the road from Melford to Extown; calculate y in terms of x .
- Set up a table to show the values of y for different values of x .
- Can you find the minimum value of y ? Then the value of x ?
- Use calculus to solve this problem.
- Use graphics calculator to verify your results.
- Use an *Excel* spreadsheet to justify your results.
- Use classic geometry for more proof.
- Use *Geometers' Sketchpad* software to illustrate your work.

Working through the problem

We need to establish a single exit (E) on a freeway to serve two towns M and C , which are located 1 km and 2 km respectively off the freeway. The road MEC should be of minimum length, in order to minimise the cost of the project.



The problem is sufficiently concrete, well connected to problems that students have previously faced in mathematics as well as interesting and solvable. I will also encourage the students to discover different aspects of the problem and to ask more relevant questions, taking advantage of the potential of the available and affordable technologies.

I found this problem in a Nelson Mathematics textbook without any instructions. They leave it to the teacher to discuss the situation with the class and to try together with the students to find a strategy to solve it. So it is up to the teacher to set up varied tasks according to the student's skills level and abilities.

In response to the question: 'Where should the exit be in order to minimise the length of the road between M and C ?' we may get the following suggestions from students:

- The exit (E) should be in the middle of AB (where A is opposite to M on the Freeway, B is opposite to C).
- The exit (E) should be closer to A .
- The exit (E) should be closer to B .
- The exit (E) should be on A .
- The exit (E) should be on B .

So, how can we make the correct decision about where this exit it should be established and where should we start?

Students should be able to draw some scenarios or plans on the class-board or on their workbooks with the assistance of the teacher and by using classic drawing equipments. They have to calculate to find all the necessary measurements, compare their plans and results, and come up with some suggestions.

The teacher may ask each student to try different possibilities for the location of the proposed exit and come up with best solution. This means that they need to do more calculations. A question will present itself here: how can we handle all the calculations and the numerical results?

A good idea is to advise students to set up a table of values to carry out all these calculations and numerical results.

Table of values

Teachers should be able to instruct students to set up a good table of values representing all the possible values of $x = AE$ and therefore to carry the numerical values of: $a = ME$, $b = EC$ and $y = a + b$.

Consider $x = 0, 1, 2, 3, 4$ respectively.

Use Pythagoras' Theorem to calculate the lengths of the parts a and b of the road.

Calculate the total length of the road ($y = a + b$) connecting M and C through the exit (E).

By using Pythagoras' Theorem we find that:

$$a = \sqrt{x^2 + 1} \quad \text{and} \quad b = \sqrt{(4-x)^2 + 4}$$

x	a	b	$y = a + b$
0	1	4.472	5.472
1	1.414	3.605	5.019
2	2.236	2.828	5.064
3	3.162	2.236	5.398
4	4.123	2	6.123

The table of values shows that when $x = 1$ the road is of the minimum length ($a + b = 5.019$).

Are we sure about that? 'Yes!' some students said.

Can we be more accurate?

Some students suggested to consider some numbers 'around 1' ($x = 0.5$ and $x = 1.5$).

So a new table of values is required, which will produce new values of $a + b$.

Perhaps we need to consider closer values of x 'around 1' to be more precise.

This means that a new table of values (maybe more than one table), more work and more calculations are needed—and students are usually not happy to do that.

The question arises: can we use technology to ease the problem solving process and to quickly find the solution? Yes!

What kind of technology is bestsuited to this?

What kind of technology is available and accessible?

Computer programs such as *Excel*, *Cabri Geometry* or *Geometer's Sketchpad* and graphics calculators are very helpful to solve this problem.

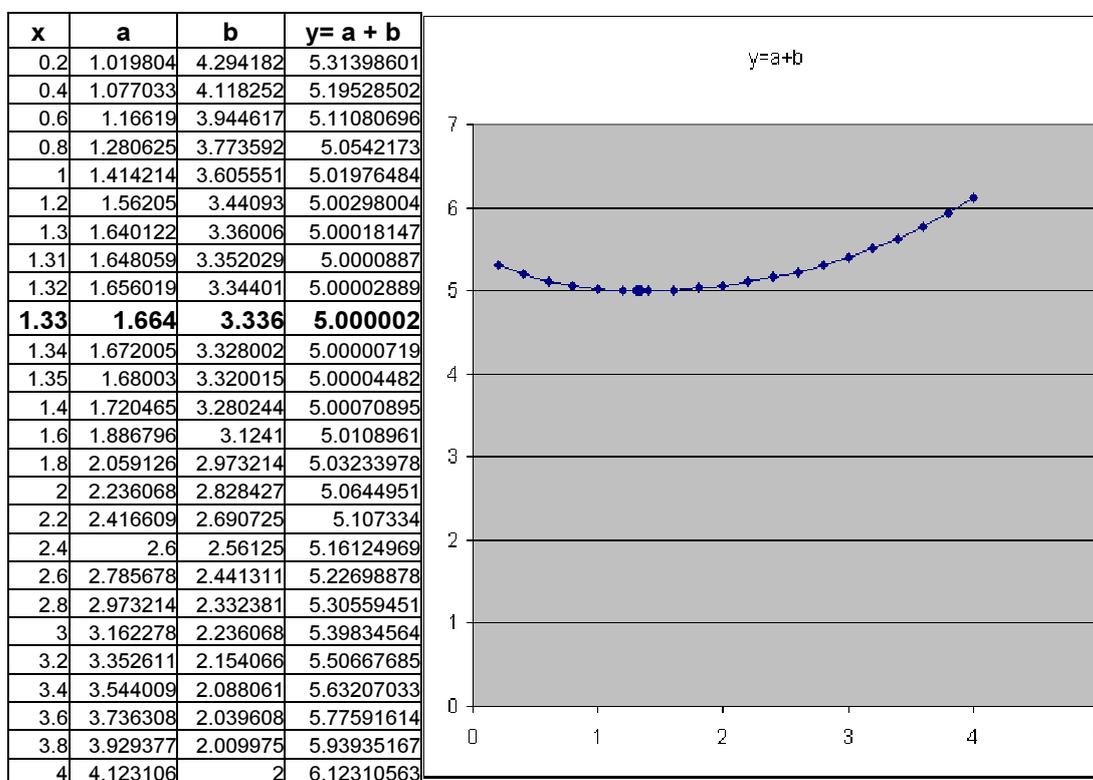
Spreadsheet

Excel spreadsheets are available on most school computers; most of the students are familiar with it and they should be easily able to conduct this problem with the support and the supervision of their teacher.

The following steps are recommended:

- Set up a spreadsheet with Excel.
- Use Pythagoras' Theorem correctly.
- Use the *Fill down* command.
- Use *Insert* command to insert rows when they are needed to be more accurate.
- Check the table of values to find the minimum length of the road $y = f(x)$ connecting *M* and *C*, and therefore decide on the best position of the exit (*E*) where $AE = x$.
- Draw the graph (using the chart wizard) to show the variation in the values of $y = f(x)$.

Where is the best exit on the freeway?



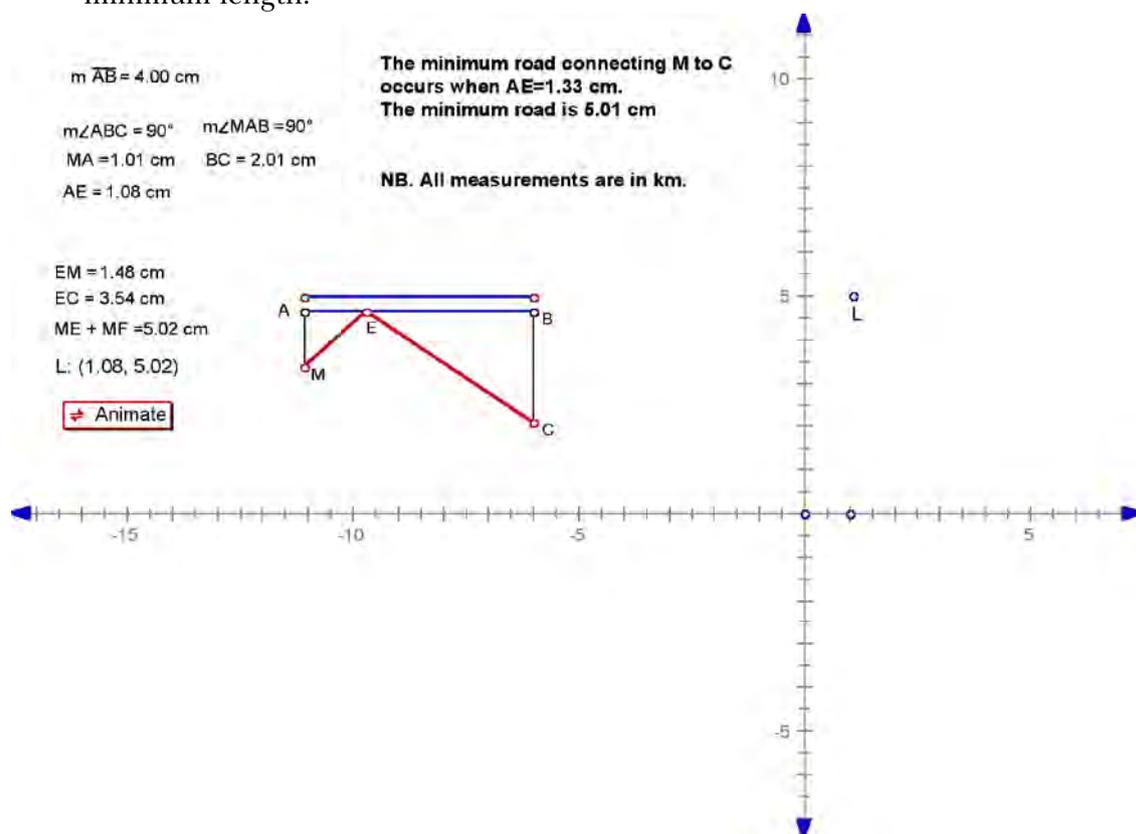
The result is quick and accurate: $x = 1.333$ km and $y = 5.001$ km.

Dynamic geometry

Another type of technology can be used to conduct the problem. *Geometer's Sketchpad* is a very interesting resource for dynamic geometry and measurements. It is available in most schools and it will give an extra bit of life to the problem.

The following steps are recommended:

- Let the students draw the diagram.
- Let them use the construction menu to construct lines, line segments, and circles, parallel and perpendicular line.
- From the display menu they may use the *Hide objects* function to hide geometric objects that you do not want to see in the diagram.
- Find and show all measurements that they need.
- Encourage them to use the animation function to check all possible situations.
- Encourage them to use the trace point function to show the variations of the roads' length.
- Find and show the minimum value of the length of the road.
- Advise students with high skills and students from Year 10 and above to plot the length of the road $y = f(x)$ versus $x = AE$, therefore allowing them to find the minimum length.



In my experience, students enjoy this kind of activities. They were very impressed with the results that they found as well as the efficiency and the time saved by using this kind of technology.

Graphics calculators

Up to this stage of analysing the problem, we are encouraging students with different abilities to take advantage of the available technology and its potential. They are encouraged to discover more aspects of the problem. So they can be more efficient and they can save lots of time.

Again, taking advantage of another available technology such as the graphics calculator, we ask students some harder questions and we will set up new tasks for them to solve.

Hints:

- Write the length of the road $y = a + b$ in term of $x = AE$.
- Use the graphics calculator to justify the work that they have done.
- Show the table of values.
- Draw the graph of $y = f(x)$ in a suitable domain.
- Use the trace function to check different possibilities of y .
- Show the minimum value of y on the graph and display it.
- Use and discover different features of the graphics calculator that you have (TI-83 is recommended).

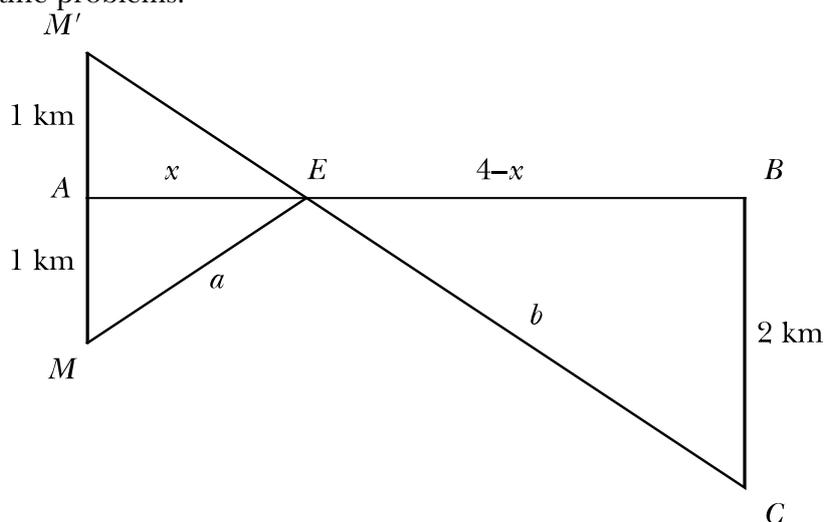
The outcomes will be just perfect and correct. Calculators can be used any time whenever needed.

Classic geometry

So far, use of technology has been the highlight of this problem; but, I would like to remind our students about some aspects of classic geometry, which are very helpful and very important to give more proof to the problem solving exercise and to confirm the results, which we have found by using various types of technology.

The classical concept of similar triangles can be used to solve the problem of the Freeway exit.

Students should have full understanding of the similar triangles properties, ratios and proportions. Also, they should be able to use their knowledge to apply related procedures to solve routine problems.



Let the point M' be the symmetric of M on the other side of the Freeway.

The right triangles $M'AE$ and CBE are similar and $\frac{x}{4-x} = \frac{1}{2}$.

Therefore $3x = 4 \Rightarrow x = 1.333$ km. That means $AE = 1.333$ km.

Therefore the straight line $M'C$ is the shortest line connecting M' and C .

Since $EM = EM'$ (E is on the axis of symmetric of MM'), then $ME + EC = M'E + EB$

Is the minimum length of the road connecting M and C ?

The Freeway exit must be on the point E .

Teachers should take into consideration that this part of geometry could be difficult for many students especially at Year 9 level. They must prepare the class adequately before commencing this task.

Calculus

Again we can set up another task and we can ask a harder question.

Use calculus to solve this problem. Only higher-level mathematics classes will be able to deal with the function:

$$y = \sqrt{x^2 + 1} + \sqrt{(4-x)^2 + 4}$$

where $y = f(x)$ represents the length of the road connecting M and C , in terms of $x = AE$.

This function has a minimum value when the derivative $f'(x)$ equals zero.

This is a very hard calculation:

$$f'(x) = \frac{x}{\sqrt{x^2 + 1}} - \frac{(4-x)}{\sqrt{(4-x)^2 + 4}}$$

$$f'(x) = 0 \text{ when } x = \frac{4}{3} = 1.333 \text{ km.}$$

This result confirms that all the work that we have done above is correct, but it needs lots of time and needs students with high skills in calculation in order to avoid any errors when using this method. Remember that the TI-83 graphics calculator is very helpful for this task. It can be used to justify this result.

Conclusion

The use of technology has changed the way mathematics is taught in the classroom. No single technology is best to achieve our goals and to solve mathematical problems, but we should take advantage of the technology that is available and accessible.

The use of scientific and graphics calculators is normal in mathematics classes; every student is expected to have a calculator in the class. Graphics calculators are handy items, allowing students to do lots of work and to save the work and come back to it later — but they cannot print without another kind of technology (computer and graphlink cable).

Students can have access to *Excel* in virtually any computer around the school or at home. They can complete lots of work by using *Excel*, especially numerical operations; they can deal with large amounts of data and produce graphs, save it and print it as a nice presentation.

When thinking of using technology in the teaching and learning of mathematics, it is important to remember that it should help to make the problem-solving easier and to make the mathematics class more interesting and more productive — making topics easier to students to comprehend. Using technology may help to maintain students' interest despite their pre-existing perceptions of that technology. Therefore, it is important to give the students the opportunity to experiment with the available technology and discover its potentials. Experimenting in a dynamic environment allows the students to explore

different aspects such as measurements or comparing and changing figures. Also with the use of the appropriate technology, students will be able to approach problems of maximum and minimum in the early school years.

Using a spreadsheet does not change the mathematics learned, but it does add other dimensions to students' work, such as speed and ability to do many numerical operations simultaneously, ability to vary numbers quickly, to compare, record and present lots of data in different ways.

So as teachers, we have to be up to date with the potential of new technology, and be able to select and prepare activities where use of technology can enhance mathematical tasks. We also need to maintain a clear vision of what is desired of the technology, and be responsive to it but not governed by it.

The Ethnomathematics website

Kay Owens

Charles Sturt University

The Glen Lean Ethnomathematics Centre in Papua New Guinea has set up a website through four countries' collaborating. It brings the diverse mathematics of this country to the world and assists its own appreciation of their culturally rich mathematics. This is the story of how the website was set up. It will be viewed during the presentation or you can look at it yourself in the address given below.

Its origin as a posthumous gift from Glen Lean

In 1994, Glendon Lean died just weeks after being bestowed with his doctorate for his life's work on the counting systems of Papua New Guinea (PNG) and Oceania. In 1999, the hardcopy collection of photocopies and some disks from Glen Lean's estate were transported back to Papua New Guinea. Half this material found its way to the University of Goroka. Glen collected data for twenty-two years and the appendices of his thesis had twenty-two volumes of data from each of the provinces and the neighbouring Melanesian countries. This is the story of bringing this wealth of information to the rest of the world.

The vision of the Ethnomathematics Centre

The Head of Mathematics and Computing Science, Dr Musawe Sinebare and two staff, Dr Wilfred Kaleva and Mr. Rex Matang both with degrees in ethnomathematics decided to set up the Glen Lean Ethnomathematics Centre. They applied and received funds from the University of Goroka (UOG) to set up the centre. It is housed in an historical building in the grounds of the University. It is pleasantly and comfortably well-equipped.

Beginning the challenges

In 2000, Chris, my husband, and I spent a month at the centre fascinated by the papers dating back into the 1800s. The oldest record is from the 1600s. The nationals who helped later were fascinated that Europeans went bravely from village to village around 1900 recording their experiences of the people and the languages. After a month, we had started up a database of the counting systems, produced a bibliography of hundreds of papers, and made a copy of the thesis from the disks. This required a visit to get coloured

copies of the maps held at the PNG University of Technology (Unitech) in Lae where Glen did his doctorate. All the staff with whom we had worked in the past assisted us as much as possible. After finally finding there were no coloured photocopies and locating the two coloured printers in town (both running out of ink during our visit), we obtained copies by scanning and printing. In our travels, to scanner was damaged and the adapter cord (used also for musical instruments) was stolen and not easily replaced. We also made copies of Geoff Smith's thesis on the counting systems of Morobe, a province of PNG.

The electronic copy of the thesis and appendix volumes was originally produced on a mainframe and then several Macintosh computers; subsequent transfers of data to an IBM-compatible computer meant that the data were not easy to work with. I learned many tricks to transfer this information into our current database. We also scanned maps and other pictures that had only been in hard copy. These pieces from the thesis I combined together to make both an electronic and hard copy of the thesis.

To make the bibliographies required sorting out who was publisher, editor and author on German or Dutch documents. This was surprisingly tricky. Chris filed all the documents alphabetically into the filing cabinets as well as developing the bibliography.

Mid-year, Geoffrey Saxe, who had visited the Oksapmin on several occasions, returned to open the Centre. He and many others like Alan Bishop and Ubitan D'Ambrosio have continued to show their interest in the centre and encourage the director Rex Matang and Wilfred Kaleva.

Beginning the database

The PNG University of Technology houses the Architectural Heritage Centre and they had produced a *Filemaker Pro* database of the carvings and other artefacts of the Sepik and Ramu river areas based on Mac Ruff's long term studies. It is beautiful. I thought that it would be good to produce a similar one of the counting systems. Thanks go to Carol, who previously maintained the MANSW database, for an initial few hours instruction.

I began the task of structuring the database. Counting systems of 700 languages have been recorded. For each language, there are from one to six sources of the counting systems with slightly varying word lists plus from half a page to five or six pages of information that Glen had collated from the original records, field visits and questionnaires completed by Unitech students and teachers. In a separate part of each volume, Glen had recorded the analyses of the counting systems. We needed the hard copies held at UOG's library to finalise our data entry. The word lists and words are complex and of course foreign to us, although I gained a good understanding of the structure of the systems and could often use the patterns of words to straighten up the records. For two short periods, staff members at the University of Goroka were employed to understand the database and assist with the data entry. Rex helped too.

The website vision

Out of the blue, I received a call from Nancy Lane, then Communications Director of Pacific Resources in Education and Learning (PREL).

'Do you know about ethnomathematics in Papua New Guinea?'

'Yes.'

'Would you like some funds to share this information on our website?'

'Yes, but I have to check if the Centre Director wants to do this. They have a website at

the University of Goroka.’

‘How much would you need?’

With approval from the Centre Director and UOG staff, we applied for the funds through PREL that came from the US National Science Foundation. Some went to UOG and some for my expenses to UWS. There was, fortunately, twice as much as we expected since the task proved to require more visits from me than we had first anticipated.

By email, Rex, Wilfred, and I worked out what we needed to do and approximate costs for equipment, travel and accommodation. By this time, they had the assistance of Kiyu, the Japanese JICA volunteer who advised on software and hardware. We had to try to retrieve the other half of Glen’s hard copy data. We needed to scan in papers. We needed to set up the website with all the necessary pages. We needed to connect the database into the website. At this stage we were using *Filemaker Pro version 5*. Kiyu said we needed to work with Macromedia’s *Flash* and *Dreamweaver* software, as well as using Javascript. He had been able to read the Japanese manuals of Filemaker and the Macromedia software and had worked out that we should upgrade *Filemaker 6* and also needed one computer dedicated to the database, and another for the website data connected to UOG server.

Multinational work on the website

In October 2002 I made a six week visit and met Kiyu. Six weeks is a bit too long in the Lodge without easy phone access but I could enjoy the company of a number of nationals (often my ex-students from Unitech), also away from their families for long periods working on Goroka projects such as the new library and the roads. Martin Imong was a temporary lecturer and able to help me and Rex with the database entry. He was both a Mathematics and Computer Science teacher. He could do the difficult work of entering the data from other languages, but complex lives interfere with time on task in PNG. Kiyu and I communicated by doing things on the computer. Tok Pisin was the common language between us all. We employed a lady to help the secretary with the scanning of papers. By the end of the six weeks, Kiyu had crafted the basis of the homepage. I had selected sections of the papers and earlier student projects to be scanned. We finished the entry of 700 language counting systems. I had a brief one night chance to use the updated version of *Filemaker Pro 6 unlimited* before the disk was returned to the post office for a three month ‘red tape’ delay about customs money, despite the software being for an educational institution.

During this visit, Francis Kari went back home to Manus to collect the remaining documents from Glen. The story goes that the salt water trip and insects had destroyed them. We proceeded to replace the data from SIL sources, the Goroka library and, on my return, through UWS database systems, but some original photocopies (probably collected in the Netherlands, England, Germany, and the US from both libraries and government archives) and other material were irretrievably lost.

A couple of three day visits were possible over the next year when I was in PNG for other projects. This brought Kiyu back on task briefly. Despite good intentions between visits, Rex and I found little time to continue work on the website. We did however write several papers over the years.

During one of these visits, we found out that UOG needed to upload all their website to the Datanet server in Port Moresby because the lines between Moresby and Goroka were unbelievably slow and often down.

Time is running out for finishing the task

Another phone call from Nancy: ‘Do you need to go back to PNG to finish off the website? Time is running out fast. We can find some more money. I’m likely to leave this job by mid-year.’

‘Yes, I do need to go back.’

Kiyu had left. The task had proved to be too difficult for him to finish despite his incredible amount of self-instruction. I had three weeks.

After breaking into his temperamental computer, I taught myself how to use *Dreamweaver* to continue the website connections and planning. I was able to organise further scanning of documents and Wilfred’s wife Roa helped collate the jpeg images of the scans while I wrote a brief description for approximately one hundred papers. I also wrote some elementary school activities.

At the end of the trip, I had some IT help back in Australia. I sent off the improvements for uploading. However, before this could be done, the disk and one of our computers, printer, and UPS were stolen.

With GLEC agreement, PREL uploaded the website onto their computers but the lack of *Filemaker Pro* connection knowledge was still a problem.

By August 2004, the new JICA volunteer, Masa, had settled in and had used the May version and remade links so the basic website was improved. I sent up a new copy of the website with a few more papers included but it did not reach Masa’s hands. In August, Nancy and Rex said, ‘Kay you must go back.’ I hesitated as I did not think I could at all help with the IT work that now needed to be done. However, my last trip, which lasted a week, brought huge success. Masa had already read the manuals and with his years of experience had both improved the UOG website (there was another JICA volunteer helping with that) but also had made the connection between the database and the main website. After a day, the two parts of the basic site were live. I had a chance to find out what was wrong with the database when it was up on the Web and to make corrections to the controlling scripts. Only the search engine had to be redone on the main website. One task was to re-enter all the counting system tables. The spacings for columns were all lost and so the complex number words made the lists unreadable on the Web. With the help of Roa, I began the tedious task of checking each of the counting system lists.

It took a week for the files to be uploaded to the Moresby Datanet site. We had more than 5000 files on the site.

Back in Australia I tested the site at a distance. I was particularly worried that the scanned papers would not be readable or printable — but they were working.

Masa proved an invaluable member of the team at the last minute. Our four nation collaboration and the stretching of our ITC skills has expanded everyone’s knowledge of Papua New Guinea counting systems enormously. There is still a huge amount of material that could still be added to the website.

During March, Rex went to his village area and interviewed a number of children learning to count in vernacular and Tok Pisin. In April, I went to a couple of elementary schools in the Highlands to trial some activities similar to those used in NSW’s Count Me In Too project. We were really pleased with how the use of the vernacular languages was helping children understand number. The website should assist teachers in elementary schools if only they had access. However, the languages in PNG are changing so much that the counting systems are rapidly changing from those recorded in Glen’s work.

The future

There is so much that we can investigate and record and reintroduce into the curriculum in other areas of mathematics besides counting. Each of the 800 cultures have different measurement and space knowledge. In PNG there are living counting systems that are not base 10 systems. The NSW Stage 4 syllabus makes particular reference to these systems.

Now you can have a go at exploring the website yourself at www.uog.ac.pg/glec.

Technical details

Counting system databases

The structure of the counting system database involves seven related databases.

1. The menu page is really just an introduction and links to the language database. It also gives more details about abbreviations and links to the other databases.
2. The languages database. This database has three layouts:
 - (a) interesting examples gives buttons to connect to 30 common or interesting examples of different cycle systems (systems are generally not base 10),
 - (b) the form layout which links every record to the general background of how Glen collected the data, a page on the province available to every language in that province, a brief summary called important notes, a map of the province in the country, and some other details such as the language classification. Each record is connected through a portal to one or more counting system pages.
 - (c) A search page.
3. There are two layouts for the counting systems database.
 - (a) The form layout lists Glen's analysis of the counting system in terms of the frame or counting words from which all other number words are formed, the patterns of number word combinations by which the larger numbers are made, and the cycles of the system (like a base). It also contains list of the counting words as collected from different sources. A large part is devoted to the details that Glen collected or recorded on the language and counting systems. In particular some languages have extensive notes on classifier systems in which different objects are counted by different words. Another section gives the links to body-parts and to cultural contexts for counting. The sources of data and updates are recorded. In order to assist the search as many different variations of the language and dialects are listed. At the moment this is an incomplete section.
 - (b) A search page.
4. Each counting system page is linked to one of the maps that gives the regions in Papua New Guinea where you can find the same cycles. It is notable that many start off with a 2 cycle although they may also have a 5 and even a 20 cycle system too. There are 4 and 6 cycles and 5 cycle, 5 and 20 cycle or 5, 10 and 20 cycle as well as body-part tally systems.
5. Each counting system is also linked to a map indicating where the system is found in the province. Neighbouring systems are indicated on the same map.
6. If the system is a body-part tally, then there is generally a picture showing which body parts are used in the tallying.

Before searching, the records need to be refreshed using the appropriate button.

The website

Dreamweaver was used to make the basic pages. Kiyu also used extensive Javascript and *Flash*. This helps in particular with the long list of counting systems given in an *Excel* format. This banner was put at the top of each main page requiring each to have the drop down menus and links activated.

Having so many lists of papers and so many papers required careful thought to ensure there was reasonable links if appropriate to Glen's work or to the centre's subsequent collections.

Several theses and key papers can also be found on the website. The GLEC team have also been captured on videotape counting in their own languages.

Where to from here

We hope that indigenous people from around the world will be encouraged to develop and preserve their mathematical understandings. We hope that the material will be available for elementary teachers in remote villages at least if they visit regional centres. We also hope that Australian and New Zealanders will recognise the rich diversity of their near neighbour. The struggle for recognition of Melanesian culture to the west in West Papua, Timor and the other Indonesian islands is a human rights issue for these people.

The site has links to the ethnomathematics sites in Australia and Hawaii. We hope other mathematics education sites might make links to it too.

Further research

Rex and the students at UOG are making efforts to continue research. There is much work needed on other aspects of mathematics. Music, measurement, space, time, sailing and traditional technologies all have extensive mathematical thinking that is quite different to our western mathematical thinking. We need to record these and maintain them before it is too late.

Refreshing ideas for secondary mathematics lessons

Cyril Quinlan

Australian Catholic University

This sharing of learning/teaching ideas will range across a variety of topics including a model for introducing indices, links between mathematics and music, sparking an interest in trigonometry, practical ideas with squares and circles, a somewhat novel way to introduce quadratics, how to avoid the trap of trying to introduce algebraic conventions from generalisations about patterns, and some models for understanding pyramid volume.

Model for indices

(See Quinlan et al., 1993)

Doubling with cubes

Class groups build a series of models showing the effect of starting with one cube (such as Cubit Cubes or Multilink Cubes) and doubling it once, twice, three times, ... , to six times. Each group makes a series of 'tents' by folding cards (rectangles about 3 cm by 6 cm) in half. On the tents they write, for example, on one side 23, on the other 8, for the case of one doubled 3 times. A preservice student last year created three-sided 'tents' and added the explicit $1 \times 2 \times 2 \times 2$ on the third side.

The groups will probably surprise the teacher when they are directed to explore relationships between the results of doubling, and use the 'tent cards' to express their discoveries in two forms of an equation. For instance, they can model $2 \times 8 = 16$ by considering two lots of the model they have for 8. When they turn tents around, they see the equation as $2^1 \times 2^3 = 2^4$. They quickly settle on the index laws for multiplication and division. Also they accept that 2^0 is what you get when you multiply 1 by 2 zero times, giving just 1. A graph of $y = 2^x$ can be generated and the process can be repeated for, say, multiplying 1 by 3 several times, and by 4 several times.

Music and mathematics

It is beneficial to link mathematics lessons to the interests and hobbies of the students. Music offers links to several mathematics topics, as is now shown.

Indices

Western musical notation usually is based on notes which are given one of seven letter names ranging from A to G, with the frequencies increasing accordingly. We will denote the next higher group of notes as A¹ to G¹, and the next lower group as A₁ to G₁. Some of these are shown in Figure 1.



Figure 1. A piano keyboard.

The eight notes from C to C¹ form an *octave* and the frequency of C¹ is twice that of C. The frequency doubles as you go up an octave and is halved when you go down an octave. We will take 256 hertz (or 256 vibrations per second) as the frequency of the note known as middle C, since it is the main frequency produced by striking the C key which is found near the middle of a piano. Students can work out the frequencies of the octaves of middle C, remembering that $256 \div 2^3$ means you double 256 three times, and $256 \div 2^1$ means you divide 256 once by 2. They could go on to frequencies of octaves above and below the note G of frequency 384.

Note that a frequency standard now commonly used has A¹ as 440 hertz and frequencies of other notes are calculated by multiplying or dividing by the twelfth root of 2. As shown in Figure 1, there are twelve intervals (when you include the black keys of a piano) within any octave.

Ratio

The timing for different types of notes is as follows, where each type of note lasts for one half of the time for the previous type in the list: semi-breve, minim, crotchet, quaver, semi-quaver, demi-semi-quaver. Appropriate questions could be:

1. If a trumpeter is playing a tune at a speed which requires that each crotchet is held for half a second, how long should the following be held:
 - (a) a semibreve?
 - (b) a quaver?
 - (c) a demi-semi-quaver?
 - (d) a minim?
2. Study the music sample in Figure 2 and answer the following questions.



Figure 2. Music sample.

- (a) Discuss how playing each of these bars requires the same amount of time (if the same pace is used for all four bars) and write notes about the notes, such as: 'In bar P, 1 minim and 2 quavers = $2 + \frac{1}{2} + \frac{1}{2} = 3$ beats'.
- (b) Tapping. Try tapping out the rhythm for bars P to S. First tap a regular 3 beats to the bar rhythm and, when ready, tap out the timing for the bars.

Musical harmony

The main frequency (or number of vibrations per second) of a sound wave determines its *pitch* — how ‘high’ or ‘low’ it is. Most musical notes are composed of a mixture of frequencies. The lowest frequency of a musical note is called the *first harmonic* or the *fundamental frequency* and the notes with multiples of this frequency are known as its *second harmonic* (twice the frequency of the fundamental), *third harmonic* (three times the fundamental frequency), and so on.

Usually many harmonics are produced by each note of musical instruments, in contrast to just the fundamental being produced by striking a tuning fork, as shown in Figure 3.

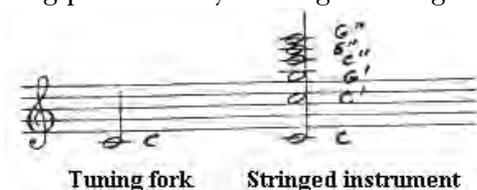


Figure 3. Contrasting note produced by a tuning fork and ‘chord’ produced by a string.

Experiments.

1. Try this experiment with a well-tuned piano, if possible. Depress the loud pedal (to lift dampers from the strings) and strike C_1 . Listen for any harmonics of C_1 . Slightly depressing the key for a chosen harmonic when striking the key for C_1 helps this stage of the investigation. Discuss.
2. Experiment with a well-tuned guitar or violin, if possible. The open strings of a lead guitar when plucked give the fundamental notes E_1 , A , D , G , B^b , E^b , while a violin is tuned to G_1 , D , A , E^b . Try the following:
 - (a) Pluck any open string, stop it vibrating, and investigate whether you can see/feel/hear another string vibrating. Is it vibrating with the fundamental frequency of that string or of the string first plucked? If not, which harmonic is it?
 - (b) Repeat with a finger on the string being plucked (e.g., on the fifth or fourth fret for a guitar).
 - (c) Set a string vibrating and lightly touch it at its centre (or some other fractional spot) and identify what you hear.

Exercises such as the following come to mind:

1.
 - (a) List the frequencies for the first 8 harmonics of C_1 .
 - (b) Identify which of these harmonics are octaves apart.
 - (c) With the help of Figure 1, match the following notes to the harmonics: C , C_1 , C^b , G^b , G , B^b flat, E .
2. Study the frequency data in Table 1.

Table 1. Frequencies of harmonics for C_1, E_1, G_1, C, E .

NOTE some only approximate	C_1	E_1	G_1	C	E	G	C^b	E^b	G^b	B^b	C^b	D^b	E^b	G^b	B^b	D^b	G^b
harmonics of C_1	128 1st			256 2nd		384 3rd	512 4th	640 5th	768 6th		1024 8th	1152 9th	1280 10th	1536 12th	1920 15th	2304 18th	
harmonics of E_1		160 1st			320 2nd			640 4th	960 6th				1280 8th		1920 12th		
harmonics of G_1			192 1st			384 2nd			768 4th	960 5th		1152 6th		1536 8th	1920 10th	2304 12th	3072 16th
harmonics of C				256 1st			512 2nd	768 3rd		1024 4th		1280 5th	1536 6th		2304 9th	3072 12th	
harmonics of E					320 1st			640 2nd	960 3rd				1280 4th		1920 6th		

- (a) Musicians know the interval from C_1 to G_1 as a *fifth*. How many harmonics, of those shown in the table, do these two notes, C_1 and G_1 , have in common (with the same frequency)? An interval of a fifth appeals to the ear perhaps more than any other interval and so is found in the music of many cultures that make music with more than one note being sounded together.
- (b) Calculate the highest common factor of the fundamental frequencies of C_1 and G_1 .
- (c) Calculate the ratio of these frequencies (128:192) in simplest form.
- (d) Calculate the lowest common multiple of the fundamental frequencies of C_1 and G_1 .
- (e) (i) Write how this lowest common multiple is related to your answer to part (a).
 (ii) Write how your answers to parts (c) and (d) are related.
- (f) Which other pairs of notes are likely to produce appealing harmonies? Why?

Introducing trigonometry

(See Quinlan, 2004a)

Surely, the most important objective when starting a class on a previously unknown branch of mathematics is to ensure that they enjoy and appreciate the significance of this new aspect of mathematics. A great way to involve the class in group work at the very start of a unit on trigonometry is to provide each group with a 45-45-90 set-square, a drinking straw, some Blu-Tack, and a metre rule, and direct them to work out some way to use all these materials to measure the height of the classroom. Hopefully, some group(s) will place the viewing instrument (with the drinking straw attached to the hypotenuse of the set-square) not on the floor but on a horizontal surface such as a desktop and adjust the position of the desk until a suitable line of sight is established. The most common next move is to measure the height of the table, and the distance to the wall from the point on the floor below the bottom end of the viewfinder. Noting that the set-square provides an isosceles triangle, the height of the room is found by adding these two measurements (Figure 4). After using this idea for several years, one group a few years ago improved dramatically on this first method. After positioning the setsquare for viewing up to the top of the wall, the group found the point on the floor where the line of sight downwards hit the floor. Then they needed just one measurement, namely the distance from this latter point to the wall (Figure 5).

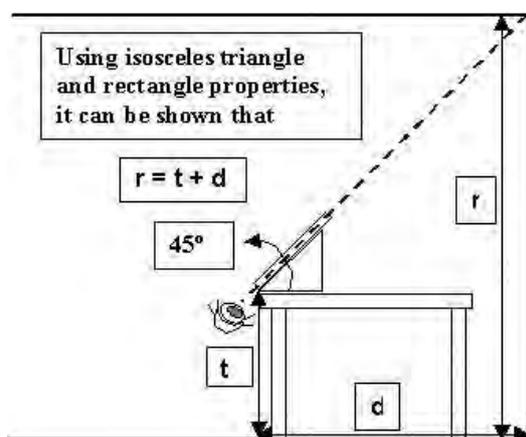


Figure 4. First method.

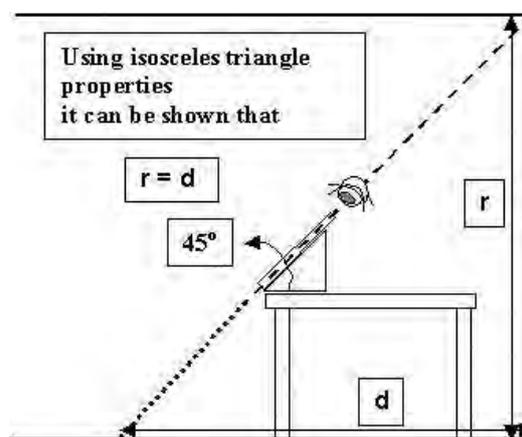


Figure 5. Second method.

Next, challenge the groups to repeat their measuring of the height of the room by providing the groups with 30-60-90 setsquares. Examining sets of similar 30-60-90 triangles should convince them that the ratio they need to use for either 30° or 60° (namely, opposite over adjacent) is constant regardless of the size of the triangles. It is time now for the introduction of the technical terms opposite, adjacent, and tangent ratio and the definition of tangent of an angle. If the students meet such terms in this reality context, there is some chance that they will appreciate the relevance and usefulness. Stay with tan for some time before introducing problems which require sin or cos.

Squares

Use a grid page as a time-saving device for this exercise. Have groups draw squares of different side lengths and measure the following two things to the nearest millimetre:

1. the perimeter;
2. a diagonal (the longest straight line to fit in the square).

They calculate the ratio perimeter/diagonal to 3 decimal places using a calculator and the results are tabulated, giving approximations to the irrational number equal to twice the square root of 2.

Circles

Circumference

Measuring circular objects and tabulating the ratio of circumference to diameter is a worthwhile introduction to π and the formula for circumference.

Recommended methods are to roll circular objects (e.g., coins, CDs, plates) along a scale, and to use a flexible tape measure to record the circumference of larger circular objects such as rubbish bins. Some of the author's preservice students last year presented the brilliant idea of lining up three 20 cent coins, touching and in a straight line, and then rolling another 20 cent coin along beside them.

Reality applications could include bicycle wheels with gears, and marking out a 400 m athletics track with two straight sections each of 85 m and two semicircular sections, knowing that athletes usually run about 300 mm away from the line markings.

Area

A very helpful exercise when studying the area of a circle is to take a tightly-rolled streamer and consider one circular face while the streamer is rolled up, and when it is cut down along a radius from the top to the centre to give a triangular shape, as shown in Figure 6 (See McSeveney, Conway & Wilkes, 2004, p. 390).

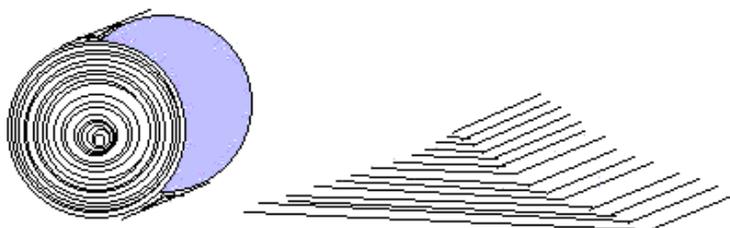


Figure 6. Insight into the area of a circle.

Angle properties

Set up a cyclic quadrilateral with one side extended using a geometry software package, as in Figure 7. By dragging points, facts and relationships about angles can be noted.

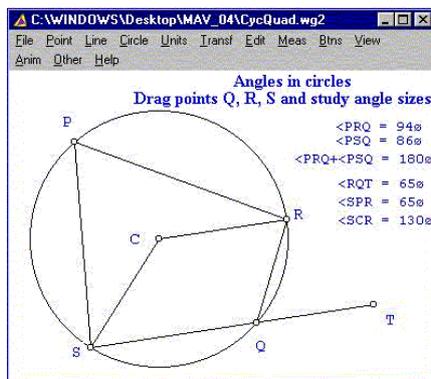


Figure 7. Cyclic quadrilateral, using Wingeom.

Introducing quadratic equations

(See Quinlan, 2004b)

1. Graphs based on $y = x^2$, with equal scales on each axis:
 - (a) On centimetre grid paper, plot $y = x^2$ for $-3 \leq x \leq 3$ (and $0 \leq y \leq 9$).
 - (b) Trace this graph onto thin card or another grid and cut out the parabola shape.
 - (c) Use the first graph to solve $x^2 = 0$, $x^2 = 1$, $x^2 = 4$, $x^2 = 9$.
(After students have a go, may suggest draw lines $y = 1$, $y = 4$, $y = 9$.)
 - (d) Using the cut-out on a set of axes with equal scales on each and allowing for $-9 \leq y \leq 16$ and $-3 \leq x \leq 4$.
Move the shape down (or up) to trace around it and draw graphs of $y = x^2$, $y = x^2 - 1$, $y = x^2 - 4$, $y = x^2 - 9$, and solve the cases for $y = 0$ for each graph.
 - (e) On the same set of axes, move the cut-out 1 unit to right, then down by the same amounts as in (d) to trace the graphs of
 - (i) $y = (x - 1)^2$ and have students check this (by substitution)
 - (ii) $y = (x - 1)^2 - 1$
 - (iii) $y = (x - 1)^2 - 4$
 - (iv) $y = (x - 1)^2 - 9$, and solve the cases for $y = 0$ for each graph.
2. Relate the graphical methods to other methods such as taking the square root of both sides, using factors for a difference of two squares, using the fact that $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$, completing the square, and, for equations of the form $ax^2 + bx + c = 0$, use $(px + q)(rx + s) = prx^2 + (ps + qr)x + qs$ to lead to the cross-product method.
3. Include real-life cases involving quadratic equations, such as analysing the equation $s = 20t - 5t^2$, where s metres is the height an object reaches, t seconds after firing it vertically.

Introducing algebraic symbols

The new *Mathematics Syllabus Years 7–10* (BOSNSW, 2002) currently stresses that students need ‘to develop an understanding of the use of letters as algebraic symbols for variable numbers of objects rather than for the objects themselves’ (p. 82). The use of a model such as the *cups and counters* model is recommended.

Research data show that very few young students succeed with a progression from pattern generalisations expressed in everyday language to expressing these in symbolic algebra (cf. Quinlan, 2000, 2001; MacGregor & Stacey, 1993). The clear message is to ensure that students are conversant with the conventions of writing algebraic expressions before being asked to translate generalisations from English to algebra. This is now a recommendation of the new NSW Syllabus: ‘The recommended approach is to spend time over the conventions for using algebraic symbols for first-degree expressions and to situate the translation of generalisations from words to symbols as an application of students’ knowledge of the symbol system rather than as an introduction to the symbol system’ (BOSNSW, 2002, p. 82).

These recommendations are incorporated into very few current textbooks, but this present author has been promoting them for some time (see Quinlan, 2002; Quinlan, Low, Sawyer & White 1993).

Pyramid volume

(See Hodges, 1982)

The Hodges article is excellent for guidance regarding suitable cardboard models to lead students to discover how to calculate the volume of a pyramid. A set of three square-based pyramids, each with vertex above one corner of the base, can be made to fill a cube with faces congruent with the bases of the pyramids. The pyramids have the same height as the cube. Similarly, a set of three pyramids with triangular bases can be shown to fill a corresponding prism. These experiments lead to the conclusion that the volume of a pyramid equals one-third the area of the base times the height.

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Let 'em fight it out: Arguing in my constructivist classroom

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We are becoming more aware of how students learn mathematics and the evidence that indicates that people construct their own understandings based on the experiences they encounter and their previous knowledge. These understandings are often full of misconceptions, particularly when students play an apparently passive role in learning. The more active students are in exploring and sharing knowledge, the more powerful their constructions and the less likely they are to be based on incorrect ideas. A learning environment that provides learning experiences which enable and, indeed, demand that each student takes an active, heuristic role is a constructivist learning environment. In this paper I will share a strategy I developed which I have found enables students to construct their mathematical knowledge and, interestingly, for me to work less hard in my classroom.

Background

I can program my old VCR — to record when I want it to! We just bought a new one the other day (with a new-fangled DVD-thingy in it) and I figured out how to program that one too. I also recently taught my seventy-two year-old aunt how to program hers, too, without having seen that model before. It is amazing what we can learn, and learn well and quickly, when we are in control of our learning — when we actively construct our own learning. This concept is fundamental to my teaching and is recognisably a constructivist approach (the Internet provides more information on constructivism).

In employing a constructivist approach in my teaching I use and devise strategies that enable students to, among other things:

- be autonomous and curious;
- use initiative;
- develop and freely use inquiring minds;
- develop deep understandings and to develop these understandings based on their previous experience and pre-existing knowledge;
- engage in dialogue with other students;
- use peer tutoring to further their own understandings and those of others;
- use co-operative learning;
- use whatever learning style suits them best;
- construct new knowledge and understandings from experiences that make particular sense to the learner.

As students construct their understandings they usually develop misconceptions which can hinder future developments. These misconceptions are best challenged by the use of *cognitive conflicts* (an Internet search on cognitive conflict can provide further information). If a student is provided with clear and irrefutable evidence that his or her understanding is incorrect, they have to reconstruct it. For example, a student may believe that when two numbers are multiplied together, the answer is larger than either of the starting numbers. When faced with the evidence that the answer to $5 \times 1/3$ is smaller than 5 their belief is challenged and they need to rethink it.

One of the precepts of the constructivist approach is the understanding that dialogue is an essential factor. So, my strategies to best enable constructivism include organising the desks in my classroom into groups of four, establishing a team ethos within these groups of four (this includes changing the composition of these groups twice a term), using collaborative techniques such as those of Barry Bennett, using authentic tasks and situations, allowing students to drive their curriculum in so far as possible, and basically allowing my classroom to be as noisy as can be. Having students sit in quiet rows working from textbooks is the antithesis of these strategies. My students spend no more than 15–20% of their class time doing that (yes, it does have a place, even in a constructivist classroom) and the rest of the time they are usually talking. If I spend more than 10% of our class time talking to the class or any one student, then I consider that lesson was probably less effective than it could have been.

As a part of this I have developed a strategy that caters for most learning styles (including those who learn best by ‘just being told what to do’) and that provides cognitive conflicts in a safe environment.

In a session given by Thelma Perso some years ago, I was introduced to the idea of cognitive conflicts as a learning tool and the use of peer tutoring to resolve these. I experimented with my classes which, at this stage, included primary extension classes, secondary classes, TAFE classes and TEE (Tertiary Entrance Examinations) classes and adapted her ideas into what I call *mathematics debating* (and quickly learned never to shorten the word ‘mathematics’ — especially not in a class of Year 9s). This is an incredibly effecting teaching strategy (for most, but not all, of my classes). I have since presented workshops on this strategy at many MAWA conferences, two AAMT conferences, the Transition Numeracy Conference, and on other occasions. I have received feedback from participants months after these sessions in which they tell me how successful this strategy has been for them.

The strategy

1. I present a problem or group of problems to the class — preferably in a relevant context and often prompted by a comment from a class member. Any problem is suitable. What is helpful is an understanding of the types of misconceptions students tend to have and an ability to recognise them. Any problem can reveal these.
2. Students discuss the problem(s) in their groups (or work on it alone — sometimes — if that is their preference).
3. Time permitting, students may be invited to check their answers with others in the class. It is an important step if there is confusion about the problem or if there is an excessive number of solutions proposed. I judge this by walking around the class and observing students.
4. I then call for answers. These are written on the board as they are suggested, with no comment from me.

5. If only one answer is suggested and the class all seem to agree with it, then it is accepted without further comment. If I want to know by what process students arrived at that answer, I give them a similar but more challenging problem.
6. If a number of different solutions are proposed, I call for speakers to defend each solution.
7. Speakers for each solution go to the board and take turns defending their answer. Each speaker is permitted to:
 - use the board;
 - speak for as long as they require;
 - speak without being interrupted;
 - refute the previous speakers arguments;
 - change their case at any time;
 - concede their case at any time (students are commended on graceful concessions — the only judgemental comment from me during the whole process).
8. When each speaker has had the opportunity both to present their case in full and to refute all other speakers, the rest of the class can participate in the discussion: asking questions of the speakers, presenting their own cases, adding further explanations to speakers' cases and so on.
9. When the class comes to a consensus on which is the right answer (if any), then the problem is considered solved.
10. If no correct solutions are proposed but the class comes to a consensus on an incorrect answer, I alert them to the fact that we need to work on it. This means they do not have the concept the problem was meant to address/consolidate/whatever. After we have done some more work towards that concept, we revisit the problem; the students themselves often prompt this.
11. This also applies if no solutions are suggested.
12. If the class cannot come to a consensus on the correct answer, I call a stalemate and revisit the problem after we have worked further on the concepts required.
13. Students are encouraged to extend the problem with 'what if...?'-type questions.
14. The process may then repeated if any 'what if...?' or other extensions are proposed.
15. Questions can be open so that more than one answer is acceptable. Students have to decide which answers are acceptable.

This process can take just a few moments or can take a whole lesson, depending on the students' requirements.

What I have to do

There are some important behaviours from me in enabling the success of this strategy. They include:

- developing a safe learning environment and modelling respect for all students. This includes valuing mistakes. We have a poster in my classroom that reads, 'This is a learning zone. ~~Mistake~~ Mistakes are welcome here.' It takes time to develop this environment; that is, one with no derisive comments or put-downs;
- allowing students control of the process. Once I initiate the process I allow it to run under student control;
- only providing the 'correct' answer if the whole class calls for it;
- not allowing students to look at me while presenting their case; that is, I avoid eye contact — they have to convince the rest of the class, not me;
- recognising that this does not work well the first few times I try it with a new class.

They need to be taught the process. For some, the challenge of thinking is a pretty big task and takes some getting used to;

- determining that my role is that of possible problem-poser (students may also do this), facilitator of discussions (before the debate), occasional crowd control, time-keeper for the debate (if the class want to put a time limit on each speaker) and sometimes adjudicator if no consensus can be reached (that is, I say, ‘We have to leave this problem as we cannot reach a consensus’);
- ensuring students learn not to ask me for help during either the discussion period or during the debate;
- not judging or commenting on students’ arguments or comments during this process;
- noting when students need more work on a concept.

Outcomes

There are a number of outcomes arising from this process. They include:

- Flaws in reasoning and misconceptions are usually recognised and acknowledged by the student themselves. The cognitive conflict arises when students thought they understood a concept and used it to derive their answer(s) but find that others disagree with them. It is highly confronting when a teacher tells a student they are incorrect but most students consider this process a part of the game. They seem far more comfortable being wrong and reconstructing their understandings from it.
- Boys love it: they call it ‘arguing’ and constantly ask for it (‘Miss, can we do arguing again?’) as they walk in the door.
- Girls love it as they enjoy talking things through.
- Far more students are engaged in the process, especially as they may perceive it as a way of avoiding ‘having to work’. In fact, I currently have one class who are convinced they never do any work but my records show significant progress in their understandings and skills. The bonus is that they are always excited to be coming to Maths.
- New concepts can be investigated and explored; I often use this to help students recognise when they need to learn something entirely new.
- Students are able to investigate new ideas and construct their understandings based on what they already know.
- Peer tutoring is employed and is highly effective for those students who ‘need to be told what to do’. For many students, this is far less threatening than when a teacher tells them.
- Students at different levels of achievement can work to their own level. Some students generate quite sophisticated reasoning; other students can follow it even if they cannot generate it.
- Students feel that they are generating the knowledge — as, indeed, they are.
- Students feel that they have control over their learning — as, indeed, they do.
- The process helps me identify what concepts the class and individuals need to work on and what they know well. I keep a record of what outcomes (from Western Australia’s *Outcomes and Standards Framework*) students demonstrate during this process.
- These discussions can prompt subsequent lessons.
- It helps build a ‘community of mathematicians’ feel to the class. Real mathematicians operate somewhat like this (usually via journals and such like).
- I can work a little less hard in class (and that’s got to be good!).

This process works with all my classes to some extent and I experiment with each class to determine how best to use it. Offering jellybeans to debaters always helps; I keep a jellybean jar in my cupboard and all those who defend an answer receive one (not those students who are just ‘mucking around — you know what I mean).

Conclusion

I will conclude with a story: two years ago I had a middle-level Year 9 class with an incredible range of achievement levels. They used this strategy extensively and thrived on it. During one lesson I was squatting down at the front of the class, helping two lower-achieving students (who needed more help than average). All of a sudden, a commotion arose at the back of the class. I looked up to see what was happening and noticed two students striding to the front of the class: they immediately began debating a problem. Meanwhile, I ducked back down, eager not to interrupt the flow. The rest of the class stopped working and watched their debate and then contributed to it. The students resolved the problem and returned to their seats. The rest of the class then returned to their own work. I played no part in the process, apart from sitting very small and still. This was one of the golden moments of my (so far) twenty-one-year career. These students were in charge of their learning and clearly thriving on their autonomy.

Finally, a quote attributed to Woodrow Wilson: ‘Use more than the brains you have: use all the brains you can get.’ This strategy uses all the brains in the classroom to promote learning.

If you try this strategy I suggest you use it first with your very best classes — those you trust the most. It is unlikely to work the first few times you try it and letting go of your teacher instincts (that urge to tell students what they need to know — it is referred to in the literature as ‘teacher lust’) is difficult. It took a couple of years for me to get it right: persistence helps. Good luck and let me know how you go; I can be reached at noemi.reynolds@det.wa.edu.au.

This paper is based on one published in the MAV 2004 conference proceedings.

Maths on a mat with Matt

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The Maths Mat is one teaching strategy that allows:

- children to construct their own mathematical concepts
- kinesthetic learning styles to be catered for
- multiple representations of ideas
- cooperative learning in a risk-taking atmosphere.

What is the mat? It is a large piece of shade cloth (7.2 m x 3.6 m) with a 10 x 5 grid painted on it. The squares are approximately 70 cm x 70 cm. All the activities involve 'doing maths' with your body — kinesthetic learning. These ideas were inspired by Doug Williams' paper in the 1993 MAV conference proceedings, 'Maths on a plastic mat'. The mat helps students to develop their own understanding of a concept, rather than just being 'told' something. Activities have been devised from early childhood to Year 12, in the areas of coordinate geometry, measurement, algebra, transformational geometry and chance and data.

I attended my first Mathematical Association of Victoria conference in 1993. Just as I left for the airport, a colleague bid me farewell with: 'I don't know why you bother going to conferences. Research shows that if you don't use an idea after 12 days, you never will.'

With that 'vote of confidence', I resolved to bring back one 'big idea' and develop it further in professional development settings in the Northern Territory. Since then, numerous classroom teachers have contributed ideas, and 'tweaked' existing ideas further.

I have found the following two principles to be powerful in guiding the development of tasks and activities for the *maths mat*:

- 'Good mathematics curriculum starts with rich mathematical tasks' (Doug Williams, Curriculum Corporation);
- 'Geometry that can be told is not geometry' (Dick Tahta, Open University, UK).

One activity (Figure 1) that challenges adults and children alike is: 'Using the elastic, make a triangle that has an area of 6 square units.'



Figure 1

This ‘simple’ task challenges the concepts a learner has constructed for themselves about area and triangles. Both students and teachers are often ‘not sure if they are right’ when first faced with this challenge. Tweaking the question to ‘make a triangle with an area of 7.5 square units’ causes students to move outside their previous understanding of how to calculate area of a triangle. Cooperation is vital, but the deliberate choice of not specifying how a group is to work together creates some valuable tension in the learning activity.

When a group of learners share a rich common experience (often a kinesthetic experience), they retain an image of the experience in their mind for a long time. Their teacher only has to say, ‘Remember when we...’ and the students conjure up the image, which allows them to re-enter the experience. Referring to a previous shared experience supports further learning, creating an active culture of learning.

For student learners, the maths mat contributes to building a richer classroom environment. Students construct their own rich images and metaphors, which helps them engage with the mathematics in different settings. For example, bilateral symmetry can be represented:

- on the maths mat
- on a geoboard
- using dotted paper
- using *Cabri Geometry* or *Geometer’s Sketchpad*
- using concrete manipulatives.

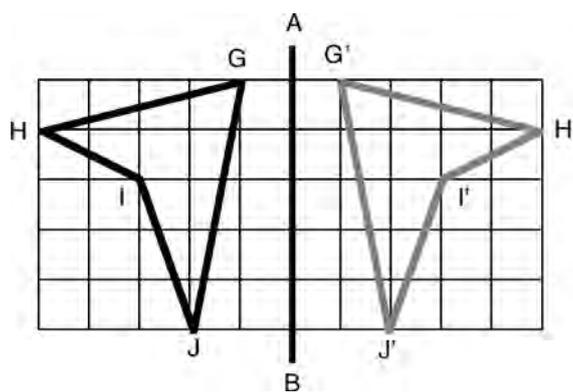


Figure 2

Figure 2 represents a challenge to students on the maths mat. Four students (GHIJ) are invited to use elastic to create a quadrilateral that they find interesting. (The teacher is able to draw attention to various features of the quadrilateral ‘on the fly’.) The only constraint offered to the students is that their quadrilateral should be in one half of the maths mat. Two more students use elastic to act as the line of reflection (AB). Four more students are then invited to use elastic to reflect the shape GHIJ about AB, to form the transformation G'H'I'. Experience in trialling this activity in a variety of settings has drawn the author’s attention to the importance of involving the rest of the class in the activity. Challenge them to give instructions to the students creating the transformation. Invariably, at least one student will be in the wrong place. By creating a climate of ‘doing maths as a community’, students are able to engage in the mathematics.

A teacher can create leverage from this community by inviting students AB to move one square to the left or right, and then invite the audience to give instructions to students G'H'I' to move, to ensure that their quadrilateral is a reflection of the original. Moving the line of reflection AB diagonally offers a challenge that creates some conflict,

which needs to be resolved. Figure 3 is an example of such a challenge, where students G'H'I'J' may have to work off the maths mat. They are not sure if they are in the right place. This dissonance is a useful tool for a teacher to intervene with some strategically placed questions.

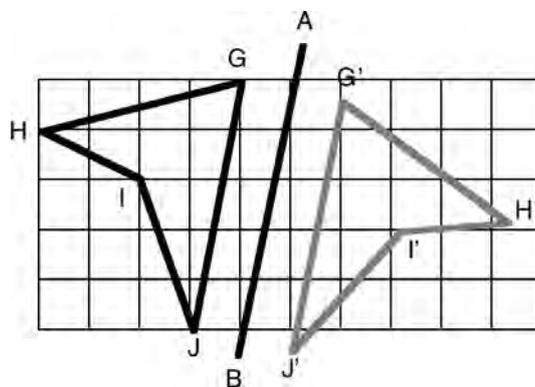


Figure 3

Each setting offers different opportunities for teachers to intervene in the learning process. In a physical setting such as 'on the mat', reciprocal obligations of teachers and students as fellow learners is brought to the forefront, with an invitation to learn, participate, conjecture and verify their mathematical experiences. Students still have to choose to engage in the mathematical discourse.

The skill of the teacher is critical, requiring:

- thorough knowledge of suitable open-ended tasks and potential ideas and concepts;
- engaging students in the mathematical challenge;
- strategic intervention coupled with critical questioning;
- a capacity to monitor multiple events, reflect-in-the-moment while maintaining the pace.

Another role the maths mat can play is as a vehicle to provoke students and teachers into considering 'boundary cases'. Challenging students to find functions that allow

$$\int_0^2 f(x) \cdot dx = 0$$

to be true has the potential to provoke a lot of discussion. Initially, obvious solutions will be offered (e.g., $y = x - 1$, $y = 2x - 1$) but with prompting, more sophisticated functions can be offered. Each function can be represented on the maths mat with some rope, helping to focus attention not only on examples that are similar, but also extreme cases or examples that are significantly different (e.g., $y = (x - 1)^3$, $y = \sin(x - 1)$).

From a professional development perspective, the maths mat is a powerful tool. Teachers cannot help showing their natural creativity by extending and adapting existing maths mat activities to their setting, and creating new activities. The maths mat captures the spirit of recognising and recording the 'collective wisdom' of teachers.

With the emerging focus on integrating learning technologies into classrooms in all school settings, activities on a maths mat represent an exciting way of introducing and reinforcing key mathematical concepts in a way that engages students. With many students spending increasing amounts of time working on computers, it is instructive observing them representing ideas in a kinesthetic domain. Having students represent mathematical concepts in multiple domains offers an opportunity to 'walk the talk' of the various multiple intelligences learning theories.

More recently I have had professional roles with a focus on supporting teachers to integrate learning technologies into their classrooms. Using activities arising from a maths

mat has been a useful window for teachers to see how students represent mathematical ideas in different mediums — multiple representation of ideas in different domains.

One example is for students making squares (or some other quadrilateral) on the maths mat with the challenge to record the ordered pairs, and enter the data into a spreadsheet. When using the X-Y scatter option in a spreadsheet, there will be mixed results, depending upon the order of the ordered pairs being entered. Some teachers may view this scenario as creating confusion, but in a conjecturing atmosphere, significant mathematical discussion can arise. Having a diagram of a ‘bowtie’ instead of a square (Figure 4), causes a degree of dissonance when success is not immediate. Posing the question, ‘What needs to change to make our figure into a square?’ has no one correct answer or approach.

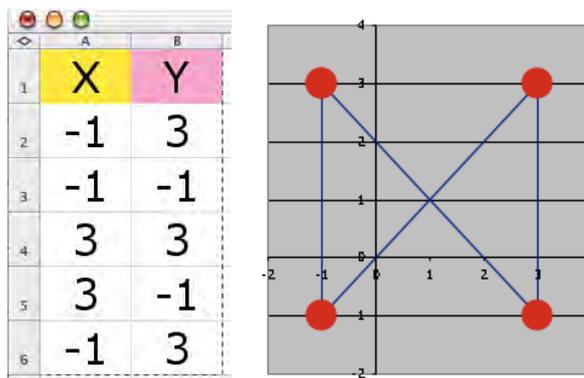


Figure 4

Similarly, a simple class survey (Figure 5) of how many siblings you have, leads to opportunities for early childhood students to begin exploring the use of spreadsheets.

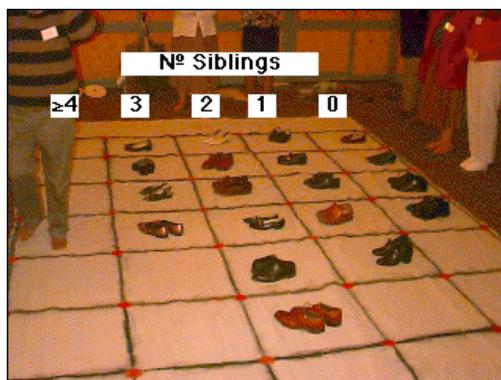


Figure 5

A powerful aspect of using the maths mat in professional development settings is that it evokes a ‘brief, but vivid description’ (Mason, 1994) that allows participants to re-enter the situation many years later. A prompt such as, ‘Remember when we did...’ can serve as a compelling stimulus.

A strategy to provoke these memories for professional development events longer than half a day or a day, is to have someone taking digital photos constantly during the sessions. During the lunchtime or overnight break, import a folder of the digital photos into *iMovie* (or similar software), and export them as a *QuickTime* movie.

While a much shorter time frame than what Mason has in mind, it the experience helps to regroup and refocus a group of teachers (or students) with a sense of cama-

raderie. For the inevitable person who has to miss some part of a session, the resource also helps to include them in what has been missed. It is also a useful record of a professional development event to leave with a school.

Another resource recently developed is a 'mini-maths mat', which fits on a typical student desk. This allows a teacher to revisit an activity that was done outside, and to consolidate particular aspects of mathematical concepts with students. If students are to develop some digital media as evidence of their mathematical understandings, then the mini-maths mat makes it easy for them to 'rehearse' their performance, prior to using the full sized maths mat.

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Benford's Law

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Introduction

If a set of non-random and naturally occurring numbers is considered, it would be reasonable to assume that the starting digit would be equally likely to be a 1 or 2 or... or a 9. Or would it? We could experimentally evaluate this with a vast variety of data ranging from populations of countries, scientific constants, lengths of rivers, etc. The validation of this by experimentation is a valuable lesson for non-academic students in establishing facts for themselves. Academic students will be able to look beyond the data and to establish a mathematical principle to the data.

Background

Benford's Law is attributed to Dr Frank Benford who was a physicist at the General Electric Company in the United States. He noticed that pages in logarithm tables varied in their 'grubbiness' with pages with a starting digit of 1 being the most 'grubby'. (It is believed that an American astronomer Simon Newcomb discovered the same phenomenon in 1881, but without any explanation or investigation it was basically forgotten). Benford embarked on an analysis of over 20 000 sets of numbers to examine the 'starting digit' data.

Benford's Law

Benford found that the number 1 occurred in over 30% of cases (unlike the 11% case where all starting digits are equally likely). The original data table by Benford from 1938 has been reproduced below.

Analysis

Even Frankcomb was able to establish that the formula for the starting digit was

$$\log\left(1 + \frac{1}{d}\right)$$

where d is the digit in the sequence 1, 2... 9.

However, it was only with the establishment of a large amount of data that Benford really staked the claim for the law to be known as Benford’s Law.

Table 1. Benford’s original data.

Col.	Title	First Digit									Samples
		1	2	3	4	5	6	7	8	9	
A	Rivers, Area	31	16.4	10.7	11.3	7.2	8.6	5.5	4.2	5.1	335
B	Population	33.9	20.4	14.2	8.1	7.2	6.2	4.1	3.7	2.2	3259
C	Constants	41.3	14.4	4.8	8.6	10.6	5.8	1	2.9	10.6	104
D	Newspapers	30	18	12	10	8	6	6	5	5	100
E	Specific Heat	24	18.4	16.2	14.6	10.6	4.1	3.2	4.8	4.1	1389
F	Pressure	29.6	18.3	12.8	9.8	8.3	6.4	5.7	4.4	4.7	703
G	H.P. Lost	30	18.4	11.9	10.8	8.1	7	5.1	5.1	3.6	690
H	Mol. Wgt.	26.7	25.2	15.4	10.8	6.7	5.1	4.1	2.8	3.2	1800
I	Drainage	27.1	23.9	13.8	12.6	8.2	5	5	2.5	1.9	159
J	Atomic Wgt.	47.2	18.7	5.5	4.4	6.6	4.4	3.3	4.4	5.5	91
K	Square roots	25.7	20.3	9.7	6.8	6.6	6.8	7.2	8	8.9	5000
L	Design	26.8	14.8	14.3	7.5	8.3	8.4	7	7.3	5.6	560
M	Reader's Digest	33.4	18.5	12.4	7.5	7.1	6.5	5.5	4.9	4.2	308
N	Cost Data	32.4	18.8	10.1	10.1	9.8	5.5	4.7	5.5	3.1	741
O	X-Ray Volts	27.9	17.5	14.4	9	8.1	7.4	5.1	5.8	4.8	707
P	Am. League	32.7	17.6	12.6	9.8	7.4	6.4	4.9	5.6	3	1458
Q	Blackbody	31	17.3	14.1	8.7	6.6	7	5.2	4.7	5.4	1165
R	Addresses	28.9	19.2	12.6	8.8	8.5	6.4	5.6	5	5	342
S	Factorials	25.3	16	12	10	8.5	8.8	6.8	7.1	5.5	900
T	Death Rate	27	18.6	15.7	9.4	6.7	6.5	7.2	4.8	4.1	418
	Average	30.6	18.5	12.4	9.4	8	6.4	5.1	4.9	4.7	1011

In graphical form that is easy to reproduce with data, it is obvious that the relative frequency of the starting digits decreases with an increase in the starting digit.

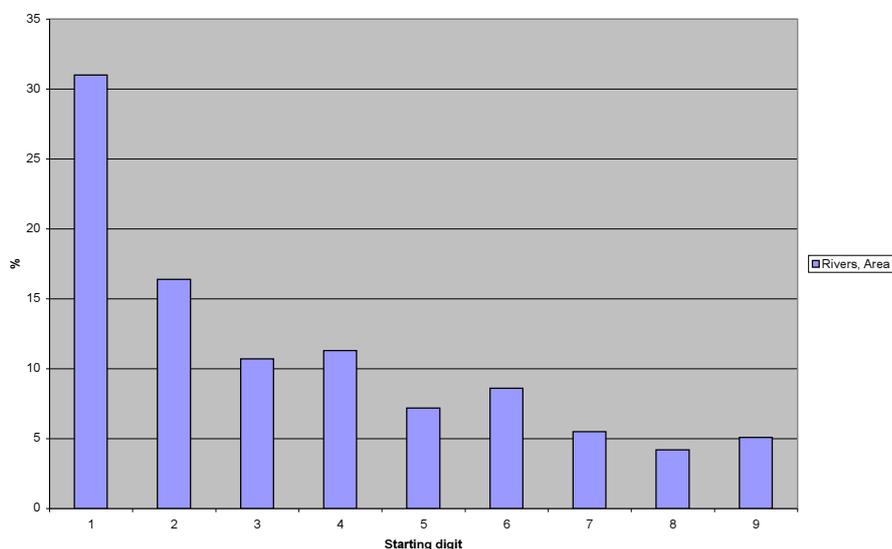


Figure 1

In order to try to shed some light on why Benford’s Law holds in a mathematical sense, it is worth considering two cases

Case 1

Consider the set of numbers from 1 to 9 (starting digits equally spread). If we double each of these 9 numbers would you expect an even spread again?

Table 2

Original starting digit	Number ×2	Frequency of digit
1	2(50%), 3(50%)	5
2	4(50%), 5(50%)	0.5
3	6(50%), 7(50%)	0.5
4	8(50%), 9(50%)	0.5
5	1	0.5
6	1	0.5
7	1	0.5
8	1	0.5
9	1	0.5

Clearly the spread is not even and in this simple case the number 1 is 10 times more likely to occur. This does not present the whole picture, however.

Case2

Consider an investment of \$1000 in the All Ordinaries Index on the Australian Stock Exchange and assume a compounded rate of interest of 10% p.a. We could then calculate how long the index is in each starting digit zone (i.e., 1000–1999 would have a starting digit of 1). If the index had been recorded daily under these conditions it would give us an indication of the relative frequencies of the starting digits. Using the compound interest formula to calculate n (the number of years) that the investment will be in each starting digit zone should also allow us to model this situation. For example, if the starting value was \$2000 and the final value was \$3000 we would use

$$3000 = 2000(1.1)^n$$

$$1.5 = (1.1)^n$$

$$\log 1.5 = \log 1.1^n$$

$$\log 1.5 = n \log 1.1$$

$$n = \frac{\log 1.5}{\log 1.1} = 4.25 \text{ years}$$

Table 3 shows this information and illustrates why the relative frequencies of the starting digits tend to decrease.

Table 3. Determining the proportion of starting digits using \$1000 investment at 10%pa interest compounded annually.

Starting digit	Starting value	Finishing value	%Increase	No. of years market is in starting digit	% of years in starting digit	$\log(k+1) - \log(k)$
1	\$1000.00	\$000.00	100.0%	7.27	30.1%	30.1%
2	\$2000.00	\$3000.00	50.0%	4.25	17.6%	17.6%
3	\$3000.00	\$4000.00	33.3%	3.02	12.5%	12.5%
4	\$4000.00	\$5000.00	25.0%	2.34	9.7%	9.7%
5	\$5000.00	\$6000.00	20.0%	1.91	7.9%	7.9%
6	\$6000.00	\$7000.00	16.7%	1.62	6.7%	6.7%
7	\$7000.00	\$8000.00	14.3%	1.40	5.8%	5.8%
8	\$8000.00	\$9000.00	12.5%	1.24	5.1%	5.1%
9	\$9000.00	\$10 000.00	11.1%	1.11	4.6%	4.6%
Totals				24.16	100.0%	

Practical applications

A practical financial application has been used for Benford's Law in the area of tax auditing. Dr Mark Nigrini from the USA tested Benford's Law on some fraud cases in Brooklyn. He assumed that tax return data should follow Benford's Law, and if it did not then it was the case that a fraud may be likely (or at the least a more comprehensive audit needed to take place).

Benford's Law is being used in Belgium to detect irregularities in clinical trials and is being trialed in a variety of countries in allocating computer disk space for greater efficiency.

Dr Nigrin has been quoted as saying, 'I foresee lots of uses for this stuff, but for me it's just fascinating in itself. For me, Benford is a great hero. His law is not magic but sometimes it seems like it'. Spoken like a true mathematician!

Further reading

New Scientist (10 July 1999) pp. 26–30.

http://www.cut-the-knot.org/do_you_know/zipfLaw.shtml

<http://mathworld.wolfram.com/BenfordsLaw.html>

<http://www.rexswain.com/benford.html>

Decide, select, perform and make sense: Computation in primary mathematics*

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Research findings will be presented on how children make and execute computation choices. Suggestions will be made for helping children make better choices, along with several ideas for assisting children to decide whether the results of a calculation are reasonable or not.

Introduction

In setting the standard for computation in Australia, *A National Statement on Mathematics for Australian Schools* (AEC, 1991) included the following comments:

All school leavers should feel confident in their capacity to deal with the computational situations which they meet daily, and number work should reflect the balance of number techniques in regular adult use... Students should develop the ability to judge the level of accuracy needed, learn to estimate and approximate, and use mental, calculator and paper-and-pencil strategies effectively and appropriately in different situations... This requires that they:

- decide what operations to perform (formulate the calculation);
- select a means of carrying out the operation (choose a method of calculation);
- perform the operation (carry out the calculation);
- make sense of the answer (interpret the results of the calculation).

(p. 108)

Curriculum documents in many Australian states and the USA have highlighted the need for students to be able to choose from a repertoire of computational tools (Curriculum Council, 1998; Education Department of Western Australia, 1998; National Council of Teachers of Mathematics, 2000), but little direction is given as to how to help children make the choice as to which form of computation to use in any given situation. The focus of this paper is on finding ways to develop children's ability to make sensible computation choices.

* This paper has been accepted by peer review.

Computation choice by children

Swan (2002, 2004) studied the computation choices students made and why they made them. Seventy-eight children from Years 5 to 7 (ages 10–12) were interviewed and asked to state how they would solve a particular computation item. The children were then invited to solve the item using their nominated method. Children were given eighteen items, some of which were drawn from previous research (Price, 1995; Reys, Reys & Hope, 1993). In these studies, children were only asked to state how they would complete a particular calculation but were not required to carry out the calculation. These parallel items are shown in Table 1 and illustrate the scope of the items.

Table 1. Items used in two studies of computation choice.

36×25	1000×945
70×600	10% of 750
29×31	$1/2 + 3/4$
33×88	0.25×800

As can be seen in Table 1, a wide range of items was provided for the children to calculate. In all, eighteen items were given and Table 2 outlines the percentage choice made by the children for various computation methods.

Table 2. Percentage distributions of initial computation for all items ($n = 78$).

Mental	Written	Calculator	Mixed	No Method
36	26	28	6	5

The general trend outlined in Table 2 suggests that students were exercising a choice. No one particular computation method dominated their response to the exclusion of other methods. An examination of the raw data indicated students varied their methods. For example, no evidence was found to indicate that any individual student had used a single computation method for all eighteen items. The data also indicate less reliance on written methods than expected given the emphasis placed on such methods in the classroom.

What was of particular interest to the authors were the reasons students gave for making particular computation choices. Generally, it was found that students made choices based on a few rudimentary criteria, such as the size of the numbers, the particular operation involved, and whether or not they knew their tables.

Helping children make appropriate computation choices

An overall goal for many teachers is to develop number sense in their children via thinking and reflecting about numbers and operations. Children with good number sense are more able to make better and more appropriate choices in computation. In this paper several strategies, for example discussing answers and methods, are suggested for helping children make better computation choices. The strategies are all underpinned with a philosophy of developing children's number sense. In this case, number sense is defined as:

...a person's general understanding of numbers and operations along with the ability

and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful and efficient strategies for dealing with numbers and operations (McIntosh, Reys & Reys, 1997).

Note in particular the reference to ‘mathematical judgements’. These judgements include choosing an appropriate method of computation for a particular calculation in context. A better understanding of numbers and operations will help children ‘see’ alternative approaches to performing a calculation. For example, a child who possibly lacks flexibility and deeper understanding of numbers, relationships and operations may choose to complete the following calculation using a standard written method:

$$36 \times 25$$

A child with a measure of number sense may realise that 36 is also 4×9 and if the child also recognises that the order in which a calculation is performed does not matter, that is, multiplication is commutative, they may rearrange the calculation to make it more manageable as a mental calculation. The question then transforms from 36×25 to $4 \times 9 \times 25$ and finally to $4 \times 25 \times 9$. A child who understands the associative property may then complete the calculation in the following order $(4 \times 25) \times 9$. The calculation then becomes relatively simple (100×9) to complete using a mental method.

Ways of ‘smashing up’ the numbers may vary between children. Some children may be more comfortable with using ‘fives’ and wish to ‘smash’ the numbers into $6 \times 6 \times 5 \times 5$ before they are rearranged to $5 \times 6 \times 5 \times 6$, that is, 30×30 . As children become more familiar, confident and skilled in ‘smashing up’ numbers they will have a genuine choice in computation method.

The above example has been used in several research studies (Price, 1995; Reys, Reys & Hope, 1993; Swan, 2002). The majority of students in each study chose to use written methods, which for the most part consisted of using a standard written method. In the most recent study, carried out in Australia, 51% of children aged between 10 and 12, used a standard paper-and-pencil to solve this item. Thirty-one percent of students made use of the calculator to find an answer. Older students were more definite in their choice, recording a 65% preference for standard written methods. Questions of this type abound in student texts and are commonly given as exercises to be completed using a written algorithm. Students may as a result gain the impression that written computation is always the best approach for solving a two-digit multiplication problem both from its preponderance and emphasis in texts, classrooms and from their lack of mental computation strategies.

A typical description of the written method used to solve this item as given by one student in the study is reproduced below. Note in particular the use of terms such as ‘carry’ and ‘put down the’ which are typical phrases used by teachers when teaching children a written procedure.

- S: Thirty-six times twenty-five and then you’d go six times five is thirty. Three times fifteen, put the zero down. Two sixes are twelve, put down the two, carry the one. Two threes are six, that’s seven.

The emphasis in the teaching of a standard procedure is placed on manipulating digits as illustrated in the commentary by the student, for example twenty is seen as 2 and thirty is seen as the digit 3. Children here are using digits rather than quantities or amounts. The written procedure is not expected to make sense.

Discuss choice in computation

Many children do not realise that they can or are expected to make a choice about which computation method to use. It is important that such discussions of appropriate choice take place. Discussions of choice should not only be about which method to use but also which strategy within that method is most appropriate for the context and the child.

The 'How did you do it?' activity involves presenting a calculation to be performed mentally and then asking the children to explain how they completed it (McIntosh, De Nardi & Swan, 1994; Sparrow, 2004). A variation on the 'How did you do it?' theme is used in the following activity:

How would you do it? 16×25

In your head?

On paper?

With a calculator?

This question is presented and children decide the method they would use to solve it. They are then asked to explain why they chose that particular method. At some point they may be asked to complete the calculation in the selected method. Another similar approach asks children to list a calculation they would perform in the head, on paper or with a calculator and to explain why such a calculation was selected.

There are several formats contained in McIntosh, De Nardi and Swan (1994), such as 'Today's number is' that also encourages children to explore and discuss mental strategies and the relationships within numbers. The 'Today's number is' activity asks children to list all they know about a particular number, for example 48: children present number sentences such as $47 + 1$, or 24×2 or the number of months in four years. After children become familiar with the format of this type of activity, the teacher can encourage children along particular paths, for example finding division sentences. This activity develops a range and variety of patterns and relationships to be connected to a particular number.

Offer the real possibility of choice

If children are to have a choice in method they will need to have developed such a range of computation methods. A teaching style that only provides instruction in standard procedures for written calculations will not provide this choice as children are only familiar with one method. There is a possibility that an emphasis on standard written methods will restrict and undermine the chance of children using any other method for calculating. A reduction in the time spent on teaching standard written methods or leaving such instruction until later, should allow more time to be spent on mental computation (not just speed recall of facts), informal written calculation methods, estimation, discussing how calculations are performed, discussing computation choices and the reasons behind making them, and learning to make efficient use of a calculator.

Focus on sensible calculator use

The following guidelines are suggested (Sparrow & Swan, 2001) to help teachers decide what constitutes sensible calculator use in the classroom. A calculator task should:

- involve more than children completing a simple, straight forward calculation;

- provide insight into children’s thinking and understanding;
- involve the development of mental mathematics or mathematical or number sense ideas;
- involve children in discussion or explanation;
- encourage children to look for more efficient ways to calculate.

The aim should also be to assist children to make better choices as to when calculator use is appropriate or not. Activities such as ‘Beat the calculator’ (Swan, 1996) may be used initially to raise children’s awareness of when to use a calculator and when to use a mental method. The discussion relating to the calculations done in the head or on a calculator is important. It should provide insight for children as to which calculations are easy for their head, even though some may initially appear to be too large. The activity can also place an emphasis on the usefulness of quick access to simple number facts.

Comparing and connecting calculations

Provide children with a table similar to the one below and ask them in pairs to complete the table and comment on their findings. Add to the chart other calculations that may be ‘done in the head’ using the information from the first two columns.

Do this on your calculator	Do this in your head	Check	Other calculations How can you use the findings in columns 1 & 2 to perform other calculations. For example:
$13 \times 29 = 377$	$130 \times 29 = 3770$		$13 \times 290 = 3770$, $377 \div 29$, $3770 \div 130 = 29$

This idea is further developed in *Starting Points*, (Swan & Sparrow, 2001). The key part of this activity lies in the discussion that ensues as a result of completing the ‘other calculations’ section. It is here that children are alerted to the extra patterns and relationships that exist in the initial calculation. The set up of the initial calculation may be altered to suit the needs of different groups of learners, for example the first two columns could contain $2 \times 2 = 4$ and $20 \times 2 =$. For many children the connections and relationships between numbers in calculations are not obvious. Part of the strength of teachers who help children to ‘see’ and connect numbers is that they develop children who start to be ‘in charge of the numbers’.

Focus on estimation

McIntosh, Reys & Reys (1997) define estimation as a method for:

...producing an approximate answer to a computation, one that is ‘close enough’ to allow a decision to be made. Estimation often involves the user in mental computation as a preliminary first step to forming an estimate (p. 322).

Computational estimation may be thought of in several different ways. These include:

- estimation as a computation choice;

- estimation as a monitoring device for exact forms of calculation; and
- estimation as a method of checking results of exact forms of calculation.

Lobato (1993) recognised the complex nature of forming an estimate when defining estimation as ‘guessing with a little bit of problem solving’ (p. 347). As with the development of problem solving ability, children can be taught some heuristics to assist when completing calculations, but eventually decisions have to be made by the children in order to tackle the calculation. Likewise, children can be taught various techniques for estimating, such as front-end methods or methods that involve rounding, but good estimators will often adjust their estimation approach depending on the numbers and the context of the calculation. For a detailed discussion of estimation techniques see Booker, Bond, Sparrow and Swan (2004).

An example of where estimation may be used is when children work with calculations involving decimals. Rather than teach children a rule for placing decimal points based on counting decimal places, children should be encouraged to make an estimate and place the decimal point according to the estimate. Try placing the point in the following examples by making an estimate first.

$$15.3 \times 17.8 = 27234$$

$$0.6 \times 9.2 = 552$$

$$1.1 \times 2.2 = 242$$

Change emphasis on basic facts

Students need to develop a bank of basic number facts that may be used to support the making of estimates, but activities that emphasise speed only may discourage students from looking for patterns or related facts. For example, many students learn basic multiplication facts such as 8×3 , without recognising related facts such as 3×8 , $24 \div 3$ and $24 \div 8$. Once basic fact knowledge is secure then games may be used to increase the speed of response.

Rather than focus on the development of specific number facts students should be encouraged to develop a bank of related facts and a strategy for generating more facts from a situation. The following routine may be used to develop a set of related facts.

If I know ... then I also know...

If I know 10×5 is 50 then...

I also know 9×5 , 11×5 , 5×5 , 10×50 10×0.5 — and so on.

Children need to show and explain how each calculation is related to the other.

Avoid teaching rules

Students are sometimes taught when multiplying by 10 all they need to do is ‘add a zero’. When dividing by 10 students may ‘take away the last zero’. Students who use this rule without understanding will often over generalise and make the following mistake:

$$4.9 \times 10 = 4.90 \text{ or } 40.9 \text{ or } 40.90$$

Consider the following transcript of a child trying to explain how multiply 70 by 600.

- I: 70×600 .
 S: 4200.
 I: You did that one in your head by the looks. How did you do it?
 S: I just took the zeros away and did 7×6 and then I added the zeros.
 I: Right so you did 7×6 and got 42 and how many zeros did you put on?
 S: Two.
 I: So you put two back on. Okay. You took three off but you put two back on. Why was that?
 S: I don't know.

Often students fail to comprehend why this 'shortcut' works but adopt the strategy because the teacher has taught it. While this strategy may work with whole numbers it can lead to misconceptions when dealing with decimal numbers as shown above.

Rather than teach rules such as 'add a zero' when multiplying by multiples of ten encourage students to observe patterns and then suggest their own mental approaches to this type of calculation. It is important that students discuss their thoughts as this will help the teacher determine whether they understand the pattern.

Write down what you notice is happening after you complete each calculation on the calculator.

$$34 \times 10 =$$

$$34 \times 100 =$$

$$34 \times 1000 =$$

$$34 \times 10\,000 =$$

Where possible it would be better to include some examples involving decimals. This will allow the opportunity to discuss problems and known misconceptions with the 'add a zero' rule.

$$3.4 \times 10 =$$

$$3.4 \times 100 =$$

$$3.4 \times 1000 =$$

$$3.4 \times 10\,000 =$$

Conclusion

There are many factors that impact on computation choice, such as the time devoted to various forms of calculation, access to calculators and the use of textbooks. Flexibility with number, an aspect of number sense, frees children to try different computation approaches. As a way forward, it is suggested that children be encouraged to take time to consider the numbers and the operation before making a computation choice. Children should be asked to consider whether an exact answer is required or whether an estimate is sufficient. If an exact method is required, then children should be encouraged to try mental methods first. Having made a computation choice, children should then be prepared to justify their choice should they be challenged by the teacher or someone else. All of this requires that children do less formal work on standard procedures, which is somewhat of a paradox, because they should learn more. This will mean children should

be given fewer calculations to complete in a lesson, but more emphasis should be placed on how the calculation was performed.

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Making ‘cents’ of spinners

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The ICT wedge

W. W. Sawyer in his book *Prelude to Mathematics* alerts us to the fact that, ‘Mental adventuresomeness is a characteristic of all mathematicians... The desire to explore, and an interest in pattern, marks out the mathematician’. Mathematics is surely an aesthetic and shared exploration of ideas and systems. It has its socially utilitarian value: a study of models that are eventually put to work in scientific endeavour, and a rationality that empowers the individual and safeguards our way of life. It is an intellectual pursuit — as Courant and Robbins (1941) put it, ‘an indispensable part of the intellectual equipment of every cultured person’. It is the study of system variability leading to deep meta-cognitive understandings. It has a rare universality: we all agree about what it is, of what its greatest discoveries were, and how its well-defined structures and language should be communicated. I stand back from 15 000 mathematics lessons offered to countless numbers of upper secondary students, and ask myself what it was that I contributed to the lives of these people and to the society they became.

Lately, large slabs of information and communication technology (ICT) have pierced into the fabric of mathematics teaching as a two-edged sword. We need to decide whether or not ICT is a phenomenon that threatens our usefulness in the classroom, or liberates us from content laden and procedurally-focussed endeavours, enabling deeper conceptual treatments of the curriculum. For me, ICT is a tool that can engage concepts like nothing else before it, but it is also a tool that can be misused by teachers and students. Just like formula-driven treatments, it can lead to shallow and erroneous interpretations of mathematical structures and processes, and devalue intellectual richness of solution. It can lead to an answer driven curriculum, and a precarious faith and dependency on an electronic screen. The domain of a function, for example, appears to begin and end where the graphic calculator displays it. The solution to a quadratic equation can reduce to numbers in an electronic display, and render factorisation and the null factor law irrelevant. Curricula can become driven, and thus in my view reduced, by ICT. For example, the use of algebra as a device for creativity, exploration and discovery into uncharted mathematical waters is threatened. Concept rich explorations of techniques developed by mathematicians down through the ages can become less valued.

However, ICT, if used carefully, can illuminate concept, empower the learner, and deliver the opportunity to become creative. It can develop deeper understandings in a fraction of the time it used to take, and allow the odd serendipity to surprise us. I can remember many classes where hours of time were wasted constructing a few frequency

histograms and polygons on a black board; times when I wished I had some way of describing quickly the effect of the coefficients of a quadratic function's sketch; times when I wanted to show students Simpson's approximating parabola laying across the particular curve I was finding areas under; times when I wanted to illuminate the amplitude dampening principle on a sound wave; times when I wanted to visually demonstrate the ratio division of a line segment as the ratio changed; times when I wanted to quickly illustrate the concept of a tangent to a curve, as the point of contact rolled along its back.

Spinners and scroll bars

It was not long ago that I happened upon a simple device that resided on, of all places, a Microsoft *Excel* spreadsheet: it was called a *spinner*. It was located in View > Tools > Forms as a 'macro' and perhaps not designed for the purpose I logically put to it. I had for some time been looking around for a dynamic inexpensive and readily accessible teaching tool that could be easily adapted to the mathematics classroom. The spinner is nothing more than an incrementing device. Its cousin, the scroll bar, sits along side it in the 'drop down' box: it also increments but has the added feature of scrolling quickly through a range of possible values. These two devices can be inserted into the spreadsheet and referred to a particular cell, say A1. Clicking the mouse over the spinner or scrollbar causes the 'integer' incrementation. I say integer because, as a mathematics teacher, that is how I saw the number. I knew at once that I could create a 'rational' incrementation in another cell, say A2, by inserting in that cell a simple formula that referred to the first cell, say = A1/10. I also figured out that the contents of the second cell could be graphed, and that upon fixing the scale of that graph (an *Excel* feature), I could observe a dynamic process. Out of this early observation has come a myriad of application programs that now reside on the Canberra Mathematics Association website (www.canberramaths.org.au). I am careful not to put too much 'front end' (I am told this is the correct computer metaphor!) on each program, because I do not particularly want to lose that 'first principles' flavour to my work. The spinner can be adapted to just about any concept in mathematics that involves some kind of variability. The spinner, and not the programs, becomes the creative tool in the hands of an adventuresome mathematician.

At the time of writing this, I have investigated variability in curve sketching (including polynomials, trigonometric, exponential and logarithmic functions), Bezier curves, Bernstein polynomials, discrete probability distributions (binomial, Poisson, geometric) derivatives (including tangents and normals), standardisation, financial mathematics (including simple interest, compound interest, depreciation and annuities), Euclidean geometry, great circles, transformations (including reflections, rotations and translations), simple coordinate geometry, arithmetic and geometric progressions, and a few small calculus investigations (parabolic and simple harmonic motion). There are many more possible. As far as running into interesting results, here are two unexpected 'tangents' that were mathematically distracting enough to warrant further investigation.

1. An inflection at 30 degrees

A student, whom I had earlier shown the device to, decided spontaneously to investigate the relationship $\tan \theta = v^2/rg$ describing the angle required to eliminate any sideways frictional force on a banked circular track. He discovered that the graph of θ against v contained an inflection that remained at 30 degrees irrespective of the radius of curvature. He then proceeded to prove this algebraically.

2. Swings and roundabouts

Upon teaching the concept of score standardisation using spinners (see CMA website), I noticed that the set of scores (100, 1, 1, 1, 1, 1, 1) produced the same Z scores as the set (n , 1, 1, 1, 1, 1, 1) for all positive values of n not equal to 1. This somewhat counter-intuitive notion was later generalised to the fact that, for a set of $m+n$ scores, m of which are of value a , and n of which (different to m) are of value b , the Z scores are completely independent of a and b . I also noticed with spinners that no set of $n+1$ positive scores can contain a Z score greater than \sqrt{n} . I posed this as a conjecture in a recent *Australian Senior Mathematics Journal* article (see Staples, 2003) but to date I have yet to be advised of a proof.

Making cents of spinners

Some of the time we will spend in the workshop will deal with investigations relating to money. There are two programs dealing directly with saving and borrowing money, namely *Mortgage Wizard* and *Financial Wizard* both available from the Canberra Mathematics site (www.canberramaths.org.au) under post-primary resources. The first program looks specifically at mortgages and the interest impact on repayments. The second program explores simple and compound interest, depreciation and annuities. The use of spinners allows dynamic changes to the interest rate, the compounding periods, and the term of the loan. The effect is immediate and understandable. Participants of the workshop will complete a short exercise relating to the *Financial Wizard* program, taken from the CMA website.

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Exact answers, old problems, and a new elementary function

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Exact answers have a certain degree of elegance compared to answers given in decimal approximation form. In this workshop a problem dating back to my own frustrations encountered as a secondary pupil in trying to express the solution to a transcendental equation in exact form, acts as the impetus for a gentle sojourn into some very modern, yet accessible, mathematics. Along the way we introduce a recently defined elementary function, now known as the Lambert W function, which allows one to solve a surprising number of simple ‘classic’ problems in mathematics related to equations where the unknown appears in the exponent.

Introduction

When I was a secondary school pupil, I was such a pedant when it came to giving exact answers to numerical problems. While my peers seemed to gain a great deal of satisfaction in being able to express their answers in decimal approximation form, I found this particularly loathsome. To me, exact answers such as $\sqrt{5}$, $\ln 2$, $\pi/3$ or $\tan^{-1}(2)$ were (and still are) the apotheosis of elegance compared to any decimal approximation.

Certain problems, however, led to frustration and angst, as it appeared as though their solutions could not be expressed in exact form, no matter how hard one tried. One particular problem that has stuck in my mind all these years concerns the equation $x + e^x = 0$. Set as a challenge question for homework, my Year 12 mathematics teacher at the time asked the class to find the solution to the only (real) root of this equation in exact form. It was with bitter disappointment that I learnt the following lesson that no ‘exact’ solution seemed to exist, the challenge question having surreptitiously served as a lead in to the topic of approximate solutions to equations. Its final numerical solution in approximate decimal form obtained using Newton’s Method only added to my despondency.

Questions on exactness, such as the above, tend to fascinate our more sagacious senior secondary pupils. Do closed-formed expressions exist as solutions to such questions? To the secondary pupil, most questions like this seem to have a negative answer and are quietly explained away as having something to do with being a transcendental equation. The pupil, despite the teacher’s vague ‘transcendental’ response, may however continue to live in hope of finding a more satisfactory answer at some later stage, once they learn a little more mathematics.

It does not take one long to realise that the availability of the so-called ‘special’ functions (like the logarithmic, exponential, trigonometric and inverse trigonometric

functions) permits the solutions to a greater number of problems to be expressed in exact form. For example, in solving the simple equation $e^x = 2$, the solution $x = \ln 2$ in terms of the logarithmic function seems to satisfy our need for an ‘answer’. Here it does a rather good job in capturing the essence of what one usually considers a final answer in the sense that it is able to be written in ‘closed-form’ and is what is commonly identified as an ‘exact’ answer.

By introducing a recently defined elementary function which is rapidly emerging as one of the important elementary functions of mathematics, I wish to show how by its elevation to the status of a standard elementary function it is not only able to provide a solution to my former mathematics teacher’s question, but finds itself at the centre of many other celebrated problems of mathematics. All this is contained in a simple, yet accessible, elementary function which could be purposefully introduced to our senior secondary pupils.

Enter the Lambert W function

Given that I now know a little more mathematics compared to when I was a secondary school pupil, what am I able to say about my past *bête noire* $x + e^x = 0$ in relation to expressing its answer in exact form? A great deal!

Let us begin our peregrination by trying to find a closed-form solution to the transcendental equation $y^y = x$, in terms of y . In doing so, we are led to the definition for the *Lambert W function*, a new elementary function which has only recently been defined (Corless, Gonnet, Hare, Jeffrey & Knuth, 1996) as the inverse of the function $f(x) = xe^x$. Denoting the Lambert W function by $W(x)$, we see that it is a solution to the equation

$$W(x)e^{W(x)} = x.$$

The above equation is called the *defining equation* for the Lambert W function and is central to the understanding of this function.

The origins of the Lambert W function date back to the mid-eighteenth century, being traceable to the initial and subsequent works of Lambert (1758) and Euler (1779) respectively; however, it seems to have been quite unintentionally overlooked until the mid-1990s when at last it was finally recognised as being sufficiently important to warrant a name of its own. While being a largely recondite function at present when compared to the familiar elementary functions one typically encounters at secondary school, it is, nonetheless, rapidly emerging as one of the important elementary functions of mathematics. Furthermore, it turns out to have a surprisingly rich mathematical structure and wide applicability, yet it is fundamentally no more difficult than that of the logarithmic function. It is surprising then to learn that it was overlooked for so long.

Since W is the inverse of the function $f(x) = xe^x$, it presents an opportunity to work with and further explore inverse relations/functions. As an illustrative example of this, $y = W(x)$ can be readily sketched by reflecting the curve $y = xe^x$ about the line $y = x$ (see Figure 1).

The Lambert W function is not too unlike the inverse trigonometric functions in that it is a multi-valued function on a given domain, and a *principal branch* needs to be defined. When x is real it has two branches. The branch satisfying $W(x) \geq -1$ is denoted by $W_0(x)$, and is defined to be the principal branch, while the secondary real branch satisfying $W(x) \leq -1$ is denoted by $W_{-1}(x)$.

Many equations which involve exponentials (or logarithms — that is, equations which

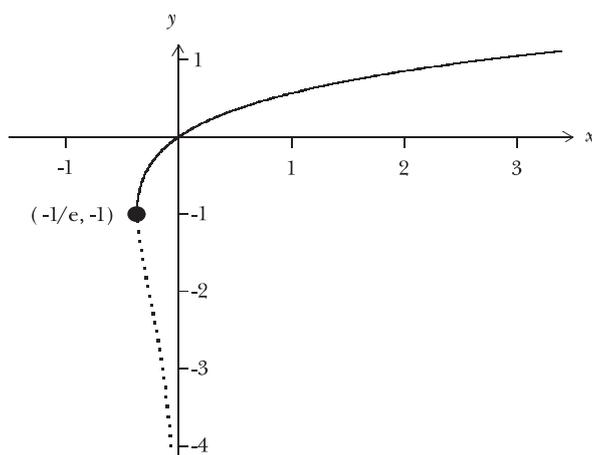


Figure 1: Plot of the Lambert W function $y = W(x)$. The solid line shows $W_0(x)$ while the dashed line shows $W_{-1}(x)$.

are exponent in form) are now able to be solved in terms of the Lambert W function, including my past *bête noire*. The general strategy to solving such equations introduces the general method of implicit solution. Here one moves all instances of the unknown to one side of the equation, make it look like the form of the defining equation, namely $f(x)e^{f(x)}$, at which point the Lambert W function provides the solution to the equation.

Consider the solution to the equation $x + e^x = 0$. As I learnt many years ago, this rather innocuous looking equation is not able to be solved in closed-form in terms of any of the known elementary (or higher) functions with which one is traditionally familiar. In the past, numerical methods have been required in order to find an approximate solution for x . If, however, we rewrite this equation as

$$x = -e^x$$

and move all instances of the unknown to the left hand side we have

$$xe^x = -1$$

Next, writing the left hand side of the above equation in the form of the defining equation, namely

$$-xe^x = 1$$

enables this equation to be readily solved in terms of the Lambert W function as

$$-x = W_0(1)$$

or

$$x = -W_0(1) \text{ } (-0.56714329\dots)$$

and is my long sought after exact answer! Substituting the infinitely more vulgar numerical value for x into the initial equation can be used to confirm the validity of this solution for any nihilist among us.

Initially, the legitimacy of a Lambert W function form of the solution will seem to many, I suspect merely sophistic and not real. The form of this solution is, however, more than a clever slight of hand designed to assuage any desire for exactness. I will show that rather than being an efficacious way of getting an answer which is exact in form to a problem where this was previously not possible, it is instead written in terms of a function which is now ineluctably part of the familiar elementary function family. Let us not forget those other seemingly 'legitimate' special functions we often make use of without a second thought: the e^2 , $\ln(2)$, $-\sin(2)$, $\tan^{-1}(2)$, and so on.

As an exact answer, $-W_0(1)$ seems to satisfy our intuitive need for what we usually regard as a 'solution' in that x is expressible in closed-form; it is explicit. Is $-W_0(1)$ any different

from, say, $\ln(2)$, other than being less familiar than the natural logarithmic function? Certainly the latter is expressible in terms of a familiar function, and after all, it is familiarity which is important since the solver must regard the function, whether it be W or \ln , as the final answer and not simply as another question. Just as a pupil is unlikely to regard the exponential function as a solution to the equation $\ln x = 2$ until the moment they formally encounter it, the same could therefore be said for the Lambert W function.

It is hoped that it will only be a matter of time before we all find familiarity and comfort in the use of W once its ubiquitous nature and wide ranging applicability is recognised. In being seen as a bona fide elementary function equal to those of the more familiar elementary functions, an entry for W is now to be found in Eric Weisstein's weighty encyclopedic tome of mathematics (2003, pp. 1684–1685), and it is included as an inbuilt library function in computer algebra systems such as *Mathematica* (`ProductLog[x]`) and *Maple* (`LambertW[x]`).

Old dogs, new tricks

A function is only as important as it is considered useful. In establishing its usefulness, it should turn up in many different contexts, particularly those where it is least expected.

A perennial problem which has been considered since the time of Euler (1748) concerns the solutions to the classic 'difficult' equation

$$x^y = y^x \text{ for } x, y > 0.$$

In solving this equation we acknowledge the trivial solution $y = x$. We expect this equation can also be solved for y in terms of the Lambert W function since it is, after all, an equation of the exponent form.

Taking the natural logarithm of both sides of this equation and rearranging gives

$$\ln y = \frac{\ln x}{x} y$$

Exponentiating both sides yields

$$y \exp\left(-\frac{\ln x}{x} y\right) = 1$$

or

$$-\frac{\ln x}{x} y \exp\left(-\frac{\ln x}{x} y\right) = -\frac{\ln x}{x}$$

Upon solving for y we have

$$y = -\frac{x}{\ln x} W_k\left(-\frac{\ln x}{x}\right)$$

where $k = -1, 0$ denotes the two real branches for the Lambert W function.

Voila! The problem is solved and may now be laid to rest. It would be a little disappointing, however, if that were the case considering the considerable attention this problem has attracted since the time of Euler. I am reminded of a quote by Barry Mazur in his book *Imagining Numbers* where he reminds his readers:

But if the problem is really good, a solution of it is nothing more than a letter of introduction to a level of interaction with the material that you hadn't achieved before...

The problem itself is an invitation, a goad, to extend your imagination. This is true... of some — perhaps all — of the famous and venerable mathematical problems. (Mazur, 2003, p. 23)

So what is the ‘letter of introduction’ the solution to this particular problem holds febrilely before us? It is the trivial solution! Taken together, these two solutions lead to a not entirely obvious simplification rule for the Lambert W function being found. Here one obtains

$$-\ln x = \begin{cases} W_0\left(-\frac{\ln x}{x}\right) & \text{for } 0 < x \leq e \\ W_{-1}\left(-\frac{\ln x}{x}\right) & \text{for } x \geq e \end{cases} \dots (1)$$

By replacing x with $1/x$ the negative signs in the above simplification rule are able to be removed resulting in the more compact result of

$$x = \begin{cases} W_0(x \ln x) & \text{for } x \geq \frac{1}{e} \\ W_{-1}(x \ln x) & \text{for } 0 \leq x \leq \frac{1}{e} \end{cases} \dots (2)$$

The above two simplification rules complement one other simplification rule which should be immediately obvious once we recall W is just the inverse of the function $f(x) = xe^x$; namely,

$$x = \begin{cases} W_0(xe^x) & \text{for } x \geq -1 \\ W_{-1}(xe^x) & \text{for } x \leq -1 \end{cases}$$

Consequently, we are immediately able to write $W_0(e) = 1$, $W_0(2e^2) = 2$, $W_{-1}(-2e^2) = -2$, and so on.

A second example making use of the Lambert W function, and which is also connected with another classically celebrated problem since Euler’s time, is the problem of (infinite) iterated exponentiation. Consider

$$x^{x^{x^{\dots}}}$$

which consists of an infinite power tower of x s such that the powers are read from the top down. Euler, in 1778, was the first to show that this iteration converges on the interval

$$\frac{1}{e^e} \leq x \leq e^{\frac{1}{e}}$$

Consisting of exponents, we suspect the problem of iterated exponentiation can be solved in terms of W on its domain for which it is convergent. Taking the natural logarithm of both sides of the iterated exponential, we can write

$$\ln h(x) = \ln x^{x^{x^{\dots}}} = x^{x^{x^{\dots}}} \ln x = h(x) \ln(x)$$

Upon exponentiating both sides of the above equation we have

$$h(x) = e^{h(x) \ln x}$$

or upon rearranging

$$h(x)e^{h(x) \ln x} = 1$$

By writing

$$-h(x)\ln x e^{h(x)\ln x} = -\ln x$$

so that it is in the form of the defining equation and solving for $h(x)$, gives

$$-h(x)\ln x = W_0(-\ln x)$$

Thus

$$h(x) = x^{x^{x^{\dots}}} = \frac{W_0(-\ln x)}{-\ln x} \quad \text{for } \frac{1}{e^e} \leq x \leq e^{\frac{1}{e}}$$

Incredible, and impressive to say the least, how the Lambert W function provides a neat, closed-form expression to the problem of iterated exponentiation, and it was on seeing this result that my interest in the Lambert W function was initially piqued.

Once again we suspect the solution to this problem is but an entrée to a deeper level of interaction with the material, and once more one will not be disappointed in what emerges. Having a closed-form expression for the iterated exponential at hand allows for some otherwise difficult questions to be readily answered.

Consider, for example, $x = \sqrt{2}$. Here

$$\sqrt{2}^{\sqrt{2}^{\sqrt{2}^{\dots}}} = \frac{W_0(-\ln \sqrt{2})}{-\ln \sqrt{2}} = -\frac{2}{\ln 2} W_0\left(-\frac{\ln 2}{2}\right) = 2$$

where use of the first of the simplification rules given by equation (1) has been made. The astute reader may object to the method used above, arguing that it is somewhat overkill, considering it is possible to solve this particular iterated exponential without any prior knowledge of W whatsoever. Similarly, without any knowledge of W, when $x = 1/4$ one can again obtain in a rather perfunctory manner an answer of

$$\frac{1}{4}^{\frac{1}{4}^{\frac{1}{4}^{\dots}}} = \frac{1}{2}$$

So has the introduction of W unnecessarily complicated the problem of iterated exponentiation? Of course not! Instead, it extends us into a realm where one previously could not venture so easily. When the final solution to the iterated exponential no longer takes on a rational form, solving such problems in exact form becomes impossible without the availability of W. However with W at hand, one is for example able to write down the following exact answers of

$$\frac{1}{2}^{\frac{1}{2}^{\frac{1}{2}^{\dots}}} = \frac{W_0(\ln 2)}{\ln 2}, \quad \sqrt[3]{3}^{\sqrt[3]{3}^{\sqrt[3]{3}^{\dots}}} = -\frac{3}{\ln 3} W_0\left(-\frac{\ln 3}{3}\right)$$

and so on, and can consider such problems as solved. And for those constantly complaining decimalists, who consider nothing as solved until they finally ‘see’ a number written down on the page, please excuse my above reticence and let me write 0.64118574... and 2.47805268... respectively!

Conclusion

Motivated by a long standing desire to find an exact answer to a transcendental equation first encountered by the author while still at secondary school, has led to what I hope you found to be an interesting sojourn into a little area of ‘modern’ mathematics. Along the way we collected a new elementary function now known as the Lambert W function and saw how it was related to the solution of a few classically celebrated problems dating back

to the time of Euler. While being accessible, I concede that any attempt to change long ingrained thinking towards what one normally regards as the standard set of elementary functions is to be regarded as largely a quixotic endeavour, I hope nevertheless to have encouraged you to think otherwise.

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Energising secondary school mathematics through modelling with the graphics calculator*

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The computer needs of schools are beyond the finances of some Australian state governments. Does this inability to provide resources necessarily mean that some children are deprived of the benefits that technology can produce in enhanced mathematical understanding? Can graphics calculators provide a low cost alternative? Our world is caught up in an information revolution. To many, information is synonymous with knowledge. However, while mastery of information is demonstrated by its reproduction, mastery of knowledge requires demonstrated novel applications. Is it possible to develop this mastery of knowledge within secondary mathematics classrooms? This paper and workshop will answer these questions.

Is the technological age for all students?

In Australia while most state governments are keen to change education, not all are equally positioned to finance the hardware requirements of information and communication technologies (ICT) in schools. There are many factors that can limit access to ICT. In some areas of Australia, neither teachers, nor their students can easily afford to purchase a computer for home use. Educational systems must cope with the cost of computers, the cost of site licenses for software, the cost of the extra security, constraints upon school timetables, lack of trained and experienced staff. The ideal world would have every school possessing an abundance of secure computer laboratories with inspiring teachers. However it is not an ideal world and thus the question arises: does the inability to provide ideal resources mean that certain children should be deprived of the benefits that technology can produce through enhanced mathematical understanding and learning?

Is ICT to be integrated or diluted?

Research (White, 2003) has produced a framework that identified and described five broad categories or metaphors of teacher response to the integration of ICT within their classroom. These metaphors describe how teachers tend to view ICT as either a demon, a servant, an idol, a partner, or a liberator.

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ICT as demon

White (2003) claimed that the evidence for this approach is observable in the teachers who actively opposed and subverted any attempt to integrate ICT into their curriculum. They are either afraid or unwilling and as a result, conduct an active or passive resistance campaign by doing the very minimum allowed. This often leads to surface integration and sometimes to inappropriate use.

ICT as servant

White (2003) claimed that this stage is observable in teachers who adopt a conservative position where ICT is being used by the teacher and students yet the pedagogy remains much the same. ICT thus is a tool for enhancing students' learning outcomes within the existing curriculum and using existing learning processes (Russell & Finger, 2003). Salomon (2000) sees this as a 'tendency of the educational system to preserve itself and its practices by the assimilation of new technologies into existing instructional practices. It fits into the prevailing educational philosophy of cultural transmission, where there is a body of important knowledge that has to be mastered' (p. 2).

ICT as idol

Here White (2003) described a stage that promotes ICT as a tool for use across the curriculum and where the emphasis is upon the development of ICT-related skills, knowledge, processes and attitudes (Russell & Finger, 2003, p. 3). Teachers are more focussed upon teaching *about* computers than *with* computers, and expect that ICT on its own will bring about change. They are seduced by the 'razzle dazzle effect' of these 'techno toys' and fail to consider the teaching and learning implications beyond a very surface level. Sir John Daniel (2002) calls this stage bias and bull: 'I urge you to be sceptical about assertions of the value of technology coming either from those who want to sell it to you or from their surrogates' (p. 13).

ICT as partner

White (2003) claimed that there are teachers who have seriously attempted integrating ICT into their classroom and tried to 'change the orientation from teaching about computers to teaching with computers' (Russell & Finger, 2003). These classrooms are where students are actively engaged in gathering data, aggregating their data with those gathered by other students, and making meaning of their results. Mathematical modelling using ICT is an example of this approach and is the focus of the second part of this paper. Here ICT is integral to the pedagogy that changes not only how students learn, but what they learn.

ICT as liberator

White (2003) presented a radical approach whereby integration is a component of the reforms that seeks to alter the organisation and structure of schooling itself. 'Among the diversity of school types will be virtual schools, where students spend part or all of their time working "off-campus", for example, from home using an online computer' (Russell & Finger, 2003, p. 3). However there are unintended and unwelcome effects of virtual schools as Salomon (2000) stated: 'not many students have the self-discipline or the sus-

tained motivation to be distance, virtual learners' (p. 4). He regards this approach as another example of technocentrism that is in danger of yielding virtual results.

Yet there are alternatives where 'more sophisticated understandings of the implications of ICTs for reforms in curriculum, pedagogy and assessment are required' (Russell & Finger (2003, p. 9).

With such a diverse range of teacher behaviour to the integration of ICT, is it possible to move teachers towards the stage of 'ICT as partner'? The writer has argued elsewhere (White, in press) that graphics calculators (GCs) not only provide an alternative to computers but they also act as 'a pedagogical Trojan horse'. He argued that it is important to introduce technology into the classroom, even if it is merely one GC connected to a display panel and controlled by the teacher. The teacher will move from a transmission style to other more student-centred styles as they gain in confidence and experience with the GC in the classroom. With this simple low cost teaching aid, the teacher is able to use the power of technology to illuminate mathematical concepts for the students. For example, the speed at which a GC can present multiple representations of a function is one of many advantages that technology has over traditional methods in the teaching of mathematical concepts. He claimed that as the teacher gained in proficiency of GC use and became comfortable with the GC supporting the teacher's usual classroom teaching strategies, then the teacher became open to accepting a range of other teaching and learning strategies.

Age of information or of knowledge?

The world is caught up in an information revolution. For many people information is synonymous with knowledge. Yet Salomon (2000) pointed out, this is far from being true and the differences have enormous implications. Salomon listed the differences between information and knowledge as:

- Information is discrete; knowledge is arranged in networks with meaningful connections between the nodes.
- Information can be transmitted as is; knowledge needs to be constructed as a web of meaningful connections.
- Information need not be contextualised; knowledge is always part of a context.
- Information requires clarity; the construction of knowledge is facilitated by ambiguity, conflict and uncertainty.
- Mastery of information can be demonstrated by its re-production; mastery of knowledge is demonstrated by its novel applications' (p. 4).

Now it is clear why Salomon's distinction is so important. If all we want from our students is information, then the old transmission models of teaching are sufficient. Teacher-centred and teacher-controlled environments where students are expected to be quiet and absorb what is being presented have been with us a long time. Commercial interests are encouraged to set up 'cram schools' where the students can go after school and spend hours memorising and rehearsing. Examinations are designed to test mainly students' ability to recall facts and procedures. Teachers are only too aware of the tyranny of the examination, which effectively drives the classroom curriculum. There is a wealth of material documenting the weaknesses and implications of such an approach, such as

a recognition that rather than training pupils in the whole mathematical process,

schools have focussed almost exclusively upon mathematical manipulations (i.e. from maths model to maths solution). This over-emphasis upon skills is seen as one of the causes of many pupils failing to see and appreciate the applicability and the potential of the mathematics they have learnt (Swetz & Hartzler, 1991, p. 8).

So, if we want our students to demonstrate and use their knowledge and not just reproduce information or skills, then we must prepare them to be able to transfer their learning to unfamiliar and unique situations. Teachers are challenged to create environments where students are able to create knowledge networks of meaningful connections between pieces of information or procedures. They need to prepare their students for the construction of knowledge by exposing them to ambiguity, conflict and uncertainty. A different style of examination is set that is designed to test student ability to apply learning to a range of challenging and realistic problems and investigations. Is there a mathematical teaching and learning classroom process that can accommodate all four aspects of knowledge construction as proposed by Salomon? The answer is, yes.

Turning information into knowledge

A number of different domains of knowledge (such as medicine) have turned to problem-based learning as an answer to developing within students an ability to construct and transfer knowledge.

Problem-based learning places emphasis on what is needed, on the ability to gain propositional knowledge as required, and to put it to the most valuable use in a given situation. It does not, therefore, deny the importance of 'content' — but it does deny that content is best acquired in the abstract, in vast quantities, and memorised in a purely propositional form, to be brought out and 'applied' (much) later to problems. Problem-based learning requires a much greater integration of knowing that with knowing how' (Margetson, 1991, p. 44).

In mathematics, a form of problem-based learning is called mathematical modelling. Mathematical modelling was strongly showcased in the early 1980s and is receiving renewed interest due to the current focus upon the process of working mathematically. The process comes widely acclaimed: 'Of the several kinds of creative activity being promoted in contemporary developments, arguably the most empowering for students is mathematical modelling' (Galbraith, 1995, p. 312). There is also considerable material available on the teaching of mathematics using modelling, with a particularly good resource book having recently been released (Goos, 2002). It is from this resource book that the following activity is taken, in order to provide an example for this discussion. The modelling cycle is usually pictured as a process cycle and there are many variations: an example is provided in Figure 1. While the secondary mathematics classroom process is never as ordered as shown in the diagram, nevertheless I will follow the process of Figure 1 while briefly investigating a relevant and topical problem in this year of the Olympics. This investigation will provide a context for students to use their knowledge and to develop new knowledge by developing further connections and learning new skills as they are needed.

Stage 1: Real world problem

World records are broken usually by only a few centimetres. For example, in the thirty-nine years from 1928 to 1967, the long jump world record rose by only 42 cm. Yet at the 1968 Mexico City Olympic Games, Bob Beamon broke the record by 55 cm. The question the class will investigate is: how good was Bob Beamon?

Stage 2: Consider assumptions

A class discussion would elicit that to be a fair contest, conditions are the same for all athletes: no wind assistance, no drugs, no altitude effect, etc.; also, that the pattern of improvement in performance would be similar from year to year. The class would have to decide that there is an assumption that the contest was fair. This assumption can be revisited if other data comes to hand. The discussion would determine that previous data for year and distance jumped will be needed. The information is condensed as follows:

Distance (m) [Date of record], 7.61 [August, 1901], 7.69 [July, 1921], 7.76 [July, 1924], 7.89 [June, 1925], 7.90 [July, 1928], 7.93 [September, 1928], 7.98 [October, 1931], 8.13 [May, 1935], 8.21 [August, 1960], 8.24 [May, 1961], 8.28 [July, 1961], 8.31 [June, 1962], 8.31 [August, 1964], 8.34 [September, 1964], 8.35 [May, 1965], 8.35 [October, 1967], 8.90 [October, 1968], 8.95 [August, 1991].

A complete and comprehensive table can be found in Goos (2002) and the GC instructions for a TI83+ are also included although the model of GC is not important. This problem can be modelled on any brand.

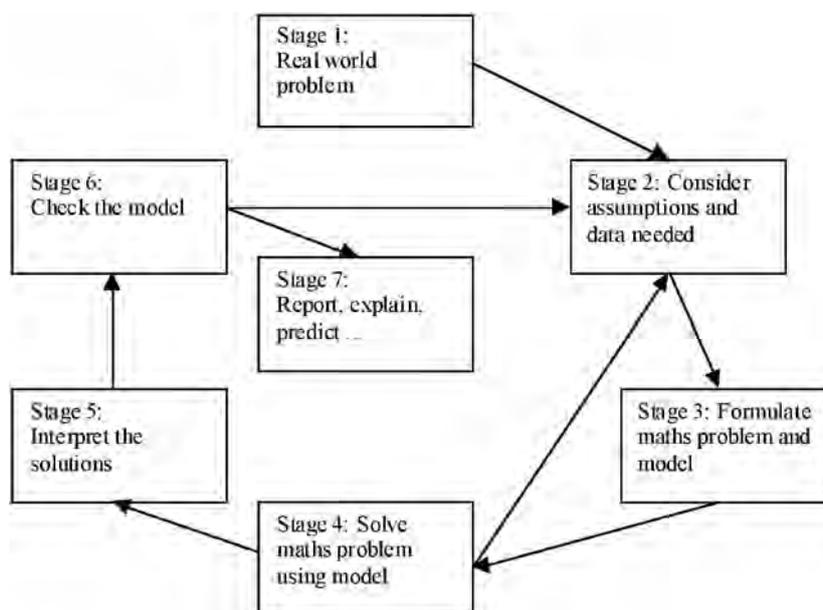


Figure 1. The stages of the mathematical modelling process.

Stage 3: Formulate the mathematics problem and construct a model.

As we have the data in table form, it is easy to enter it into the GC with years in list 1 and heights in list 2. We then ask the students to decide the model that they want to use with this data. A statistics plot seems to be the best option and is easily achieved with a GC. The

trace function can then be used to interrogate the points. A discussion would result in a straight line being used to best represent the data. This would be done using their knowledge of how to find the equation of the line joining two points.

Stage 4: Solve the mathematics problem using the model

Using the points (1901, 7.61) and (1967, 8.35), the gradient of the line joining them can be calculated. Substituting this value and the coordinates of one of the points into $y = mx + c$ gives an estimated equation of $y = 0.0112x - 13.68$. This equation can then be plotted over the data points; a discussion should follow about why the use of only two data points is not generally appropriate.

Stage 5: Interpret the solution

Students may wish to fine tune their equation until they are satisfied with the line of best fit. This provides opportunities for them to explain how gradient and intercept alterations affect the graph. Tracing along the line allows us to find the year in which a jump of approximately 8.90 metres is predicted at 2016, the year in which previous world record trends indicate such a jump would be expected.

Stage 6: Check the model

This is an important stage in the process but one that is often rushed or over-looked. This is the stage where the class considers the model and solution against the initial problem and the assumptions. If the model needs modifying then the class must go through the process again (stage 2). If the class is satisfied then they can move to stage 7. At this stage the GC is very useful by using the linear regression option available. A good discussion will elicit that the two outliers (1968 and 1991) should be removed before completing the regression. We can then enter the new equation $y = 0.0116x - 14.5106$ to see if it makes any difference.

It is important to stress that a model is only as good as the assumptions that it is based upon. Students must make two decisions. First, whether the assumptions have been violated or if other variables should be included; and second, should they continue investigating or stop and write a report of their findings. They began with the assumption that it was a fair contest, yet many studies were published searching for explanations of Beamon's record jump. One popular theory attributed his performance to the altitude and rarefied air of Mexico City (2250 metres above sea level). However, the students may point out that the other contestants were also jumping at this height and that at Tokyo in 1991 the record was again broken, so this assumption has not been broken. Otherwise this variable would need to be included and we would develop a method in our model for compensating for altitude.

What could really develop into an interesting investigation is rejecting the second assumption of linearity, but that is the substance for another paper.

Stage 7: Report, explain and predict

So in our case, as the class are happy with the model, they make the statement that Bob Beamon was forty-eight years ahead of his time, based on their available evidence. You can read more about Bob Beamon's story at the internet site:
www.sptimes.com/News/121699/Sports/Beamon_jumps_into_recs.html.

Conclusion

This paper has raised a number of questions in attempting to give a sketchy perspective on the secondary mathematics classroom and some of the forces acting upon it. It has been deliberately provocative in challenging some common current teacher behaviours. To those who say that they do not have computers, it answers by saying start small and use a graphics calculator. To those who say their current teaching methods help students to master information, it answers by saying teach them to construct knowledge. To those who say teachers still teach in a transmission way, it answers by saying give them a graphics calculator and panel, as it will help to change the teacher. Finally, to those teachers who have never experienced the enthusiasm and creative surge within a classroom of students immersed in modelling with graphics calculators, it sympathises by saying that they do not know what they have been missing.

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Mathematical simulations using a spreadsheet

Paul White

ACU National

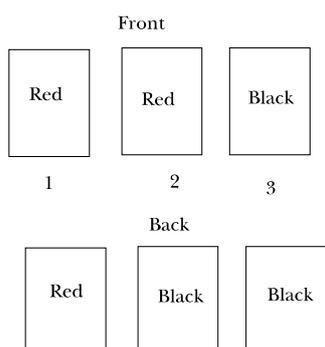
The random number generator and logic functions on a spreadsheet provide a way to simulate situations involving chance and to investigate mathematical relationships which model real world phenomena. This paper looks at a number of such simulations for which spreadsheets can be constructed. Spreadsheets mentioned are available from the author.

One form of mathematical modelling which allows for investigations of real situations is simulation. Simulation can take a number of forms according to the mathematics concerned. For example, it is particularly powerful in highlighting common misconceptions about chance events. Simulation can also be employed to investigate mathematical relationships in real world phenomena. By simulating the relationship for a simple model, inferences can be drawn about realistic situations involving living creatures. Some examples follow.

Chance situations

Events may be simulated using a re-enactment process, which is sometimes effective, but other times is quite clumsy. A more efficient alternative is to use spreadsheets (with the random number generator features and certain logic functions). Both types are considered. Spreadsheets which are mentioned are available from the author.

Three card risk

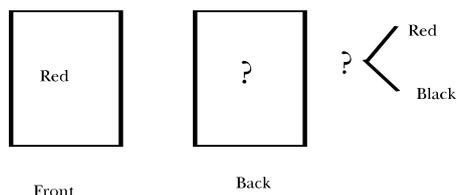


The game is played between a dealer and a player using three cards. One card is black on each side, one card is red on each side and the third card has one side black and the other side red.

The dealer hides the cards in a hat and allows the player to pick one. The player lays it face down on the table, showing one side face up without looking at the other side. The dealer wins when the card has the same colour on both sides (i.e., is the red-red or black-black card).

The argument is that this is a fair game. For example, if the

card laid down has red face up, the card cannot be the black-black card and so is either the red-red or red-black card. Therefore, there are two outcomes: one where the two faces have the same colour and one where they are different.



Hence, there is a 50% chance of beating the dealer. Is this a fair game? Modelling by acting out here is quite efficient and usually quickly shows that it is not a fair game and that the dealer has a two thirds chance of winning.

The day you were born

Do you think one seventh of Australians were born on a Monday? Tuesday? It is not possible to ask all Australians, but it is possible to ask a sample and to calculate the day for any individual. With a large enough sample, some conclusion can be drawn. How to calculate? Use the formula:

$$W = 2 + D + 2M \left(\frac{2(M+1)}{5} \right) + N + \left(\frac{N}{4} \right) - \left(\frac{N}{100} \right) + \left(\frac{N}{400} \right)$$

W = day in week; e.g., Sunday = 1, 8, 15... Monday = 2, 9, 16... Saturday = 7, 14, 21...

D = day of month; e.g., $D = 30$ means the 30th day of the month.

M = number of month (March = 3, April = 4... December = 12, January = 13, February = 14). Note: 1 and 2 are not used as values for M .

N = year; e.g., $N = 2004$ means the year 2004. Note: when the month is January or February, use the previous year for N , e.g., for January and February 2004, use $N = 2003$.

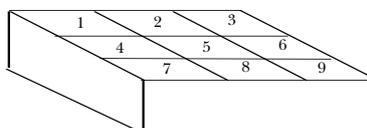
$[x]$ means the largest integer not greater than x ;
e.g., $[1.5] = 1$, $[-1.6] = -2$, $[3] = 3$, $[-2.4] = -3$

Clearly, to calculate this manually will require a great deal of work. The random number generator on a spreadsheet and the use of logic functions makes it much more accessible.

Then using the dates of known people, a reasonable data base can be created.

Show game

A square box is divided into nine equal compartments 1 to 9 for a game called 'Three in a row'. The object of the game is to throw three softballs into the box. Three balls in a line make a winner. For example, 1-4-7, 3-7-5 or 6-4-5 are winning results, but 1-5-8, 2-9-4 and 7-8-1 are not winning results. Any throw which does not land in a compartment is re-thrown. What are the chances of winning?



The number of winning combinations can be easily calculated as eight. However, the total number of possible combinations (9 choose 3) is not easily calculated by most students. The situation can be modelled using the random number generator on a calculator. (Use the three digits to correspond to the outcomes of the three throws. For example, 735 corresponds to the balls landing in compartments 7, 3 and 5. Any random number with a zero or repeated digits is ignored.) This last step can make simulation by hand time consuming, but a spreadsheet can eliminate this obstacle and give a sizable sample very quickly. The perceptions about chances here (about 11%) can also be erroneous.

Birthday problem

A famous one: how many people do you need in a group before the chance of at least two of them having the same birthday is 50% or better? Usual estimations are 180 or so; the actual answer is 23 and can be easily demonstrated on a spreadsheet. A simpler version is to use *month* instead of *year*. For example, in a group of four people, what is the chance that two or more will have their birthday in the same month? This can be simulated manually by using a pack of cards: Ace = January, 2 = February, ... 10 = October, Jack = November, Queen = December. Four cards are dealt. When two or more of the same numbers appear, these people are considered to have the same birthday month (success). When all four cards show different numbers, there are no common birthday months (failure).

Again, a spreadsheet allows for a large number of trials in a short time.

Sample size

A town has two hospitals. On average, each day there are thirty babies born in one and ten babies born in the other. In one year, each hospital records the number of days the number of girls born was 60% or more of the total number of deliveries. Which hospital is more likely to record the most 60%+ days for girls?

To simulate the above situation for the larger hospital use the random number generator thirty times on a calculator with < 0.5 being female and ≥ 0.5 being male (note that it is possible to have the number .000 appear randomly and this scores as a female.)

To simulate the above situation for the smaller hospital, use the random number generator ten times on a calculator with < 0.5 being female and ≥ 0.5 being male.

To obtain enough data to show that the smaller hospital is more likely could take a long time by hand. A spreadsheet provides a quick analysis.

Surface area and volume

Heating up, dehydration and heat loss depend on the relationship

$$\frac{\text{surface area}}{\text{volume}}$$

The larger this comparison, the faster a person will dehydrate or lose heat because there is a greater area from which moisture or heat can escape compared to the amount of moisture or heat contained in the body.

In general, the effect of increasing size of this relationship can be explored in the simple case of a cube as shown.

Side length	1	2	3	4	5	6	8	10
Surface area	6	24	54	96	150	216	384	600
Volume	1	8	27	64	125	216	512	1000
Value:1	6	3	2	1.5	1.2	1	0.75	0.6

The general result for the cube is found easily enough by hand, but can be explored more extensively by spreadsheet. Furthermore, graphs of side length – Value:1 can be plotted to show the asymptotic nature of the pattern. Consider as well side length compared to successive ratios.

Side length	1	2	3	4	5	6	8	10
Value:1	6	3	2	1.5	1.2	1	0.75	0.6
Successive ratio *		2	1.5	1.33	1.25	1.2	1.17	1.14

Extending this table on a spreadsheet and plotting the resultant points shows the curve

$$y = 1 + \frac{1}{x}$$

which can be confirmed by algebra.

One other aspect relating to heat loss is that all warm blooded animals use energy to keep warm and the energy required is proportional to how easily the animal loses heat. The following shows the energy used per day by some different animals.

Animal	Mass in kg	Kilojoules per day
Guinea Pig	0.7	656
Rabbit	2	487
Human	70	9700
Elephant	4000	218 400

The rate

$$\frac{\text{kilojoules per day}}{\text{mass}}$$

provides a comparison of mass to food consumption. This relationship is quickly explored on a spreadsheet. An index for each pair of animals can be established on the basis of their mass and kilojoule ratios.

Using the information from simulating surface area and volume for a cube allows for inferences about living creatures. For example:

- It explains why large animals have short stocky legs, big animals like elephants have trouble getting rid of body heat and so often like water or may have large ears to increase their surface area.
- It shows that gravity can be a killer for larger animals. A human would certainly be killed from a fall of 500 metres. However, a mouse would walk away from such a fall. On the other hand, surface tension is not a problem for larger animals, but is a serious danger for smaller animals. An elephant, hippopotamus, etc. are not troubled by the amount of water which sticks to their skins. However, a mouse has to carry its own weight in water and can easily drown. Insects have no hope. This is why the surface of swimming pools often has dead insects floating on the top and small animals which swim have very strong legs.
- The tallest person in medical history was the American Robert Wadlow (1918–1940). He grew to a height of 272 cm and had a shoe size of 37. How does

his shoe size to height compare to other people of more normal height? Why might he have such big feet? As a youth, Robert had enjoyed good health, but his large feet had troubled him for many years. He had little sensation in his feet and did not feel any chafing until blisters formed. While making an appearance in July 1940, a fatal infection set in when such a blister formed.

- If you doubled your height, width and depth, would you feel any different?
- What might the giant in Jack and the Beanstalk look like?
- Do thinner/tall people feel the cold more than heavier set/shorter people?

Conclusion

The use of simulation can provide interesting, meaningful mathematics which require the adaptation of 'school mathematics'. Simulation can also be used as a tool to investigate (often) misleading perceptions.

Mathematics Challenge for Young Australians

Sue Wilson

Australian Mathematics Trust

The Mathematics Challenge for Young Australians is a program run by the Australian Mathematics Trust. This program aims to encourage students in a greater interest in mathematics and a desire to succeed in solving interesting mathematical problems. At the same time, it also aims to provide teachers with interesting and accessible problems and solutions, detailed discussion and extension materials, and statistics of students' achievements in the Challenge.

It is directed at the top ten per cent of primary school students in Years 5 and 6, and secondary students in Years 7 to 10. It may be particularly useful in schools where teachers may be working with talented students spread out over a number of classes.

The Challenge provides materials so that these teachers may help talented students reach their potential. Teachers in larger schools also find the materials valuable.

There are three stages in the Mathematics Challenge for Young Australians: the Mathematics Challenge stage, the Mathematics Enrichment stage, and the AMOC Intermediate Contest. Each is an independent program and this workshop will demonstrate the first two: the Challenge and Enrichment stages.

Mathematics Challenge Stage

Students from all states of Australia, as well as New Zealand, Hong Kong and Singapore attempt the Mathematics Challenge stage. In 2004 there were over 14 500 entries from 585 schools.

The Mathematics Challenge stage (held during three consecutive weeks around April) comprises four problems for students in primary schools and six problems for secondary school students. There are separate problem sets for primary (Years 5–6), Junior (Years 7–8) and intermediate (Years 9–10) students. In the junior and intermediate levels two problems can be discussed in pairs before individual submission of solutions while the other problems are to be attempted individually.

Participating schools will receive student problem booklets and a teachers' reference book containing solutions and marking schemes, with teaching, discussion and extension notes in problem solving. Within Australian schools, Mathematics Challenge Directors who wish to pay to have their Mathematics Challenge stage scripts marked, can do so. In approximately late July, each school will receive comprehensive Australia-wide statistics of students' achievements, as well as Certificates of High Distinction, Distinction, Credit or Participation for their students.

Mathematics enrichment stage

Students from all states of Australia, as well as New Zealand, Hong Kong and Singapore attempt the Mathematics Enrichment stage. In 2004 there were over 6600 entries from 414 schools. The most popular series is the Euler series, followed by the Gauss series. The Newton and Dirichlet series, which were introduced more recently, are growing in popularity. Currently, each has approximately 1000 entries. The Australian Mathematics Trust has received excellent feedback from primary schools about both series, including a recent email from a teacher who had students queued at the door to join the program. The program can be organised at the school by a teacher or an interested parent.

The Mathematics Enrichment stage is a six-month enrichment program. It consists of six different, parallel series of comprehensive student and teacher support notes. Students in this stage of the Mathematics Challenge for Young Australians work through their chosen series notes during a flexible sixteen-week period between April and September. Each student receives student notes, including topics relevant to the series they are participating in, and a student problem booklet containing up to sixteen problems. The school will receive a teacher set for each relevant series consisting of a teacher's reference book containing solutions and marking schemes, with teaching, discussion and extension notes, plus the student problem booklet and student notes.

The Mathematics Enrichment stage does not depend on the earlier Challenge stage. They both provide challenging mathematics problems for students, as well as support materials for teachers.

The six series are:

- *Newton*
This series comprises a number of introductory topics in geometry, counting and numbers. It introduces polyominoes, fast arithmetic, polyhedra, pre-algebra concepts and divisibility as well as chapters on problem solving. Although it is written for advanced Years 5 and 6 students, the series is also most appropriate for use with Years 7 and 8. There are eight questions in the series.
- *Dirichlet*
This series is designed for students in Years 6 and 7. This series has chapters on some problem solving techniques, tessellations, base five arithmetic, pattern seeking, rates and number theory. There are eight questions in the series.
- *Euler*
This series comprises elementary number theory, geometry, pigeonhole principle, elementary counting techniques and miscellaneous challenge problems, mainly for Year 8 and outstanding Year 7 students. There are twelve questions in this series.
- *Gauss*
This series comprises elementary geometry, similarity, Pythagoras' Theorem, elementary number theory, counting techniques and miscellaneous challenge problems, mainly for Year 9 and 10 students and those who have already done the Euler series. This series consists of material independent of the Euler series and develops problem solving techniques such as counting and the use of spreadsheets. There are twelve questions in this series.
- *Noether*
This series is designed for students in the top five to ten per cent of Year 9 who have taken the Gauss series in another year, and are not yet ready for the Polya series. This series consists of material on problem solving, algebra, geometry and number theory. There are sixteen questions in this series.

- *Polya*

This series consists of notes on deductive reasoning (Euclidean geometry) and algebra. It was designed specifically for the top five per cent of Year 10 students and outstanding students in lower years. Schools have found that this series gives a sound base for students who wish to specialise in Years 11 and 12 mathematics. There are sixteen questions in this series.

The workshop will include a display of the materials from the two stages. Participants will have the opportunity to attempt some of the problems and to examine the teacher support materials, and to trial the procedure for marking student answers.