MATHEMATICS:
LAUNCHING FUTURES

Proceedings of the 24th Biennial Conference of
The Australian Association of Mathematics Teachers Inc.

Edited by Sandra Herbert, Julie Tillyer & Toby Spencer

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PREFACE

'Mathematics: Launching Futures' was chosen as an appropriate theme for this, the 24th Biennial Conference of The Australian Association of Mathematics Teachers Inc., for several reasons.

This will be the first national conference on mathematics education since the advent of the Australian Curriculum. The national focus of this curriculum marks the start of a new chapter in Australian education and the ways in which mathematics is taught. There are also considerations about the support educators will need when implementing this curriculum—matters which will be discussed in depth at this conference.

There has been much public discussion of people's lack of mathematical knowledge as they leave the school system, as well as the low numbers of students studying higher level mathematics at school, and even fewer continuing that study at university. This has been repeatedly stated as a cause for alarm for many professions, including the teaching profession itself, with the number of appropriately qualified mathematics teachers at record lows. Mathematics as a discipline can be viewed as the launch pad of many careers.

Of course, one of the aims of this conference is for educators to share knowledge that will help shape their teaching and their future careers, especially those new to the teaching profession. Of late, AAMT as an association has begun to look closely at its membership and what it can do to encourage new members and support teachers at all stages of their careers. Early career teachers and those who are attending their first AAMT conference are especially welcome and it is hoped that you find your experiences and new contacts beneficial.

This conference features a joint day with the 36th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA). The joint day aims to highlight teacher–researcher collaboration and some of the wonderful outcomes and understandings—for teachers and students—that can result when classroom practitioners work closely with academics. This is not the first time that this has happened at an AAMT conference, and delegates at previous conferences have expressed that it is a great opportunity for professional learning. It is hoped that such professional collaboration will continue into the future.

In short, it is hoped that all delegates—classroom teachers, pre-service teachers, researchers and those who work in the various education systems—and others who did not attend but read this work post-conference, can learn from each other as we shape the future of Australian mathematics education together.

Review process

Presentations at AAMT 2013 were selected in a variety of ways. Keynote and major presenters were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or a workshop. Seminars and workshops were selected as suitable for the conference based on the presenters' submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals approved for presentation at the conference were also invited to submit written papers to be included in these proceedings. These written papers were reviewed without any author identification (blind) by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed the review process have been collected in the 'Research Papers' section of these proceedings. Papers that were not regarded as acceptable as peer-reviewed research but acceptable for publication have been included in the 'Professional Papers' section.

The panel of people to whom papers were sent for review was extensive and the editors wish to thank them all:

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Editors: Sandra Herbert, Julie Tilyer & Toby Spencer

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KEYNOTE PAPERS

LAUNCHING FUTURES: YOU CAN'T DRIVE BY LOOKING IN THE REARVIEW MIRROR

MIKE ASKEW

Monash University

mike.askew@monash.edu

It has become a cliché to say we are living in times of great change, that does not make it any less true and it does pose challenges for us as mathematics educators. In this presentation I focus on the issue how mathematics taught today might best support launching learners into an unknown future. I will invite us to step back from the day-to-day concerns of how best to cover the curriculum and help learners be prepared for short term success (I'm looking at you, NAPLAN) and to consider why mathematics is held to be important in the school curriculum and whether it still deserves to hold such status (hint: I think it does). But I shall argue that we need to pay closer attention to how we teach mathematics (not just what mathematics) and the impact this can have on what students learn not only about mathematics but also about themselves as learners and citizens, and why this might be the best 'launch pad' for their futures.

Introduction

"The problem of keeping knowledge alive, of preventing it from becoming inert...is the central problem of all education." (Whitehead, 1967 [1929], 5)

It has become a cliché to say we are living in times of great change, but that does not make it any less true and the rate of change poses challenges for us as mathematics educators. In this paper I explore how mathematics as taught today might best support launching learners into an unknown future. In doing so I step back from the day-to-day concerns of how best to cover the curriculum and help learners be prepared for short term success—important and valid concerns though these are—because alongside these concerns is our responsibility to nurture in as many pupils as possible an interest in and love of mathematics, not only so that they may go on to continue to study is but also so that they become the stewards of mathematics for future generations.

While the theme of this conference is launching futures, I am framing thoughts around looking forward by looking back to the work of the philosopher of education, Alfred North Whitehead. Taking up Whitehead's challenge of how to keep mathematical knowledge alive involves (at least) three questions:

- How best to teach mathematics for learner success?
- How best to teach mathematics so that learners believe they can learn mathematics?

- How best to teach so that learners develop a desire to continue to learn mathematics?

The bulk of this paper addresses the third question, as that is central to keeping mathematical knowledge alive: no amount of attention to raising standards is sufficient if the result is to turn off learners. First, however, I look briefly at the other two questions.

How best to teach mathematics for learner success?

This question is what the majority of books, articles and research papers in mathematics education set out to address. Despite this wealth of advice to teachers I am not convinced that we will ever have the definitive answer to this question, because the nature of teaching is an adaptive challenge, rather than a technical problem (Heifetz, Linsky & Grashow, 2009).

The difference between these is, as Heifetz and colleagues argues, that we can find solutions to technical problems through our current expertise. Technical problems involve familiarity with the steps to go through to solve them and as such are rooted in a logic of complicated systems: systems with multiple interacting parts, which are highly predictable in that a 'tweak' to one part of the affect other parts along the logic of 'if … then'. Clocks are the archetypal complicated systems.

Policy initiatives often treat schools and classrooms as complicated systems and hence the teaching and learning of mathematics as a technical problem: that we already know from current practices the solution to engaging children and raising standards. Such a view has a long history that can be traced back to the curriculum theorist Ralph Tyler.

Writing around the same time as Whitehead, Tyler, has been described as writing one of the most influential books on curriculum thought and practice (Schubert, W. H., 1986). In 'Basic Principles of Curriculum and Instruction' Tyler (1949) raised four questions that are still familiar and current:

1. What educational purposes should schooling seek to attain?
2. How can learning experiences be selected to be useful in attaining these objectives?
3. How can educational experiences be organised for effective instruction?
4. How can the effectiveness of learning experiences be evaluated?

The Australian Curriculum and NAPLAN may not consciously owe a debt to Tyler, but so wide reaching was the influence of his work that one can hear the echoes of it down the years.

The Tyler Model continues to hold sway through its appeal to belief in the power of a prescribed and predictable curriculum, with the accompanying sense of security and promotion of 'standards' that logically leads to certain classroom practices. Curriculum and instruction are technical problems.

An alternative is to view mathematics teaching as an adaptive challenge. In contrast to technical problems, adaptive challenges require solutions that have yet to be found—solutions that will be different from current practices, solutions that we may be able to imagine. That means trying out new ways to teach and in particular allowing pedagogies to emerge rather that imposing them.

Adaptive challenges fit within a theoretical framework of complex systems. Like complicated systems, complex systems are made up of many parts acting in relation to each other. How complex systems change, however, is not as predictable as in complicated systems. Change in complex systems is not random, but it is only with hindsight that the chain of cause and effect can possibly be traced. The weather and gardens are typical complex systems. In complex systems solutions to adaptive challenges have to emerge, they cannot be engineered into being, because complex systems have the capacity to learn as they develop. Schools and classrooms are complex systems and the sort of teaching that is appropriate for five, ten years time has not yet emerged. I return to this issue of emergence later in considering what this might mean for teachers and researchers.

This does not negate the fact that, literally, tomorrow children across the globe are going to go into mathematics lessons. We cannot use pedagogies that have not yet been invented to teach them—we can only work with what we have got. We have to work with our current knowledge, but that we need to treat that knowledge as 'conditional'. It works to the best of our current knowledge within the current conditions of teaching. But we must be cautious of claims for descriptions of current 'good mathematics teaching' being what is needed in the near, or far, future.

How best to teach mathematics so that learners believe they can learn mathematics?

We must look beyond teaching that effectively addresses 'standards'. Any pedagogy, including the particular case of mathematics, actually teaches far more than the 'content': students learn more than just mathematics in mathematics lessons. They learn a lot about themselves and about their peers and about relationships. What students learn about themselves and others directly impacts on the learning of mathematics at the same time as their experiences in mathematics lessons itself influences this other learning, in particular learning whether or not they are 'good' at mathematics.

Some current practices establish norms about different abilities in mathematics and enact these through practices such as sorting pupils into high medium and low groups and labelling individuals—below average, gifted and talented, and so forth. Research now challenges the long held view that mathematical ability is innate and fixed (Shenk, 2010). While there will always be natural variation in learners taste and talent for mathematics, if mathematical ability is more malleable than we consider it to be then many differences in attainment largely come about from social practices, particularly how we attribute mathematical ability on the basis of learner attributes that may be independent to any real differences. The research Carole Dweck has explored extensively the implications of seeing oneself (and others) with either a 'fixed' or 'growth' mindset towards ability in mathematics and the advantages of the latter (Dweck, 2000).

How best to teach mathematics so that learners want to continue to learn mathematics?

As indicated, I take this question as central to keeping mathematical knowledge alive and explore it through Whitehead's tripartite model of learning. He argued that all learning goes through three stages:

- precision,
- generalisation,
- romance.

In taking this model, I part company with some of Whitehead's writing suggesting that these are three distinct stages that learners pass through in turn. I see them as interwoven and iterative, not separate and sequential. While I consider them separately here, the reader should bear in mind that they cannot be teased apart, nor that attending to one of the three can, in reality, be done independently of the other two.

Precision

Precision is the stage most valued by many sections of society and, with mathematics in particular, precision is often seen as the main aim of teaching. With our increasingly test-oriented society—locally, nationally and globally—precision is highly prized. I am not arguing against precision—the precision of mathematics has allowed us many great human achievements (and some dubious ones) and doubtless will continue to do so.

A 'litmus test' question of one's beliefs about teaching mathematics is "should children should learn 'the tables' or the 'standard algorithm' for, say, long multiplication?" But this question assumes that there are two 'camps' in mathematics education—those that think such things are important (the precision/procedural camp) and those that do not (the understanding camp). The evidence is that we need both. Precision in certain aspects of maths is important, but on its own does not guarantee understanding. On the other hand, understanding is hindered if certain processes take up too much working memory, and attention is diverted from thinking about the bigger mathematical picture. For example, research evidence shows that success in being fluent and precise in basic number calculations in the early years of primary school is strongly correlated with later mathematical success (Cowan et al., 2012). That should not, however, be taken as an indication that rote learning of precision in basic number calculations is the key to promoting later success, but neither should becoming fluent in basic number bonds be left to chance on the assumption that children will eventually come to be fluent without explicitly attending to it.

An emphasis on arriving at precise answers should not be conflated with drilling learners in mindless procedures: arriving at correct answers involves choosing methods and procedures, and working flexibly depends on the calculation to hand. For example, a student might mentally calculate 3004 − 2997 by counting on from 2997 to 3004, or by partitioning 3004 into 3000 and 4, and using retrieval of 7 + 3 = 10 to figure out that 2997 + 3 = 3000, so the total difference must be 3 + 4 = 7. Applying a similar approach to 2005 − 8 (counting on from 8 to 2005 or adding 1992 to 8 and then another 5) may not be a sensible approach. Working flexibly would, in this latter case, mean counting back, or partitioning 8 into 5 + 3, taking 5 from 2005 and then subtracting 3. Reaching a precise answer cannot be separated from awareness of generalisations about the relationship between the numbers and understanding subtraction both as ‘taking away’ and ‘finding the difference’.

Generalisation

In research I carried out with colleagues at King’s College in London, a sample of around 2000 10- and 11-year-olds were told, and had written down in front of them, that 86 + 57 = 143. They were then asked a series of calculations all of which could be answered using this number fact, for example, 57 + 86, 143 – 86, or 860 + 570. The assessment was set up to tune the students into the fact that they were going to be answering calculations to which, with a bit of reasoning, they could write down the answers without a great deal of calculating.

We expected a high percentage, if not almost all children at the end of primary school, would to be able to answer such calculations correctly, and just over 90% of our sample could write down that 57 + 86 would also be 143. But twenty percent of the children in our study could not figure out that 143 – 86 would be 57; a fifth of the children who were about to leave primary school and go to secondary school did not display appreciation of the inverse relationship between addition and subtraction. I suspect those students would experience difficulty with algebra. How do you begin to make sense of questions like ‘find x if $3x + 5 = 2x - 4$’ if you do not understand inverse relationships? And given the time in primary schools spent teaching place value, it was worrying that around 25% of our sample could not correctly answer that 860 + 570 would be 1430.

Findings like this suggest that even though there have been moves in the curriculum to encourage students to reason about and see generalisations in mathematics many learners in primary school are not coming the subject with that frame of mind and are coming to expect that every calculation has to be worked out from scratch.

Another of Whitehead’s proposals was that education should focus on teaching only a few, main, ideas that we “throw these ideas into every combination possible”. In mathematics, inverse relationships and place value are certainly ‘main’ or big ideas, but we may need to ‘throw’ these into more combinations. Deep learning in mathematics hinges on generalising and it is important that we go beyond a focus on whether or not children get correct answers in mathematics and to encourage them to explain how they arrive at answers and the need to justify their solutions. In doing so they build up networks of big ideas, a topic that Di Siemon and colleagues have written about (Siemon, Bleckly & Neal, 2012).

Coming to understanding mathematical generalisations comes about through the act of generalising: it is not enough to simply tell learners that addition and subtraction are inverse operations—they have to come to that awareness through exploring operations, articulating their insights and repeatedly applying them. In Vygotskian terms, development follows learning. In other words, that being able to generalise is not a developmental ‘stage’ that children grow into, but that classroom environments and tasks that go beyond simple right/wrong solutions provide the ground through which children’s generalising develops.

Romance

The Vast and Endless Sea
If you want to build a ship, don't drum up the men to gather wood, divide the work and give orders. Instead, teach them to yearn for the vast and endless sea.

Antoine de Saint-Exupery (1900–1944), "The Wisdom of the Sands"

Setting aside the sexist language, Saint-Exupery captures an essence of Romance—the sense of yearning. I am not sure how many students I have taught have ended up yearning for maths, but there are two elements that I consider contribute to Romance, that I think can be bought into being in classrooms: presence and playfulness.

Presence

Sitting in on a Grade 6 class recently, a boy was working on finding two different ways to fill in the blanks on:

$16 \times [\ ] = 8 \times [\ ]$

He had written down

$16 \times 4 = 8 \times 8$

and

$16 \times 2 = 8 \times 4$

but then had crossed this second answer out, and was sitting there looking stumped.

MA: Why have you crossed this one out?
B: Because it's not the same.
MA: How do you mean, not the same.
B: Well here (pointing to 16×4) it equals sixty-four and sixteen times two is thirty-two, not sixty-four.

This is a great example of what Eleanor Duckworth calls a 'wonderful idea' in her claim that intellectual growth comes from everyone, irrespective of their level of development, having their own, unique, "wonderful ideas". "The having of wonderful ideas, which I consider the essence of intellectual development, would depend … to an overwhelming extent on the occasions for having them" (Duckworth, 2006, p. 181).

I think this boy's response was a wonderful idea as it had not occurred to me that the instruction to give two solutions could be interpreted as meaning each solution had to come to the same product, and, although that interpretation sets up an impossible situation, it showed that the boy was thinking deeply about how to interpret the mathematics, and not just plugging in numbers to get a solution. The provision of an open-ended challenge had provided the opportunity for this wonderful idea to emerge, an opportunity that a more closed question may not have provided. This emergence of wonderful ideas is linked to both the teacher and the pupil being 'present', both to other people and to the mathematics. In this instance, the boy was present to the mathematics—actively engaged with thinking about how best to make sense of the challenge. Looking back on this, I was aware of how often I have been more present to the mathematics, and how I might have pointed out the boy that his answer was fine, that 16 × 2 was the same as 8 × 4, instead of being present to him and his ideas and so being able to go on and have a conversation about what ‘two different solutions’ might mean.

The writer and therapist Stephen Grosz (2013) describes seeing the quality of presence in watching Charlotte Stiglitz, an eighty-year-old remedial reading teacher, engage with a four-year-old who was drawing:

> When he stopped and looked up at her—perhaps expecting praise—she smiled and said, ‘There is a lot of blue in your picture.’ He replied, ‘It’s the pond near my grandmother’s house—there is a bridge.’ He picked up a brown crayon, and said, ‘I’ll show you.’ Unhurried, she talked to the child, but more importantly she observed, she listened. She was present. (Grosz, 2013, p. 21)

Presence, Grosz argues, helps build children’s confidence through indicating that they are worthy of the observer’s thoughts and attention. That we have a desire—a yearning—not only to understand but also to be understood.

I still regularly teach in schools and it is part of my practice that problem-solving lessons end with some children, carefully selected, coming to the front to explain their solutions to the class. Over time, I can see the learners’ grow in confidence and in their engagement with the mathematics. I think this is down, in part, to the old saw that you never really understand anything until you’ve taught it, but I also think it satisfies this desire to be understood.

I also work with classes on what it really means to listen to someone-else’s explanation, through inviting other learners to re-explain what they heard, and to check if they are correct with the learner giving the explanation, not with me, the teacher. Everyone is thus encouraged to be ‘present’ with that particular learner and the feeling that the others in the class are trying to understand them is, I think, more powerful for learning than praise for right answers.

The core of Duckworth’s pedagogical philosophy is “to listen, to have our learners tell us their thoughts” (Duckworth, 2006, p. 181) which requires treating the content as “explorable, and the pedagogy asks the students to express their thoughts about it” (Duckworth, 2006, p. 261). Such a pedagogy of listening challenges teaching mathematics based on direct instruction, whereby the teacher models what to do and learners have to replicate this.

In a similar vein, Zhang Hua (2012) writes of the importance of establishing a “listening” pedagogy at the centre of education, through teachers’ listening and the mutual listening between teachers and learners, and among learners. Doing so establishing learning as arising from cooperatively creating knowledge. But over and above that, it makes ‘ “good listeners” i.e., persons with freedom as the aim of education, who integrate morality and creativity.’ (p. 57)

Being present is similar to Ellen Langer’s concept of being mindful in learning (1997). Mindful is a term redolent with overtones of Zen and meditation but Langer uses it to mean learning with awareness and of being ‘mindful’ of the nature of the knowledge in the sense of not simply taking it as unquestioned givens. A key element of being mindful, Langer argues, is an awareness of the conditionality of much knowledge. Treating knowledge as conditional—that it holds under certain conditions—makes, she argues, for more powerful learning experiences. For example, in one of her experiments Langer presented images of ambiguous pictures to two groups of high school students’.

"We presented the pictures either conditionally ('This could be ...' or with absolute language ('This is ...') and asked the students to remember them. Tests of recall and recognition of the objects in a new context revealed that conditional learning resulted in better memory." (Langer 1997, p. 80)

But isn't most of mathematics absolute and not open to being conditional? Children have to know that 2 + 2 = 4 and not think that it could be 5 or a cabbage. Well, yes and no. Pour two glasses of water into a jug, followed by another two and you don't end up with four glasses of water. More seriously perhaps, take an example like, what 'could' be the answer to 13 ÷ 4? Three, three remainder one, three and a quarter and four could all be sensible answers depending on whether one is considering

- How many tables for bridge would thirteen card-players need?
- What happens when putting thirteen apples into bags of four?
- How much toast does each of four hungry children if they share 13 slices fairly
- How many taxis (each carrying four people) do thirteen travellers need to book?

Playfulness

Many mathematicians and physicists talk of adopting a playful state of mind in thinking about their disciplines. Einstein, for example, described his insights as arising from "combinatorial play," and the role that playfully imagining racing after and catching a beam of light played in his development of the theory of relativity.

Teachers who encourage a playful approach to the mathematics report that learners don't want to stop when the lesson ends, they want to continue to explore the ideas being played with. Adopting playful approaches facilitates learning, creativity, and problem solving. They are inhibited by evaluation or expectation of rewards.

I agree with Peter Gray's assertion that 'In all of us, the capacity for abstract, hypothetical thinking depends on our ability to imagine situations we haven't actually experienced and to reason logically based on those imagined situations. This is a skill every normal child exercises regularly in play.' (Gray, 2013 p. 140). The whole of mathematics is predicated on 'abstract, hypothetical thinking' and arriving at mathematical generalisations means reasoning about imagined situations.

Adopting a playful approach towards learning mathematics is that encourages a free exchange of ideas—play promotes dialogue rather than discussion. Dialogue means that students engage with the multiple, connected, senses of mathematics made by others in the community—leading to growth in collective meaning making. While discussion is often characterised by holding tight to one's position—trying to establish that one's ideas are correct at the expense, sometimes, of others ideas—dialogue has the ideas at the centre, not those persons putting forward the ideas. Playfulness promotes a mathematics-centred classroom, a classroom where ideas are welcomed, held up to be examined, built upon and refined. It is through dialogue and playing with ideas that depth of understanding comes about in the mathematics.

It is through play that we discover our talents and interests. I am not suggesting that play is a panacea for students' aversion to mathematics, nor that play will turn everyone on to mathematics, but play is likely to help more students find that they have a taste and talent for mathematics and that they want to continue to be a player.

Again, the benefits of dialogue may extend beyond simply improving standards in mathematics. As Robin Alexander point out, dialogue is at the heart of developing caring learners:

> Dialogue requires willingness and skill to engage with minds, ideas and ways of thinking other than our own; it involves the ability to question, listen, reflect, reason, explain, speculate and explore ideas; to analyse problems, frame hypotheses, and develop solutions; ... Dialogue within the classroom lays the foundations not just of successful learning, but also of social cohesion, active citizenship and the good society. (Alexander, 2006)

Playfulness is not the same as 'fun'. For me, tickling, roller coasters and candy floss are fun—short lived, engineered by someone else and leave you feeling somewhat nauseous. Playfulness is more of an internal attitude. Fun may be a by-product of playfulness but not the starting point. The laughter of playfulness is of a different quality to the laughter of fun, which sometimes verges on the hysterical.

Playfulness promotes a mathematics-centred classroom where ideas are welcomed held up to be examined, built upon and refined. Playing with ideas brings about depth of understanding the mathematics. Take an example like:

12 + 15 = [ ] + 14

Precision involves adding 12 and 15, figuring out what to add to 14 to make 27. To move towards generalisation, learners could be asked to articulate what these equations are all examples of:

10 + 12 = 11 + 11
27 + 24 = 26 + 25
48 + 37 = 50 + 35

A learner may notice that these are all examples of compensation—that increasing one number in a sum has to be compensated by decreasing the other by an equivalent amount if the sum is to remain the same. And the noticing may stop at that.

A playful approach however would go beyond this and ask questions like:

Does this always work?

4657 + 3458 = 4660 + 3455?

56.75 + 34.75 = 57 + 34.5?

What about three addends?

Subtraction?

Multiplication?

The emergence of new pedagogies

Finally I want to return to the question of how we—teachers, researchers—might go forward with new pedagogies being treated as adapted problems the solutions for which need to emerge. Davis and Simmt (2003) suggest that there are five conditions necessary for emergence:

- Diversity
- Redundancy
- Enabling constraints
- Neighbour interactions
- Distributed control

How might these apply to us as communities of practice developing mathematics teaching?

Diversity means there needs to be variation amongst the participants, to provide the possibilities for novel responses. If all our teaching become homogenised then the chance for new practices to emerge is reduced. We saw this in a five-year study of the introduction of the national numeracy strategy in England. Classroom observations before the introduction of the strategy revealed a wealth of practices, which could provide rich opportunities for teachers to share, debate and build on—to have a dialogue about. After the strategy was introduced, virtually all the lessons had the same 'three part' structure, thus reducing the opportunities for innovation.

Redundancy is the other side of the diversity coin: members of a community have to have sufficient common ground, rules and assumptions to be able to work together. Davis and Simmt suggest that for emergence of new ideas, redundancy is helpfully thought about in terms of proscription—what we do not do round here—rather than prescription—we only do it this way.

Enabling constraints sounds like an oxymoron but these provide focus to activity whilst still allowing for diversity. For example, exploring ways in which students have to work in pairs on a problem imposes a constraint (paired work) that enables ways of working to emerge.

Neighbour interactions means more than simply teachers working together. In schools and networks of schools it means the sharing of ideas, hunches, questions, records of teaching practices. It means having more dialogue about the outcomes of teaching, the evidence for these and what we value, than planning the inputs of teaching.

Distributed control is probably the one area that most challenges current trends—that increased centralization of curriculum and testing may militate against emergence of new pedagogy. Local, distributed control is essential; else the dangers that Paolo Freire (1996) warns of may emerge: "Leaders who do not act dialogically, but insist on imposing their decisions, do not organize the people—they manipulate them. They do not liberate, nor are they liberated: they oppress."

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A TIME TO REFLECT BEFORE WE LAUNCH FORWARD

KATHRYN PALMER

Melton Network Numeracy Coach
Numeracy consultant: Every Child Counts Numeracy Consultant

palmer.kathryn.k@edumail.vic.gov.au

Reflection, or consciously thinking about our experiences, is the key to powerful learning. Reflection allows us to analyse our experiences, make informed changes based on our mistakes, maintain successful practices, and build upon or modify our past understandings based on new and emerging knowledge. This paper describes my journey through education in Victoria and provides insights into the elements I have identified as integral to successful mathematics education as we launch into the future.

Introduction

As teachers we are more than just educators, we also become the best “thieves”, “samplers” and “borrowers”. Much of our growth stems from looking at an idea, resource and activity then using our professional judgement and knowledge to adapt it to suit the students in our class, the school we’re working in or for the professional learning of teachers. We grow from this culture of sharing: the sharing of ideas, research and enthusiasm for our endeavour to provide our students and teachers with the best of Mathematics Education.

I have been extremely fortunate over my teaching career to have participated in some valuable numeracy projects and had the privilege of working and learning from a community of “dedicated sharers”.

In this paper I will discuss the projects and programs that have most strongly influenced the way in which I communicate mathematics: the *Teaching and Learning Coaching Initiative*, *Implementing structured problem-solving mathematics lessons through Japanese Lesson Study* and specific programs including *YuMi Deadly Maths* and *Back to Front Maths* (*Problem-based Maths)*.

Teaching and learning coaching initiative

In 2007, the Department of Education and Early Childhood Development (DEECD) worked extensively with Professor Richard Elmore on evaluating the School Improvement Practices in Victorian Government schools. Elmore (2007) noted that human investment was the key strength of the education system in Victoria and suggested that teachers should view their teaching practice as one in which new and powerful ideas are public goods, rather than private practice. Teachers should be exposed to coaching and mentoring others as early as possible in their careers.

In 2009 I was one of 200 teaching and learning coaches seconded to the *Teaching and Learning Coaches Initiative* and I commenced working in the Western Metropolitan Region as part of the school improvement team. The initiative was underpinned by the key understanding that student achievement is determined to a significant extent by the knowledge and skills of teachers in individual classrooms.

The aims of the initiative were to improve:

- student learning outcomes, especially in the areas of literacy, mathematics and/or science, for students in identified schools;
- teacher knowledge and skills related to effective literacy, mathematics and science teaching; and
- teacher capacity in the use of ICT, particularly for online curriculum planning, assessment and delivery in preparation for the ultranet school capacity to support improved student learning outcomes (DEECD, 2013).

The purpose and intention was to provide intensive assistance to identified schools to bring about changes in classroom practices necessary to improve student outcomes and build teacher capacity. Six key elements emerged as the ongoing focus for the coaches. The elements were:

- building professional relationships within the schools;
- building teacher capacity to establish priorities, analyse student results, measure student progress and use collected data purposefully;
- improving the quality of learning and teaching through purposeful instruction by modelling, observing and providing feedback;
- providing substantive conversations with teachers to elicit goals, prompt inquiry and support reflective practice;
- developing school improvement by working with the school leadership team and professional learning teams; and
- continuing one's own self development.

This represented a significant change in focus for school improvement policy to one that more directly supported teachers within the classroom and shifted the onus of accountability to the individual. For schools the rationale and incentive for teacher coaching was based on research that increasing teacher capacity had the most direct impact on improving student achievement (Hattie, 2003).

Coaching offers differentiation of professional learning for teachers within schools and recognises teachers are at different stages of their careers, and possessing varied levels of knowledge and skill. It successfully overcomes the difficulty of external professional learning transferring into their classroom. With many coaching models to choose from, the region I was working in adopted the Gradual Release of Responsibility framework. In this framework, the responsibility for the new learning in this case gradually shifts from the coach to the teacher who is being coached. This shift leads to the embedding and sustaining of change by the classroom teacher.

Teachers in primary schools require a deep understanding of mathematics for teaching and this is a key component in improving student learning outcomes (Hill, Rowan & Ball, 2005). My coaching has helped me to identify a common element that is fundamental for successful teaching in mathematics; teacher content knowledge, and the associated understanding of the continuum of learning that this provides. Where teacher content knowledge is poor, it stands to reason that so too is student learning. Developing both the practical and theoretical aspects through coaching leads to powerful gains for all stakeholders. It is widely accepted that teachers of mathematics require appropriate strength in both content and pedagogical content knowledge. In my role I am trying to address this issue by offering timely professional learning to teachers around a specific key mathematical idea. The purpose of these sessions is to give teachers some valuable kinaesthetic activities while building their own content knowledge and highlighting links to class management and organisation ideas.

Whilst initially many teachers viewed coaches suspiciously and as an inconvenience, this soon changed as they recognised the value to themselves and the children, and most schools across the region employed their own School Based Coaches in both Literacy and Numeracy. My role as a regional coach became that of the main resource for the school based numeracy coaches. At the end of 2011 any lapsing government projects were not re-funded and unfortunately the coaching initiative was one of these. The network that I was working in had pooled their National Partnership Funding to support school improvement and to distribute resources. In 2012 the network funded me to continue the work as Network Numeracy Coach for the 22 schools within the network. As a result my role within the network allows teachers to work collaboratively and engage in reflective practices together. There has been a cultural change where professional learning is embedded into all schools I work with. The model I have developed within my network has been to divide School Based Coaches into small working groups, who meet regularly to achieve a specific goal. As the network is extremely diverse (with P–9, primary, secondary and small rural schools) this model allows me to meet the needs of all schools I'm working with and supports a collective approach.

I am extremely passionate about my profession and everyday enjoy the challenges my job. Coaching has allowed me to focus on my greatest passion: the teaching of mathematics. I wouldn't say I'm a great a mathematician (I am not) but I am a good coach. In my position I have the rare opportunity of going into a school and working closely with teachers and leadership teams. I am a critical observer who isn't involved in the politics of the everyday running of the school and therefore can make suggestions. Building relationships is a crucial component of my work. Teaching is a very social occupation, interacting with students, parents, teachers and the wider community. Coaching can be an isolated position and for me the rewards come when I'm working with a teacher who makes the ideas that I had shared with them previously their own. When they are so excited to see me the next time I'm in their school to share something that worked well, that enthusiasm becomes infectious.

Implementing structured problem-solving mathematics lessons through Japanese Lesson Study

Lesson Study is an ongoing, collaborative, professional development process that was developed in Japan. Teachers systematically examine their practice in order to become more effective instructors. *The Trends in International Mathematics and Science Study* (TIMSS) identified Lesson Study as a powerful, ongoing process for improving the quality of teaching in mathematics. (Stigler & Hiebert, 1999).

Lesson Study involves a number of steps.

- Teachers select a *research theme*.
  This theme focuses on a broad research question regarding their students that involves skills or attitudes they would like to foster.
- The research team selects *a goal and a unit of study* on which to focus.
  They research their students' abilities and needs within this unit of study. The team researches and shares "best practice" ways to teach this.
- The team *creates the lesson*.
  Teachers select a lesson within the unit to develop, and follow an established lesson plan template. This template focuses on how the lesson fits within the broader school curriculum, linking the lesson topic and skills to previously learned content, and to content that will be learned in future grades. This lesson plan template also focuses on ways to assess student thinking during the lesson.
- The *lesson is taught* by a member of the group and observed by the other members, as well as other teachers in the school and usually some outsiders. The focus of the observation is on student thinking not on the teacher's abilities.
- The group then gets together to *discuss the lesson and their observations*.
  This is usually done on the same day. They evaluate the components of the lesson.
  - Who was the lesson just right for?
  - Who was too challenged?
  - Who was under challenged?
  - What would we do differently next time?
  - What are our next teaching steps for these students?

  Revisions are made to the lesson, based on these observations and analysis, and another member of the group may possibly be selected to teach the lesson again. This experience can lead to valuable insight into student thinking, strengths and weaknesses.
- At the end of this process, the group may produce a *report* that outlines what they learned in regards to their research theme and goal.

One of the fundamental elements of *Lesson Study* is that it is an ongoing process. The process focuses on the key actions of collaborating, planning, teaching, observing, reflecting and revising.

All lessons are informed by research and pre-reading. The team of teachers share the planning and responsibility of the lesson. The team matches the lesson to their students and agrees on roles for the lesson.

I was fortunate to be able to attend the IMPULS Lesson Study Immersion Program in Tokyo, Japan this year as a result of my involvement in the Deakin University *Implementing structured problem-solving mathematics lessons through lesson study* project over the past year. The professional learning of teachers is an ongoing process of knowledge building and skill development in effective teaching practice (NPEAT, 2003). This project became the vehicle to build teacher pedogogical content knowledge. The collaboration between the schools and teachers involved, as well as the support from both our university collegues at Deakin and the network, all resulted in teachers involved being more reflective of their practice. A highlight of the project was watching the confidence of the teachers in their teaching of mathematics build over the 12 months and being seen in their own schools as leaders of numeracy.

Influential programs

Numerous mathematics programs, texts and support materials emerge, seemingly constantly. One of the challenges faced by teachers is to select programs that will work successfully for their students and for themselves. I found two programs, *YuMi Deadly Maths* and *Back to Front Maths*, particularly useful for creating connections and encouraging genuine student dialogue.

YuMi Deadly Maths

In 2010 I was invited by the Sunshine/Deer Park Network to attend the YuMi Summit at Queensland University of Technology as a critical friend, to offer insights and help evaluate whether this project would be beneficial to the students in our schools. The summit was an opportunity for schools involved in the project to professionally engage, share and showcase their YuMi Deadly Maths experiences and journeys with colleagues.

> "YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used with permission in this project to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used to mean smart, in terms of being the best one can be in learning and life. (QUT 2010)

The YuMi Deadly Maths Program is based on two imperatives: first, that mathematics can empower all people's lives if understood as a conceptual structure, life-describing language and problem-solving tool; and second, that all people can excel in mathematics if taught kinaesthetically, contextually, with respect and with high expectations. It centres around a framework focussing on *Reality, Abstraction, Mathematics,* and *Reflection* (RAMR).

Reality

- Ensure existing knowledge prerequisite to the idea is known.
- Construct kinaesthetic activities that introduce the idea (and are relevant in terms of local experience).

Abstraction

- Develop a sequence of representational activities (physical-virtual-pictorial-language-symbols) that develop meaning for the mathematical idea.
- Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
- Allow opportunities to create own representations, including language and symbols.

Mathematics

- Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
- Facilitate students' practice to become familiar with all aspects of the idea.
- Construct activities to connect the idea to other mathematical ideas.

Reflection

- Set problems that apply the idea back to reality.
- Lead discussion of the idea in terms of reality to enable students to validate and justify their own knowledge.
- Organise activities so that students can extend the idea (use reflective strategies—being flexible, generalising, reversing, and changing parameters).

The project has involved 12 schools in Victoria being trained using a 'train the trainer' and research based models. My role has been to support trained schools and facilitate a professional learning team with all trained schools to continue the learning and ongoing implementation. In 2011 I accompanied six schools from Victoria to the sharing summit in Brisbane. I was able to highlight the network approach to implementing the program into Victoria. In May this year I coordinated all schools in the project to showcase their learning and highlight best practice with delegates from schools not involved in the project at the Victorian sharing summit.

The YuMi Deadly Maths project identified key elements that I believe to be important to maths education. The train the trainer model and research based professional learning supported the need to improve teacher pedagogical content knowledge. The ideas shared where kinaesthetic and engaging to both the teachers and students, but to me the "aha" moment came when I saw children in Queensland doing the abstraction component of the RAMR framework. If an idea can be represented in a diagram, table or graph then it could be modelled kinaesthetically and then students make their own representation of where they were in the reality with symbols and language. In my experience of observing many schools across Victoria, the abstraction component was the missing link, what we weren't doing. And yet it was so simple. For example, in a class we get students to use their bodies to get into 'groups of 3 or 4' and discuss how many groups have been made. If we then we don't get them to draw or represent what that looked like, we jump straight from the reality to the maths. We expect somehow as a child sits back at their table that they have made the link from the engaging 'groups of' activity to the multiplication worksheet in front of them.

Back to Front Maths

Last year I received a phone call from a numeracy coach I had met in Queensland telling me she had just attended an extremely valuable professional learning. With my passion for great maths teaching I was inspired to find out more and discovered *Back to Front Maths* (Kennedy, 2010). It is a problem-based teaching resource and contains a series of novel or unfamiliar problems that are used to introduce new topics, uncover student misconceptions, stimulate interest and experimentation and ultimately lead to building new mathematical understanding in students.

I am extremely fortunate that I get to observe numerous problem solving lessons in various settings. A constant issue during the lesson introduction is that the teacher tells the students how to get the answer and they lose the opportunity for students to develop their own capacity for logical reasoning and analytical thought. I try to highlight to teachers that I work with "never tell a student something they can find out for themselves". *Back to Front Maths* supports teachers to start asking students to solve a problem that they don't yet know how to solve. This requires the students to think mathematically, and experiment to try and work out a solution. Next, they look for student misconceptions (where the student has a fundamental misunderstanding of a concept), and help the students to analyse their ideas to see if they really work. When students self-correct their misconceptions, their mathematical understanding deepens and they learn concepts far more quickly (Kennedy, 2010).

Finally, and possibly most importantly, the teacher's primary role is one of asking really searching questions that encourage students to think deeply about a problem, access their prior knowledge about it, experiment with different ideas and then analyse how well these work when applied. Teachers help students to focus on the fundamental principles and patterns in mathematics, therefore enabling deep understanding and developing less need for repetition and memorisation.

As educators, "we understand something if we see how it is related or connected to other things we know." (Heibert et al., 1997) Students formulate and solve problems when they use mathematics to represent unfamiliar or meaningful situations, when they design investigations and plan their approaches, when they apply their existing strategies to seek solutions, and when they verify that their answers are indeed reasonable. When students are told rather than being able to explore they tend to develop a fragmented set of rules and procedures that do not represent what the teacher intended (Williams, 2010).

At a facilitators training with Tierney Kenney, the mastermind behind *Back to Front Maths,* I observed teachers having those 'aha' moments—calling in unison "we tell our students too much". The training is unique. Within the first hour of a two day training, teachers observe a *Back to Front* lesson in a classroom. By the completion of the lesson the observers have many insights into what these students know and don't know; the purpose of the problem based task is to identify student misconceptions. The second day of training involves participants trialing a problem based lesson with students they have no prior knowledge of and in a school which isn't their own. This form of professional learning gives participants a rare opportunity to trial their learnings and reflect on their own practice with the assistance of the facilitator and other participants.

Conclusion

Reflecting on this range of diverse and high quality projects has allowed me to isolate the key elements they have in common, to develop purposeful and genuinely effective maths teaching. The Gonski Review (2012) highlights the need for extra specialist teachers in the area of literacy and numeracy, and a need to support teachers to sift through the numerous resources and projects available. I believe specialist coaching in mathematics is vital if we are to build teacher capacity. Lessons need to be engaging and challenging whilst making connections to prior concepts and student interests. I believe these projects have gained momentum because I was a network resource to coach and mentor teachers at the point of need to address issues and support trials.

As we launch to the future I can see the importance of not "throwing out the old and in with the new". As teachers begin to implement an Australian Curriculum, they should take with them best practice and keep borrowing and trialling those elements that engage our students, by making links to previous learning, other contexts and to experiences inside and outside the classroom. It is my belief that if we are to improve the quality of teaching and learning in mathematics, we need to be part of a community that shares ideas. Therefore we don't need to start from scratch ourselves but can learn from the teacher next door or the teacher on the other side of the world or the international guru. As Isaac Newton wrote (quoting an earlier scholar), "we can stand on the shoulders of giants".

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PARTICIPATING IN LESSON STUDY TO SHARE NEW VISIONS AND IMMUTABLE VALUES: A JAPANESE APPROACH TO LIFELONG DEVELOPMENT OF MATHEMATICS TEACHERS

YOSHINORI SHIMIZU

University of Tsukuba, Japan

yshimizu@human.tsukuba.ac.jp

Lesson study is a Japanese approach to improve teaching and learning mathematics through a particular form of activity by a group of teachers. It provides teachers with key learning opportunities in working collaboratively with their colleagues to study subject matter, students' thinking and learning, and how to change classroom instruction. By participating in lesson study teachers learn new visions in education and immutable values of school subjects, with and from their experienced colleagues. This paper describes how Japanese mathematics teachers learn these key elements for improving their teaching through continuous participations in lesson study.

Introduction

Becoming a teacher in the Japanese educational system means not only to finish a teacher preparation course and to pass an examination to be recruited, but also to learn more about subject matters and teachers' key roles in teaching and learning of the subjects, informally with and from their colleagues (Shimizu, 2010). Beginning teachers in Japan are expected to keep learning the subject matter and teachers' key roles through informal interactions with their experienced colleagues, in addition to a formal educational system. Experienced teachers are also expected to develop their teaching competence gradually and continuously throughout their career. From this perspective, learning to teach in the classroom is regarded as a lifelong process, which is closely tied to participating in social and cultural activities related to the community of teachers (Stigler & Hiebert, 1999).

Lesson study, which is a literal translation of 'jugyo kenkyu', means a Japanese approach to develop and maintain quality classroom instruction through a particular form of activity by a group of teachers (Fernandez & Yoshida, 2004; Shimizu, 2002). Lesson study serves as an approach to professional development whereby a group of teachers collaboratively develop and conduct lessons to be observed: examining the subject matter to be taught, how their students think and learn the particular topic in the classroom, how to incorporate new methods for improving classroom instruction, and so on. The activity of lesson study includes planning and implementing the 'research lesson' as a core of the whole activity, followed by a post-lesson discussion and reflection by participants, with revisions of conducted lessons. Lesson study raises opportunities for teachers to learn new visions on educational movements and curriculum changes and immutable values of school subjects, with and from their experienced colleagues, as well as some practical ideas and methods to be used in the classroom.

This paper describes how Japanese mathematics teachers learn key elements for improving classroom instruction by participating in lesson study, and how they utilise the learning opportunity for the development of their capabilities as teachers. Key characteristics of what is learned by teachers by participating in lesson study are described in three different levels to illustrate the mechanism and impact of it; the values in education pursued in the long term, pedagogical terms shared among teachers, and teachers' classroom actions recommended by and shared among Japanese teachers. The author concludes the paper by raising the issues in adoption of the approach to other contexts, and points out the challenges to be resolved in research and practice to pursue its possibilities.

The origin and the current state of lesson study

Although Fernandez and Yoshida (2004) mention that the origin of lesson study can be traced back to the early 1890s, it seems to have appeared earlier. At the beginning of the modern era, the Japanese government established normal schools, where teachers set the goals of the lesson, prepared experimental lessons, and conducted those lessons in actual classrooms while other teachers were observing them. In the late 1890s, teachers at elementary schools affiliated to the normal schools started to study lessons by observing and examining them critically. Makinae (2010) argues that the origin of Japanese lesson study was influenced during the late 1880's by U.S. books for educators which introduced new approaches to teach. He points out that a book by Sheldon (1862) describes methods to learn about new teaching approaches, called 'criticism lesson' and 'model lesson'. This may be the beginning of Japanese lesson study. In fact, Inagaki (1995) argues that 'criticism lesson' was already practiced among elementary schools affiliated to the normal schools in Japan as early as the late 1890s. Teacher conferences utilising criticism lessons were conducted by local school districts in the early 1900s. Some of these conferences were already called 'lesson study conferences', or jugyo-kenkyu-kai in Japanese (Makinae, 2010). In this sense, lesson study has a history of more than a century.

In the early stages of development of Japanese lesson study, 'criticism lesson' (Sheldon, 1862) included a particular function of studying lessons, carefully examining the effectiveness of teaching, and publicly discussing ways to improve teaching and learning. The term 'research lesson', or kenkyu-jyugyo, might come from this particular function of lesson study with its major focus on producing a new idea, or testing a hypothesis in the form of an operationalised teaching method or teaching materials. On the other hand, 'model lesson' (Sheldon, 1862) included another function of studying lessons; demonstrating or showcasing exemplary lessons, or presenting new approaches for teaching. For this purpose, the lesson should be carefully planned and based on research conducted by a teacher or a group of teachers. Participants can observe and discuss actual lessons with a hypothesis, instead of simply reading papers or handouts that describe the results of the study. The two different functions of lesson study—'criticism lesson' and 'model lesson'—can be the original model of a variety of lesson study practiced around the county.

Despite the long history of lesson study in their own country, Japanese mathematics educators, and researchers in other areas have not been much interested in studying lesson study itself until recently. After the publication of *The Teaching Gap* (Stigler & Hiebert, 1999), followed by a Japanese translation (Minato, 2002), Japanese educators, often deeply involved in lesson study, 'found' the importance of this particular cultural activity.

Today, lesson study takes place in various institutions and contexts (Lewis & Tsuchida, 1998; Shimizu, 2002). Pre-service teacher training programs at universities and colleges, for example, include lesson study as a crucial and challenging part in the final week of student teaching practice, which usually lasts three or four weeks. In-service teachers also have opportunities to participate, held within their school (konai-kenshu), outside their school but in the same school district or city, at the level of prefecture, and even at the national level for several objectives. Teachers at public schools may just participate in lesson study in their school to develop their teaching skills, since the school is their working place. Other teachers may play the major roles in planning and conducting research lesson, for testing critically their hypothesis in the use of particular method for teaching mathematics. Teachers at university-affiliated schools that have a mission to developing a new approach to teaching, often open their lesson study meeting for demonstrating an approach or new teaching materials they have developed. Thus, we can still see two major functions of lesson study that seems to have arisen from the original form of it.

Key elements of lesson study

Components and the cycle of lesson study

Lesson study is a problem solving process whereby a group of teachers, in many cases a school as a whole, work on the problem as identified, with research questions related to the particular theme important to the group. The theme can relate to the examination of the ways to teach a newly introduced content or to the use of new teaching materials in relation to the revision of national curriculum guidelines, or to the way to enhance students learning a certain difficult topic in mathematics, such as common fractions or ratio.

Thus, the first step of lesson study is defining the problem. In some cases, teachers themselves pose a problem to solve, such as how to introduce a concept of common fractions, or an effective way to motivate students to learn mathematics. Second, planning the lesson follows after the problem is defined. The groups of teachers collaboratively develop a lesson plan in this context. A lesson plan typically includes analyses of the task to be presented, and of the mathematical connections both between the current topic and previous topics (and forthcoming ones in some cases) and within the topic. It identifies anticipated students' approaches to the task, and the planned instructional activities based on them. The third step is to implement the research lesson in which the teacher teaches the planned lesson, with colleagues as observers. In most cases, the observers take a detailed record of teacher and students utterances to use in the post-lesson discussion. Evaluation of the lesson follows in a post-lesson discussion, focussing on such issues as the appropriateness of the tasks presented, students' responses to the tasks, the roles of teachers questioning and so on. Based on the evaluation of the lesson, a revised lesson plan is developed to try the lesson again. The entire process forms a cycle of lesson study (Table 1, Shimizu, 2002).

Table 1. Key components of lesson study.

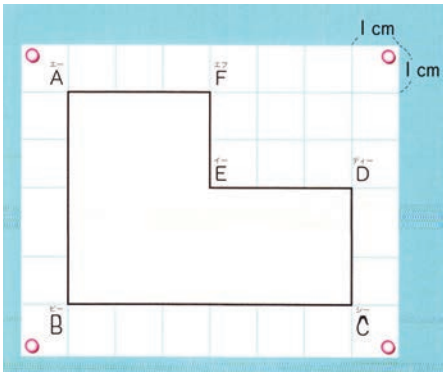
Before	- Deciding a theme - Selecting a particular unit and topic for the study - Writing a lesson plan (analysing the topic to be taught, assessing students' learning, examining the task to be posed, considering teacher's roles, etc.) - Discussing and revising the lesson plan(s) - Trying out the lesson by other teachers or in other classes - Reflecting on the lesson and revising the plan
During	- Teaching/observing the lesson - Recording what the teacher and students said, how students worked on the task during their seat work, and what was written on the chalkboard - A self reflection by the teacher - Extensive discussion on the lesson - Discussion on the task, students' response, teacher's roles, and so on - -Comments and suggestions by a mathematics educator or an experienced teacher
After	- Ideas are used in the following lessons - A report of the lesson is sometimes shared by teachers in other schools - Next cycle starts and a new theme may be identified

Analysing subject matter in relation to students' thinking

One of the key areas of teachers' work during lesson study is examining the subject matter in the form of problem posed in the classroom with students' thinking. A lesson plan, which incorporates the problem with various types of anticipated students' response as major parts, plays a key role as a medium for the teachers to share and discuss the ideas and hypotheses to be examined through the process of lesson study. Leaning how to write, and how to read, a detailed lesson plan is thus one of the key elements in the entire process. The importance of students' thinking is emphasised throughout the process of writing a lesson plan, as well as in conducting a lesson, as necessary elements to be incorporated into the developing and implementing the lesson.

The following example illustrates how anticipated students' thinking can be incorporated in planning a lesson. The problem (Figure 1) is a typical one that appears in most textbooks in Grade 4 (Fujii, et al., 2010).

The problem is to be presented to grade four students after they have learned the concept of area and how to find the areas of square and rectangles. In particular, students are expected to use their prior knowledge of area of rectangles in the new situation of a composite figure.



Find the Area of “L shape”.

Figure 1. The Problem (Grade 4).

In the textbook, one of the methods shown is finding the area by dividing the shape into two rectangles, together with a number expression (Figure 2). Then, another method of finding the area, by dividing the figure into two parts again but in a different direction, is presented without any number expression (Figure 3).

We need to be careful here with the difference between the two similar methods which are, however, different in terms of the direction of division. Also, the addition of a number expression does matter when planning a lesson with the problem. For example, a teacher may ask students to write a number expression for solution 2, after confirming the meaning of each number and operation in the case of solution 1. Or, the teacher may just delete the number expression from Figure 2 in their lesson plan in order to ask the original question in a more open way.

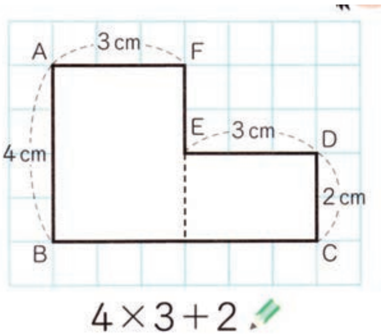


Figure 2. Solution 1.

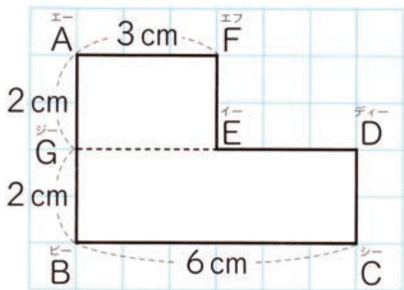


Figure 3. Solution 2.

The textbook then shows two more methods, neither of which include any dashed line that suggests a division. These figures are shown with number expressions and invite students to think of the idea of subtracting a small rectangle on the upper right corner from an ‘entire’ rectangle (Figure 4), and the idea of moving a part to form a new single long rectangle (Figure 5) respectively. It should noted here that the length of AB (4 cm) is given as double of DC (2 cm) intentionally for inviting the idea of forming a single rectangle represented in Figure 5.

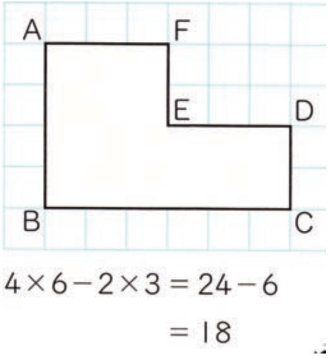


Figure 4. Solution 3.

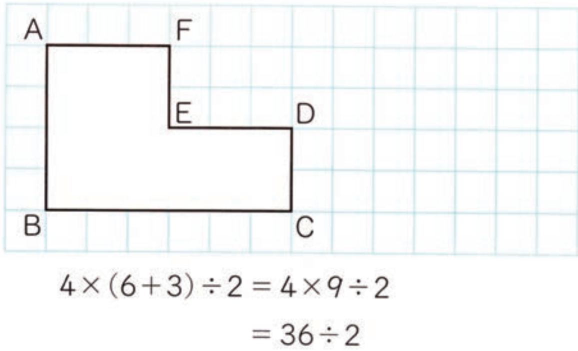


Figure 5. Solution 4.

Then, the teacher may ask those students who solved the original problem on their own in these ways to explain their method, using everyday language and/or number expressions. Or, if none in the classroom found these methods, the teacher presents the figures and number expressions and asks the students to explain how we can interpret the method of finding the area and show the ideas by using the figures. These considerations are necessary for writing a lesson plan with the original problem.

Finally, in the textbook, a different shape is presented (Figure 6). The new figure is meant to be the place for applying and expanding what students are supposed to have learned. Teachers can invite students to pose a problem they themselves develop.

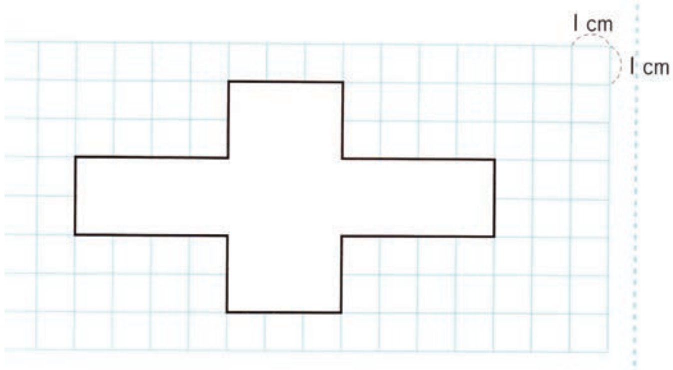


Figure 6. A new problem with different shape.

In writing a lesson plan for a lesson with the original problem, multiple considerations of subject matter are required. We need to anticipate students' alternative solutions methods and deal with them in a certain order to integrate the different ideas behind them. We also need to consider the way ideas are presented, everyday language, number expressions, figures, and so on. The final 'applying and expanding' phase may or may not be included in the same lesson, depending on the reality of students' ability in the classroom. A whole consideration is just a part of the analyses of the topic taught in relation to anticipated students' response to the problem.

The role of outside experts

In lesson study, an outside expert is often invited as an advisor who facilitates the post-lesson discussion and/or makes comments on the possible improvement of lesson from a broader viewpoint (Fernandez and Yoshida, 2004; Shimizu, 2008). The expert may be an experienced teacher, a supervisor at the local board of education, a principal of a different school, or a professor from the nearby university. In some cases, the group of teachers may meet with the expert several times prior to conducting the research lesson to discuss issues such as reshaping the objective of the lesson, clarifying the rationale of a particular task to be presented in the classroom, expanding the range of anticipating students' response to the task, and so on. In this context, the outside expert can be a collaborator who shares responsibility for the quality of lesson with the teachers, not just an outside authority directing the team of teachers.

The university professor invited as an outside expert is expected, as a researcher, to provide new visions on curriculum reform and teaching practices, trends and issues in local and national educational policies. They can also provide some concrete suggestions for improving daily classroom practices, as well as commenting on what was observed in the research lesson. Given the tradition of lesson study, mathematics educators have often challenged by school teachers who deeply engaged in lesson study whether research results are useful for improving classroom practices. The expectations from teachers who are actively involved in lesson study are quite high. If the professor would not meet the standards for their needs, he or she may not be asked to come to the school as an advisor. In such cases, teachers may say the advice is "too theoretical" or "little relevance to our school", for example.

Leaning with and from their experienced colleagues

Sharing new visions and immutable values

Coping with new visions in education

In lesson study a group of teachers collaboratively study the subject matter, how their students think and learn the particular topic in classroom, how to incorporate a new method for improving classroom instruction, and so on. New trends in the revision of national curriculum standards, revised roughly every 10 years, have strong impacts on the implementation of lesson study. We cannot neglect the connections of any objective of lessons with the goals and emphases described in the national curriculum standards.

In the current national curriculum standards (MEXT, 2008), for example, such classroom activity like communication, discussion, explaining, and writing are strongly valued and emphasised as 'activities with languages' in all the subject areas. Then, ongoing lesson studies in public schools often focus on introducing peer dialogues, small-group discussions, writing, and so on, in addition to the study of certain subject matters. Also, the revision of national curriculum standards that introduces some changes in the scopes and sequences of mathematical content eventually influences the choice of the themes to be pursued through lesson study. The introduction of an earlier conceptualisation of a common fraction is taught in the second grade in the new curriculum, while in the former national curriculum standards the concept of a common fraction was introduced in fourth grade. Teaching common fractions throughout the elementary school curriculum can then be a 'hot topic' to be examined through lesson study. Learning to cope with these new visions in mathematics curriculum, teaching methods, and general emphasis in education is one of the major aims for teachers who are involved in lesson study.

Sharing views on a 'good' lesson and an 'excellent' teacher

Teachers who participate in lesson study learn other aspects, which go beyond the new visions and trends in education. Participating in lesson study provides opportunities for teachers to learn immutable values of teaching mathematics as a school subject, with and from their experienced colleagues. Such values are related to teachers' views on a 'good' lesson and an 'excellent' teacher.

Sugiyama (2008) argues that at least three different levels in the excellence of mathematics teachers can be identified in relation to the goal of teaching and their teaching competencies. School mathematics is often dominated by learning only with 'instrumental understanding' (Skemp, 1977), where teachers just tell the right procedure or algorithm in classroom, like "turn it upside down and multiply" for a division of a fraction. On the other hand, those teachers who care about students constructing mathematical meanings of an operation may explain with figures or number lines the need to turn the divisor upside down and multiply it to get the dividend. Further, an excellent teacher will aim to foster students' ability for thinking independently by inviting them to think of division as an inverse operation of multiplication; presenting particular numbers such as 8/15 divided by 2/5, which can be calculated by dividing the numerator and denominator respectively. Or, the teacher may want to support students in making connections between the division of common fractions and the division of decimal fractions. Then, the teacher may reconsider the teaching of division of decimal fractions (taught one year prior to the teaching of division of common fractions) as an opportunity for introducing such a property of the relationship among divisor, dividend, and quotient: $a \div b = (a \times c) \div (b \times c)$ to be applied with division of common fractions.

As the example above illustrates, three different levels of teacher's competence can be identified.

- Level 1: a teacher who can tell important basic ideas of mathematics such as facts, concepts, and procedures
- Level 2: a teacher who can explain meanings and reasons for important basic ideas of mathematics in order for students to understand them
- Level 3: a teacher who can provide students opportunities to understand these basic ideas by themselves, and support their learning in order for students to become independent learners.

The teacher at each level differs in their interpretation of the topic taught in relation to students' thinking. Raising teachers' awareness of children's mathematical thinking provides teachers with a basis for their instruction and also for their own continued learning (Llinares & Krainer, 2006). There is a shared view on teacher's excellence in terms of their capability of interpreting the topic taught in relation to mathematical background and anticipated students' thinking on it. The views on teachers' excellence are derived from our views on a good lesson.

Japanese teachers think that any lesson should include at least one climax (Shimizu, Y. (2006). The point here is that all the activities, or some variations of them, constitute a coherent system called as a lesson that hopefully include a climax. Among Japanese teachers, a lesson is often regarded as a drama, which has a beginning, leads to a climax, and then invites a conclusion. The idea of 'ki-sho-ten-ketsu', which originated in the Chinese poem, is often referred to by Japanese teachers in their planning and implementation of a lesson. It is suggested that Japanese lessons have a particular structure, a flow moving from the beginning (ki, a starting point) towards the end (ketsu, summary of the whole story). If we take a story or a drama as a metaphor for considering a good lesson, then a lesson needs to have a highlight or climax based on the students' activities guided by the teacher in a coherent way. Although they are both implicit in the post-lesson discussion of lesson study, once we talk about issues further beyond each lesson observed, the views on a good lesson and an excellent teacher appear as an important basis of lesson study.

Joining in the discourse of the community of teachers

Learning key pedagogical terms used in lesson study

As mentioned above, the activity of lesson study includes careful planning and implementing the research lesson as a core of the whole activity, followed by post-lesson discussion and reflection by participants. In the discourse of teachers in planning, implementing, and reflecting on lessons, particular pedagogical terms are often used in the contexts of examining classroom instruction (Shimizu, 1999). Through the participation in lesson study, beginning teachers learn these terms together with values attached to them.

'Hatsumon' with posing a problem

'Hatsumon' means asking a key question for provoking students' thinking at a particular point in a lesson. At the beginning of the lesson, for example, the teacher may ask a question to probe or promote students' understanding of the problem. On the other hand, in a whole-class discussion the teacher may ask, for example, about the connections among the proposed approaches to the problem or the efficiency and applicability of each approach.

'Kikan-shido' during problem solving by students

Structured problem-solving approach includes time for students to work on the problem on their own. 'Kikan-shido', which means an 'instruction at students' desk', includes a purposeful scanning by the teacher of students' problem solving. The teacher moves about the classroom, monitoring students' activities silently, doing two important activities that are closely tied to the whole-class discussion that will follow. First, he or she assesses the progress of students' problem solving. In some cases, the teacher suggests a direction for students to follow or gives hints to the students for approaching the problem. Second, he or she will make a mental note of several students who made the expected approaches or other important approaches to the problem. They will be asked to present their solutions later. Thus, in this period of the purposeful scanning, the teacher consider questions like "Which solution methods should I have students present first?" or "How can I direct the discussion towards an integration of students' ideas?" Some of the answers to such questions are to be prepared in the planning phase but some are not.

'Neriage' in a whole-class discussion

This is a term for describing the dynamic and collaborative nature of a whole-class discussion in the lesson. The term 'Neriage' in Japanese refers to 'kneading up' or polishing up'. In the context of teaching, the term works as a metaphor for the process of polishing students' ideas and getting an integrated mathematical idea through a whole-class discussion. Japanese teachers regard Neriage as critical for the success or failure of the entire lessons. Based on his or her observations during Kikan-shido, the teacher carefully calls on students, asking them to present their methods of solving the problem on the chalkboard. The teacher encourages students to find the mathematical connections among alternative solution methods and leads the discussion on them.

'Matome', summing up, at the final phase of lessons.

'Matome' in Japanese means summing up. Japanese teachers think that this stage is indispensable to any successful lesson. At the Matome stage, Japanese teachers tend to make a final and careful comment on students' work in terms of mathematical sophistication. In general, the whole-class discussion is reviewed briefly and what the students have learned is summarised by the teacher.

'Bansho' as providing a bird's-eye view of entire lesson

For a research lesson, teachers carefully plan and implement how to organise writing on the chalkboard. 'Bansho' means writing on the chalkboard in front of classroom. The chalkboard may be divided into a few parts, such as the problem for today, students' alternative solutions, and the summary of what the class learned. In some cases, teachers may use a smaller white board, which will be incorporated into the large picture, to invite students to write their ideas to present to their classmates later. Bansho can provide the afterword, a bird's-eye view of entire lesson and then referred quite often during the post-lesson discussion.

Associated values with pedagogical terms

It is important to note that these pedagogical terms are used in the discourse in the particular contexts embedded in a whole system, to describe a particular style of teaching. Structured problem solving is often mentioned to describe the system with an emphasis on students' thinking on problem posed. Japanese mathematics teachers often organise an entire lesson by posing just a few problems, with a focus on students' various solutions to them. Educating teachers about lesson plans includes understanding key pedagogical terms. By using and listening to these terms, preservice and inservice teachers gradually become members of the community of teachers. The informal aspects found in the process of teacher education support the formal systems of teacher education programs in Japan.

Learning practical methods for daily classroom practices

Practical ideas shared by Japanese teachers

I have been involved in lesson study of various forms and in various contexts in the past decades. Teachers engaged in lesson study have made numerous suggestions for improving classroom teaching in mathematics (Shimizu, 2009). Among these suggestions, the following five items are especially noteworthy as concrete ideas shared through the process of lesson study and then disseminated among teachers all around the country.

Suggestion 1: Label students' methods with their names

During the whole-class discussion of the students' solution methods, each method is labelled with the name of the student who originally presented it. Thereafter, each solution method is referred to by the name of student in the discussion. This practical technique may seem to be trivial but it is very important to ensure the student's "ownership" of the presented method and makes the whole-class discussion more exciting and interesting for the students.

Suggestion 2: Use the chalkboard effectively

Another important technique used by the teacher relates to Bansho, the use of the chalkboard. Whenever possible, teachers put everything written during the lesson on the chalkboard without erasing. By not erasing anything the students have done and placing their work on the chalkboard in a logical, organised manner, it is much easier to compare multiple solution methods. Also, the chalkboard can be a written record of the entire lesson, giving both the students and the teacher a bird's-eye view of what has happened during the lesson.

Suggestion 3: Use the whole-class discussion to polish students' ideas

The teacher carefully calls on students, asking them to present their methods of solving the problem on the chalkboard, selecting the students in a particular order. The order is quite important to the teacher both for encouraging those students who found naive methods, and for showing students' ideas in relation to the mathematical connections among them that will be discussed later. In some cases, even an incorrect method or error may be presented, if the teacher thinks it would be beneficial for the class. Students' ideas are presented on the chalkboard, to be compared with each other with oral explanations. The teacher's role is not to point out the best solution but to guide the discussion by the students towards an integrated idea.

Suggestion 4: Choose the context of the problem carefully

The specific nature of the problem presented to the students is very important. In particular, the context for the problem is crucial for the students to be involved in it. Even the numbers in word problems are to be carefully selected for eliciting a wide variety of student responses. Careful selection of the problem is the starting point for getting a variety of student responses.

Suggestion 5: Consider how to encourage a variety of solution methods

What else should the teacher do to encourage a wide variety of student responses? There are various things the teacher can do when the students come up with only a few solution methods. It is important for the teacher to provide extra encouragement to the students to find alternative solution methods in addition to their initial approaches.

Hirabayashi (2002) argues that there are two major functions of lesson study. One function is as a way of doing research with a hypothesis, in the form of conducting lesson. Another function is as a place for presenting and discussing new findings based on the classroom practice. These functions relate to the issue of why lesson study can be suitable for professional development; for teachers to study the effectiveness of mathematics teaching and learning in their own classrooms. The following suggestions are accumulated findings based on the classroom practices of many teachers.

Discussion and conclusion

In the previous sections, aspects of how Japanese mathematics teachers learn key elements for improving classroom instruction by participating in lesson study and utilise the learning opportunity for the development of their capabilities as teachers were described. Three different levels of characteristics of the mechanism and impact of lesson study were provided: sharing new visions and immutable values in education, joining in the discourse of the community of teachers, and learning practical methods for daily classroom practices. It is important to note that these elements work together in particular contexts in which lesson study is conducted.

The physical arrangement of the school also promotes the interaction among colleagues. All teachers share a large room in each school, the 'teachers' room', where each teacher has a desk. They spend time in there, besides their classroom teaching. This situation allows for them to share information about students, ideas about subjects matter topics, and instructional materials. Also, teachers in public schools are required to move within their local prefecture from one school to another several times in their careers, possibly every three to ten years,. In some prefectures, teachers even move from elementary schools to lower secondary schools, and vice versa. Teachers also move within a school from one grade to another each year. This move may be beneficial for beginning teachers as it helps them to be familiar with the content to be taught in different grades.

Throughout the process of lesson study, a lesson plan is used as a vehicle with which teachers can learn and communicate about the topic to be taught, possible students' approaches to the problem presented, and important teachers' roles.
Writing a lesson plan is supported by the use of a form.

Improvement of teaching and learning through lesson study took place in the Japanese education system more than a century ago. The continued efforts by teachers conducting lesson study can partly be attributed to clear learning goals for students shared among teachers in relation to the national curriculum standards, as well as teachers' voluntary hard efforts in schools and collaborative supports from outside the schools. Further, importantly, the environments surrounding teachers, contexts and needs for lesson study, and forms embedded in the cultural activities, support teachers' works in a lesson study cycle.

For more than a decade, in particular after the publication of *The Teaching Gap* (Stigler & Hiebert, 1999), educators and researchers in the field of mathematics education have been interested in lesson study as a promising source of ideas for improving education. For a Japanese mathematics educator who has been deeply involved in lesson study for more than two decades, this 'movement' has provided an opportunity for reflecting on how lesson study as a cultural activity works as a system, embedded in the society and community with values shared among teachers. The classroom practices are socially and culturally situated; shared values and beliefs of teachers are the key for continuous development of quality teaching. Japanese regular public schools have a clear mission: committed teachers, time and environments for teachers to work together, and feedback cycles that lead to continuing improvements. Learning new visions in education that are explicitly discussed in various phases of lesson study, and immutable values that are implicitly pervasive in the community of teachers through the participation in lesson study, are crucial driving forces for professional developments in this particular context.

After researchers outside Japan introduced lesson study to their own mathematics education community, the term ‘lesson study’ spread among researchers and educators around the world. By now, schools and teachers in some different countries have been trying to implement lesson study into their own education systems. The central question to the possibilities of adoption of one approach to another context is raised from a perspective of teaching as a cultural activity. Improvement of teaching and learning through lesson study over a long period of time took place in the Japanese education system within the context where clear learning goals for students are shared among teachers in relation to the national curriculum standards, and with teachers’ voluntary hard efforts. There are challenges to be resolved in research and practice and possibilities to be explored in each context of different cultures.

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LAUNCHING MATHEMATICAL FUTURES: THE KEY ROLE OF MULTIPLICATIVE THINKING

DIANNE SIEMON

RMIT University

dianne.siemon@rmit.edu.au

Access to multiplicative thinking has been identified as the single, most important reason for the eight-year range in mathematics achievement in Years 5 to 9. While elements of multiplicative thinking are variously represented in the Australian Curriculum, the connections between these and how they contribute to the development of multiplicative thinking over time is not entirely clear. Two aspects of Hanna Neumann's internationally respected reputation as a mathematician, teacher, researcher, and mentor will be used to frame this presentation. The first is her commitment to making the abstract accessible. The second is her passionate interest in reforming school mathematics curricula. Examples will be used to demonstrate how the abstract might be rendered accessible in the context of school mathematics and, conversely, how abstracting the everyday can help challenge long-held beliefs about learning mathematics. But the major part of this presentation will be concerned with the critical importance of multiplicative thinking in launching mathematical futures and its representation in the Australian Curriculum.

Introduction

It is an honour to have been asked to do the Hanna Neumann lecture at this, the 24th Biennial Conference of the Australian Association of Mathematics Teachers (AAMT). Hanna and Bernard Neumann came to Australia in 1963 to take up positions at the Australian National University—Hanna as a professorial fellow until her appointment to the chair of pure mathematics in 1964, and Bernard as Professor of Mathematics in the newly formed research Department of Mathematics at ANU. Convinced that mathematics education in Australia was 'lagging behind the rest of the world to a frightening extent" (Fowler, 2000), Hanna became actively involved in the Canberra Mathematical Association providing courses for secondary mathematics teachers and contributing to the discussions on the new senior secondary mathematics syllabuses in NSW.

Previous lectures have been given by those who knew Hanna personally or at least heard her speak, for example, Dr Susie Groves and Dr Peter Taylor both of whom could claim 0 degrees of separation. I can claim 1 degree of separation having undertaken my Honours year in Pure Mathematics at Monash with Colin Fox and Steve Pride both of whom went on to ANU to study under either Hanna or Bernard and complete PhDs in

Mathematics. Bernard was a capable musician and Colin—now a broadcaster for ABC Classic FM—remembers fondly his many visits to the Neumann household for musical get togethers where Hanna, always working, would join them at the end of the evening for coffee.

We are indebted to Hanna for the very many legacies she has left behind but as a mathematics educator there are two that I would like to take up in this address—the first is her deep commitment to making the abstract accessible. The second was her passion for reforming school mathematics curricula (Fowler, 2012; Newman & Wall, 1974).

A commitment to making the abstract accessible

According to Newman and Wall (1974), Hanna Neumann developed a style of teaching that made the "acquisition of very abstract ideas accessible through judicious use of more concrete examples and well-graded exercises" (p. 4). She regularly offered lectures on topics that were not considered formally in University courses but served to convey her own joy in mathematics, participating in the model-building group and introducing undergraduate students to 'new mathematics' in creative and innovative ways. She also took an active interest in the professional development of secondary teachers of mathematics. For example, in 1971 she addressed a regional meeting of teachers at Wodonga Technical School on 'Modern Mathematics—Symbolism and its importance at the secondary and tertiary levels'. An issue, many would agree, we are still grappling with today.

Having completed my undergraduate degree in pure mathematics at Monash, I had no idea at the time just how 'modern' the mathematics courses were at Monash. All I know is that when we arrived as naïve first year students having done reasonably well under the 'old mathematics curriculum', we were deep-ended into the 'new mathematics'. My first semester was a blur—nothing looked or felt like anything we had done before. We were introduced to sets, fields, and groups and were required to use a very different type of mathematical language—but by October or so, it all magically fell into place and I remember being carried away by the sheer beauty and connectedness of it all—Hanna would have been proud.

As a teacher of secondary mathematics and fledgling mathematics educator I was again lucky to be in the right place at the right time. The Study Group for Mathematics Learning (SGML) was set up by an enthusiastic group of mathematics teachers[1] who were keen to apply the 'new mathematics' in schools. Zoltan Deines spent some time in Melbourne around this time and the SGML workshops familiarised us with the use of structured materials to support a different approach to mathematics teaching and learning (e.g., Multibase Arithmetic Blocks, Attribute Blocks, Logic Tracks and games involving multiple representations and embodiments).

Exploring group theory

Some years later, encouraged by what I had learned, I incorporated my own version of Dienes' 'games' (e.g., Dienes, 1960) to illustrate the properties of mathematical groups. I was the Year 9 and 10 Coordinator at Mater Christi College in Belgrave and we were

1 Notably Vic Ryle and Ken Clements

keen to see if we could encourage more girls to continue with mathematics into Year 11 and 12, I set up an elective called 'Advanced Maths' which was open to all and designed to explore mathematics that was not in the curriculum through games and activities. We worked with Boolean Algebra using logic tracks, vector mappings using a rectangular courtyard, and I designed a line-dancing type of routine based on Dienes' games to tease out the properties of cyclic groups. Each set of four girls formed a square ABCD and the moves were limited to:

N: no move, stay in the same place.
R: move one place right (A to B to C to D to A)
L: move one place left (A to D to C to B to A)
C: move to the diagonally opposite place (A to C to A, B to D to B)

Table 1. Table of moves.

	N	**R**	**L**	**C**
N	N	R	L	C
R	R	C	N	L
L	L	N	C	R
C	C	L	R	N

We also explored Modulo Arithmetic, otherwise known as clock arithmetic, in this context, specifically, {0, 1, 2, 3; + mod 4} where, for example, 2 + 3 ~ 1 mod 4, and 2 x 3 ~ 2 mod 4. This resulted in a similar table to the one above and facilitated a discussion about patterns and commonalities. In particular that each combination of moves or combination of digits under addition generated another element of the set (closed property), that for every element of each set there was another element that when combined/added resulted in the original element (identify element), that for each element there was another element that when combined with the original element resulted in the identity element (inverse), and that order of combination or addition did not matter, that is the operation in each case was commutative. This lead us to consider other properties and eventually a better, deeper understanding of the real numbers.

The case of 9C

In my second year at Mater Christi I found myself teaching a Year 9 class of girls who were intent on leaving school as early as possible (the 'C' stood for 'commercial'). Financial mathematics was a core component of their 'modified' program but the available text treated these topics in a particularly procedural way. I decided to try a different approach. I asked the students to pinch the pages of the respective chapters between their forefinger and thumb (it amounted to about half a centimetre) and said, 'I'm going to let you in on a secret. All of the problems in these pages are of the type n% of $m = p$'. Over the course of two weeks we selected and solved problems according to type, that is, (i) n and m known, (ii) n and p known, or (iii) m and p known, and with little regard for context (i.e., profit and loss, simple interest, discount, etc.). This might seem counterintuitive, but it worked. At the end of the two weeks not only were they happy to sit the test, they all passed with flying colours and asked if we could do more maths like that—I obliged and we used this technique to explore Pythagoras' Theorem, which was not in their course, but it demonstrated to them that they could perform equally as well as the girls in 9P (P for 'professional'). This taught me a valuable lesson about the clarifying power of mathematical structure, and that 'real-world' contexts can sometimes get in the way of learning mathematics.

Making the everyday abstract

Sometimes, making the everyday abstract can be a useful strategy to focus on the learning involved. For example, many years ago in an effort to convince parents that rote learning the multiplication 'tables' was ultimately counter-productive, I developed a set of * tables, where * was an operator defined as follows: $a * b = ((a + b) \times (a \times b))/(b - a)$. Exerts from two * tables were provided (see Table 2) and all but a small group of parents were asked to learn these by whatever means they chose in preparation for a test in 10 minutes time. The small group were taught the meaning of the operator in terms of the rule: 'sum multiplied by product, divided by the difference reversed' and encouraged to practice applying the rule to any fact in the one and two * tables including the related facts (e.g., 4 * 1 as well as 1 * 4).

*Table 2. List of * facts.*

0 * 1 = 0	0 * 2 = 0
1 * 1 = undefined	1 * 2 = 6
2 * 1 = –6	2 * 2 = undefined
3 * 1 = –6	3 * 2 = –30
4 * 1 = –6.666...	4 * 2 = –24
5 * 1 = –7.5	5 * 2 = –23.333...
6 * 1 = –8.4	6 * 2 = –24

The test involved five 'facts', 3 * 1, 4 * 2, 6 * 1, 1 * 2 and 2 * 3. Not surprisingly, most parents remembered at least three of the facts and the best anyone, not involved in the small group did, was four out of five. Some assumed * was commutative, others complained that it was not fair as 2 * 3 was not in the list provided. By contrast, all of the small group were able to achieve five out of five correct. This prompted an extremely robust and valuable discussion about the importance of understanding the operator involved and not just relying on memory however effective this was in the short term.

Another example of the benefits of making the everyday abstract arose from inviting primary pre-service teachers to construct a completely new set of names and symbols for the digits 0 to 9 then brainstorm what might be involved in teaching these to a group of five-year-olds. This also led to a robust discussion on the nature of mathematics learning and our assumptions as teachers. I have not tried it, but extending this activity to another number base and generating multi digit numbers might also be worthwhile.

Reforming mathematics curriculum

In 1964 Hanna Neumann was actively engaged in the discussions on the new, senior secondary mathematics syllabuses in NSW and, as Newman and Wells (1974) point out, "[it] was undoubtedly her work in evaluation of the draft proposals and her energetic work on suggestions for improvements, which earned for the Canberra Mathematical Association a reputation for trenchant and constructive criticism" (p. 6). It is in this same spirit that I offer the following commentary on mathematics curriculum in general and the role and place of multiplicative thinking in particular.

The crowded curriculum and the lack of succinct, unambiguous guidelines about the key ideas and strategies needed to make progress in school mathematic have been the concern of teachers of mathematics for many years. This is particularly the case for Number which successive mathematics curricula and text books have tended to represent as long lists of disconnected 'topics' that value the reproduction of relatively low-order skills and competencies rather than promoting deep understanding of key ideas, generalisation and problem solving (Siemon, 2011a).

While the importance of focussing on 'big ideas' is widely recognised (e.g., Charles, 2005), there is little agreement about what these are or how these are best represented to support the teaching and learning of mathematics. For example, what might be a 'big idea' from a purely mathematical perspective (e.g., set theory), may not be a 'big idea' from a pedagogical perspective. That is, 'big ideas' need to be both mathematically important and pedagogically appropriate to serve as underlying structures on which further mathematical understanding and confidence can be built (Siemon, Bleckly & Neal, 2012). The *Curriculum Focal Points for Pre-Kindergarten through Grade 8 Mathematics: A Quest for Coherence* (NCTM, 2006) go some way towards achieving this goal by providing a more detailed account of 'important mathematics' at each grade level for K to 8. But big ideas are notoriously difficult to accommodate in curriculum documents as Hanna Neumann experienced in her endeavours to introduce the big ideas of 'modern mathematics' into the NSW mathematics curriculum in the sixties. But this does not mean we should not engage with this slippery notion. Big ideas serve a useful purpose in that they operate as a test of curriculum coherence and serve as interpretive lenses through which skill-based content descriptors can be examined in more depth.

Big ideas in mathematics

For the purposes of the *Assessment for Common Misunderstandings* (Department of Education and Early Childhood Development, 2007; Siemon, 2006) and the *Developmental Maps* (Siemon, 2011b), which were developed for the Victorian Department of Education and Early Childhood Development, a 'big idea' in mathematics:

- is an idea, strategy, or way of thinking about some key aspect of mathematics without which, students' progress in mathematics will be seriously impacted;
- encompasses and connects many other ideas and strategies;
- serves as an idealised cognitive model (Lakoff, 1987), that is, it provides an organising structure or a frame of reference that supports further learning and generalizations;
- cannot be clearly defined but can be observed in activity (Siemon, 2006, 2011b).

The big ideas identified for this purpose are shown in Table 3. The rationale for the choice of number and for considering multiplicative thinking in particular will be addressed in more detail below.

Table 3. Big Ideas identified for the Assessment for Common Misunderstanding *Tools*

By the end of	'Big Idea'
Foundation	*Trusting the Count*—developing flexible mental objects for the numbers 0 to 10 (2 tools)
Year 2	*Place-value*—the importance of moving beyond counting by ones, the structure of the base 10 numeration system (4 tools)
Year 4	*Multiplicative thinking*—the key to understanding rational number and developing efficient mental and written computation strategies in later years (6 tools)
Year 6	*Partitioning*—the missing link in building common fraction and decimal knowledge and confidence (7 tools)
Year 8	*Proportional reasoning*—extending what is known about multiplication and division beyond rule-based procedures to solve problems involving fractions, decimals, per cent, ratio, rate and proportion (8 tools)
Year 10	*Generalising*—skills and strategies to support equivalence, recognition of number properties and patterns, and the use of algebraic text without which it is impossible to engage with broader curricula expectations at this level (4 tools)

Multiplicative thinking

The capacity to think multiplicatively is crucial to success in further school mathematics. It underpins nearly all of the topics considered in the middle years and beyond, and lack of it or otherwise is the single most important reason for the eight-year range in mathematics achievement in Years 5 to 9 (Siemon, Virgona & Corneille, 2001). Hence the choice of number for the 'big ideas' listed above.

Multiplicative thinking involves recognising and working with relationships between quantities. In particular, it supports efficient solutions to more difficult problems involving multiplication and division, fractions, decimal fractions, ratio, rates and percentage. Although some aspects of multiplicative thinking are available to young children, multiplicative thinking is substantially more complex than additive thinking and may take many years to achieve (Vergnaud, 1983; Lamon, 2007). This is because multiplicative thinking is concerned with processes such as replicating, shrinking, enlarging, and exponentiating that are fundamentally more complex, rather than the more obvious processes of aggregation and disaggregation associated with additive thinking and the use of whole numbers (Siemon, Beswick, Brady, Clark, Faragher & Warren, 2011).

The *Scaffolding Numeracy in the Middle Years* (SNMY) research project (see Siemon, Breed, Dole, Izard & Virgona, 2006) was designed to explore the development of multiplicative thinking in Years 4 to 8. Multiplicative thinking was seen to be characterised by:

- a capacity to work flexibly and efficiently with an extended range of numbers (i.e., larger whole numbers, decimals, common fractions, ratio and per cent),
- an ability to recognise and solve a range of problems involving multiplication or division including direct and indirect proportion, and
- the means to communicate this effectively in a variety of ways (e.g., words, diagrams, symbolic expressions, and written algorithms).

The SNMY project used rich tasks in a pen and paper format to test a hypothetical learning trajectory for multiplicative thinking in Grades 4–8 (Siemon et al., 2006). Item response theory (e.g., Bond & Fox, 2001) was used to identify eight qualitatively different categories of responses, which subsequently lead to a *Learning and Assessment Framework for Multiplicative Thinking* (LAF) comprised of eight 'zones' representing increasingly sophisticated levels of understanding (see Table 4). Rich descriptions were developed for each zone and teaching advice was provided in the form of what needed to be *consolidated and established* and what needed to be *introduced and developed* to scaffold multiplicative thinking to the next zone.

Table 4. The Learning Assessment Framework for Multiplicative Thinking (Siemon et al., 2006)

Zone 1: Solves simple multiplication and division problems involving relatively small whole numbers but tends to rely on drawing, models and count-all strategies. May use skip counting for groups less than 5. Makes simple observations from data and extends simple number patterns. Multiplicative thinking (MT) not really apparent as no indication that groups are perceived as composite units, dealt with systematically, or that the number of groups can be manipulated to support more efficient calculation
Zone 2: Counts large collections efficiently—keeps track of count but needs to see all groups. Shares collections equally. Recognises small numbers as composite units (e.g., can count equal groups, skip count by twos, threes and fives). Recognises multiplication needed but tends not to be able to follow this through to solution. Lists some of the options in simple Cartesian product situations. Some evidence of MT as equal groups/shares seen as entities that can be counted.
Zone 3: Demonstrates intuitive sense of proportion. Works with useful numbers such as 2 and 5 and intuitive strategies to count/compare groups (e.g., doubling, or repeated halving to compare simple fractions). May list all options in a simple Cartesian product, but cannot explain or justify solutions. Beginning to work with larger whole numbers and patterns but tends to rely on count all methods or additive thinking (AT).
Zone 4: Solves simple multiplication and division problems involving two-digit numbers. Tends to rely on AT, drawings and/or informal strategies to tackle problems involving larger numbers, decimals and/or less familiar situations. Tends not to explain thinking or indicate working. Partitions given number or quantity into equal parts and describes part formally. Beginning to work with simple proportion.
Zone 5: Solves whole number proportion and array problems systematically. Solves simple, 2-step problems using a recognised rule/relationship but finds this difficult for larger numbers. Determines all options in Cartesian product situations involving relatively small numbers, but tends to do this additively. Beginning to work with decimal numbers and percent. Some evidence MT being used to support partitioning. Beginning to approach a broader range of multiplicative situations more systematically
Zone 6: Systematically lists/determines the number of options in Cartesian product situation. Solves a broader range of multiplication and division problems involving 2-digit numbers, patterns and/or proportion but may not be able to explain or justify solution strategy. Renames and compares fractions in the halving family, uses partitioning strategies to locate simple fractions. Developing sense of proportion, but unable to explain or justify thinking. Developing capacity to work mentally with multiplication and division facts
Zone 7: Solves and explains one-step problems involving multiplication and division with whole numbers using informal strategies and/or formal recording. Solves and explains solutions to problems involving simple patterns, percent and proportion. May not be able to show working and/or explain strategies for situations involving larger numbers or less familiar problems. Constructs/locates fractions using efficient partitioning strategies. Beginning to make connections between problems and solution strategies and how to communicate this mathematically
Zone 8: Uses appropriate representations, language and symbols to solve and justify a wide range of problems involving unfamiliar multiplicative situations, fractions and decimals. Can justify partitioning, and formally describe patterns in terms of general rules. Beginning to work more systematically with complex, open-ended problems.

What the data that underpins this research-based framework shows is that the transition from additive to multiplicative thinking is nowhere near as smooth or as straightforward as most curriculum documents seem to imply, and that access to multiplicative thinking as it is described here represents a real and persistent barrier to many students' mathematical progress in the middle years of schooling (Siemon & Breed, 2005; Siemon et al., 2006).

To become confident multiplicative thinkers, children need a well-developed sense of number (based on trusting the count, place-value and partitioning) and a deep understanding of the many different contexts in which multiplication and division can arise (e.g., sharing, equal groups, arrays, regions, rates, ratio and the Cartesian product). The transition from additive strategies to meaningful, mental strategies that support multiplicative reasoning more generally requires a significant shift in thinking from a count of *equal groups* and a reliance on repeated addition, to the *for each* and *times as many* ideas for multiplication that underpin all further work with multiplication, division and rational number. While the *array* and *region* ideas for multiplication can be used to support a count of equal groups, their power lies in the fact that they can be used to underpin this important shift in thinking and, ultimately, the *factor–factor–product* idea that supports the inherently multiplicative operations of equipartitioning, replicating, enlarging, shrinking and a more generalised understanding of the relationship between multiplication and division. In addition, the *region* and *for each* ideas for multiplication are also critically important in the interpretation and construction of fraction representations (for a much more detailed discussion of these ideas see Siemon, Beswick, Brady, Clark, Farragher & Warren, 2011).

The difference between additive and multiplicative thinking

The essential difference between additive and multiplicative thinking relates to the nature of the units under consideration. For addition and subtraction, "all the number meanings … are directly related to set size and to the actions of joining or separating objects and sets" (Nunes & Bryant, 1996, p. 144). In these situations it is possible to work with the numbers involved as collections that can be aggregated or disaggregated and renamed as needed to facilitate computation.

While it is possible to use repeated addition to solve multiplication problems and repeated subtraction to solve division problems, these are essentially additive processes—the only difference is that the sets being added or subtracted are the same size. Multiplicative thinking involves much more than this and "it would be wrong to treat multiplication as just another, rather complicated, form of addition, or division as just another form of subtraction" (Nunes & Bryant, 1996, p. 144). For example, consider the following Year 4 responses to the problem: how many muffins could be made with 6 cups of milk if 2/3 cup of milk produced 12 muffins?

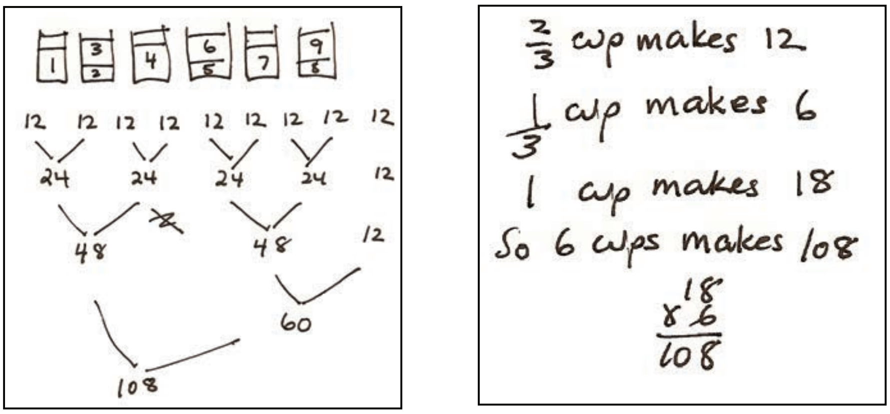

Figure 1. Two solutions to the Muffin problem (Siemon et al., 2011).

The first response uses repeated addition to determine how many two-third cups are in 6 cups, and then counting to find the total number of muffins. The second solution recognises the proportional relationship between the quantity of milk and the number of muffins. Both strategies produce the correct answer, but the first is additive whereas the second is multiplicative.

For multiplication, it is necessary to simultaneously recognise and coordinate the number of groups (multiplier) and the number in each group (multiplicand) (Anghileri, 1989; Jacob & Willis, 2001; Nunes & Bryant, 1996; Vergnaud, 1983). According to Steffe (1992), for a ‘situation to be established as multiplicative, it is always necessary at least to coordinate two composite units in such a way that one composite unit is distributed over the elements of the other composite unit’ (p. 264), resulting in a composite unit of composite units (e.g., see Figure 2).

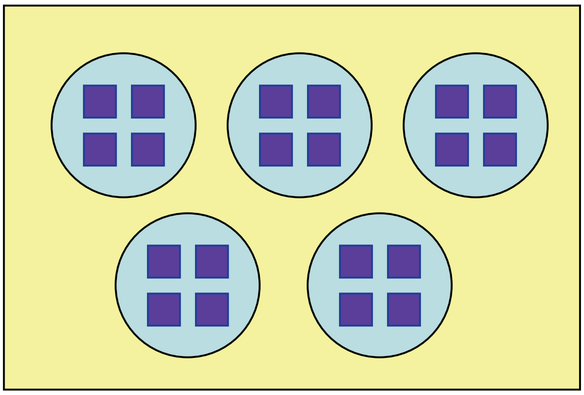

Figure 2. A composite unit of composite units.

Recognising and working with composite units introduces the distinction between *how many* (the count of composite units) and *how much* (the magnitude of each composite unit). This distinction is often overlooked in the rush to symbolise, with the result that many children interpret *3 groups of four* as successive counts of four ones rather than *3 fours* which emphasises the distribution of one composite unit over another. This has important implications for the development of multiplicative thinking and children’s capacity to understand fractions. By distinguishing between the count and the unit, children are more likely to recognise the multiplicative nature of our number systems For example, the digits in the numeral 34 are both counting numbers (i.e., how many numbers), but their location or place determines the unit (i.e., how much). In the fraction ¾ the numerator indicates how many but the denominator indicates how much. They are also more likely to recognise the relative magnitude of different units (e.g., that 3 quarters is larger than 3 eighths) and the inverse relationship between *how many* and *how much* (e.g., the larger the number of shares/equal parts, the smaller each share/part).

Multiple pathways to multiplicative thinking

Nearly all of the research-based developmental frameworks for multiplication are framed in terms of counting-based strategies that ultimately terminate with a reference to the use of number fact knowledge (e.g., Department of Education & Early Childhood Development, 2010; Department of Education & Training, 2007; van den Heuvel-Panhuizen, 2001). This is not surprising given the almost exclusive focus on *equal groups* and repeated addition in the early years. However, an increasing number of researchers (e.g., Confrey, Maloney, Nguyen, Mojica & Myers, 2009; Downton, 2008; Nunes & Bryant, 1996; Schmittau & Morris, 2004) suggest that there is a parallel path to the development of multiplicative thinking based on young children's capacity to share equally and work with one-to-many relationships. For example, having explored the 'Baa-Baa Black Sheep' rhyme in literacy, a teacher posed the following question to her class of 5 and 6 year olds: 'I wonder how many bags of wool would there be if there were 5 sheep?' While most decided that there would be 15 bags of wool, what was interesting was the number of children who constructed abstract representations, in particular, representations that connected each sheep with three bags of wool (e.g., see Figure 3).

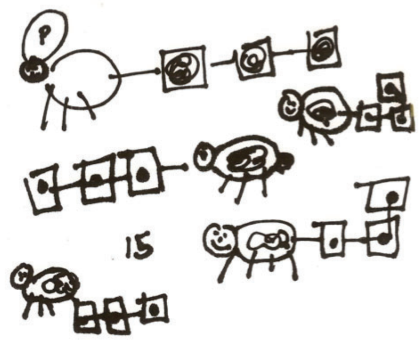

Figure 3. Five year olds solution to the Baa-Baa Black Sheep problem (Siemon et al., 2011).

This suggests that the children understood the situation in terms of *for each* sheep there are 3 bags of wool. This is essentially a ratio or times as many idea (e.g., 3 times as many bags of wool as sheep) and is quite distinct from the *equal groups* idea, even though the children invariably counted by ones to arrive at the solution of 15 bags of wool altogether.

The representation of multiplicative thinking in the Australian Mathematics Curriculum

In most English-speaking countries, multiplication and division are introduced separately, with multiplication typically considered before division. Given what is known about multiplicative thinking, young children's experience with sharing, and evidence to suggest that simple proportion problems can be solved earlier than generally expected (e.g., Confrey et al., 2009; Nunes & Bryant, 1996; Schmittau & Morris, 2004; Siemon et al., 2006), the introduction of the *Australian Curriculum: Mathematics* [ACM] provided an opportune time to reconsider when and how we introduce these important ideas. How well has it fared?

A detailed analysis of the ACMs potential for developing multiplicative thinking is included in Appendix A. Here, I shall draw on some of the observations made previously (see Siemon, Blecky &Neal, 2012) in relation to the presence or otherwise of the key ideas and strategies mentioned above—the codes in brackets refer to the content descriptors in the ACM.

While *sharing* is mentioned in Foundations (ACMNA004) the only other reference to any of the key ideas discussed above is in Year 2 where students are expected to recognise and represent "multiplication as repeated addition, groups and arrays" (ACMNA031) and "division as grouping into equal sets" (ACMNA032). This reference to division is ambiguous as it could refer to *quotition* division (where the divisor refers to size of group) or *partition* division (where the divisor refers to the number of equal groups). However, grouping a collection into equal sets and working with arrays is no guarantee of multiplicative thinking unless the focus of attention is shifted from a count of groups of the same size (additive) to a given number of groups of any size (Siemon et al., 2011). Importantly, the *region* idea is not mentioned at all and yet this underpins the *area* (*by* or *factor*) idea of multiplication (i.e., each part multiplied by every other part) which is needed to support the multiplication of larger whole numbers (e.g., 2-digit by 2-digit multiplication), the interpretation of fraction diagrams (e.g., thirds by fifths are fifteenths), and, ultimately, the multiplication and division of fractions and linear factors.

In Year 3 students are expected to "recall multiplication facts of two, three, five and ten and related division facts" (ACMNA056). This wording together with the previous (AMN026) and subsequent (AMN074) references to number sequences implies that the multiplication facts are learnt in sequence (e.g., 1 three, 2 threes, 3 threes, 4 threes, 5 threes, etc.) rather than on the basis of number of groups irrespective of size (e.g., 3 of anything is double the group and one more group).

Factors and multiples are referred to in Year 5 (ACMNA098) and Year 6 (ACMNA122), indices in Years 7 and 8 (ACMNA149 & ACMNA182), and solving problems involving specified numbers and operations across year levels (e.g., ACMNA100, ACMNA101 and ACMNA103). However, there is no suggestion of the connections between them or that something other than a repeated addition model of multiplication is needed to support a deep understanding of factors and indices (Confrey et al., 2009).

In the early years, the ACM refers to the capacity to "recognise and describe half as one of two equal pieces" (ACMNA016) and "to recognise and interpret common uses of halves, quarters and eighths of shapes and collections" (ACMNA033) but no mention is made of the important link to sharing which provides a powerful basis for the creation of equal parts and the link between fractions and partitive division (Nunes & Bryant, 1996). In Year 3, students are expected to be able to "model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole"

(ACMNA058). This suggests that fraction symbols are expected at this stage, which is problematic given the well-known difficulties associated with interpreting fraction symbols and representations (e.g., Lamon, 1999). Also, the reference to counting fractions at Year 4 (ACMNA078) could lead to an over-reliance on additive, whole number-based approaches to locating fractions on a number line at the expense of multiplicative approaches such as equipartitioning (Confrey et al., 2009; Lamon 1999).

The ACM does not refer to proportional reasoning explicitly until Year 9 where reference is made to solving problems involving direct proportion and simple rates (ACMNA208) and enlargements, similarity, ratios and scale factors in relation to geometrical reasoning (ACMMG220 & ACMMG221). While many of the prerequisite skills are included in Years 6 to 8, these appear in the form of disconnected skills. For example, "find a simple fraction of a quantity" (ACMNA127) at Year 6, "express one quantity as a fraction of another", "find percentages of quantities and express one quantity as a percentage of another" (ACMNA 155 & ACMNA158) at Year 7, and solve a range of problems involving percentages, rates and ratios (ACMNA187 & ACMNA188) at Year 8. Importantly, there is nothing to suggest how these skills relate to one another or their rich connections to multiplicative thinking more generally.

As the above discussion and the analysis in the Appendix shows, the content descriptors of the ACM have the potential to support the development of multiplicative thinking. But the extent to which this potential is realised is heavily dependent on how the descriptors are interpreted, represented, considered and connected in practice. Content descriptors do need to be in a form that is clearly assessable but, if these are taught and assessed in isolation with little attention to student's prior knowledge and the underpinning ideas and strategies, there is a substantial risk that access to multiplicative thinking will continue to elude many. On the other hand, if the content descriptors are taught and assessed in conjunction with the proficiencies, that is, conceptual understanding, procedural fluency, mathematical reasoning and mathematical problem solving, the chances of increasing access to multiplicative thinking in the middle years can be greatly enhanced.

Conclusion

Hanna Neumann left a valuable legacy to mathematics and mathematics education both here and abroad. As one of the founding members of AAMT, it is fitting that we acknowledge her contributions to school mathematics in the biennial lecture that bears her name. Her commitment to making the abstract accessible and her passion for reforming school mathematics curriculum framed this presentation. In demonstrating how group theory might be explored in the context of dance and clock arithmetic and what can be gained from working with mathematical structures, I hope you too might be prompted to consider how you might make the abstract accessible and the everyday abstract. My comments on the place of multiplicative thinking in the *Australian Curriculum: Mathematics*, are offered in the same spirit and with the same motivation that Hanna offered her suggestions and feedback on the NSW senior secondary mathematics syllabuses in the sixties—that is, the need to recognise and focus on the 'big ideas' in mathematics so that all learners have the opportunity to experience the joy of doing mathematics and to access the future that it affords.

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Appendix: Opportunities for multiplicative thinking in the ACM

Year Level	Selected ACM Content Descriptors (ACARA, 2013)	Relationship to Multiplicative Thinking
F	Subitise small collections of objects (ACMNA003)	Helps establish the notion of composite units—children 'see' a collection of 4 ones as 'four' without having to count
	Represent practical situations to model addition and sharing (ACMNA004)	Sharing helps establish notion of equal shares, equal groups and relationship between the number of shares (how many) and the number in each share (how much)
1	Develop confidence with number sequences to and from 100 by ones from any starting point. Skip count by twos, fives and tens starting from zero (ACMNA012)	Skip counting, while essentially additive, if used as a strategy for physically counting large collections, helps establish one-many relationships and notion of composite units. Risk: limited to number naming sequences
	Recognise, model, read, write and order numbers to at least 100. Locate these numbers on a number line (ACMNA013)	Locating numbers on a number line—if open—invites the use of multiplicative or equipartitioning based on benchmarks (e.g., it's about half)
	Recognise and describe one-half as one of two equal parts of a whole. (ACMNA016)	Introduces multiplicative partitioning and halving. Risk is that parts will not be seen in relation to the whole
	Investigate and describe number patterns formed by skip counting and patterns with objects (ACMNA018)	Potential to support notion of composite units. Risk is that this will be limited to additive or repeating patterns rather than multiplicative or growing patterns
	No further reference to sharing	
2	Investigate number sequences, initially those increasing and decreasing by twos, threes, fives and ten from any starting point, then moving to other sequences. (ACMNA026)	Suggests a count of twos, threes, etc. Risk: limited to number naming sequences, preferences a count of groups as basis for multiplication facts
	Recognise and represent multiplication as repeated addition, groups and arrays (ACMNA031)	Key representations. Risk: interpretation limited to *equal groups*, count of groups
	Recognise and represent division as grouping into equal sets and solve simple problems using these representations (ACMNA032)	Inclusive of both forms of division (*quotition* and *partition*) . Risk: limited to count of groups, sharing not generalised to 'think of multiplication"
	Recognise and interpret common uses of halves, quarters and eighths of shapes and collections	Potential to engage students in equipartitioning, Risk: Fraction names seen as labels for parts rather than relationships. No involvement in equipartitioning, partitioning strategies, teaching may not deal with core generalisations
	Place value appears to be treated additively	
3	Recall multiplication facts of two, three, five and ten and related division facts (ACMNA056)	Implies memorisation of facts, unclear as to how these are represented (e.g., counts of 2 or 2 of anything). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
	Represent and solve problems involving	Potentially supportive of multiplicative thinking if

	multiplication using efficient mental and written strategies and appropriate digital technologies	strategies based on *arrays* and *regions* and shift of thinking from size of group (count of equal groups) to number of groups (*factor* idea)
	Model and represent unit fractions including 1/2, 1/4, 1/3, 1/5 and their multiples to a complete whole (ACMNA058)	Strongly multiplicative where students engaged in equipartitioning strategies to construct their own fraction models and representations. Risk: focus on the name of parts not the relationship to the whole
	No reference to representations of multiplication and division	
4	Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075)	Implies memorisation of facts, unclear as to how these are represented (see above). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
	Develop efficient mental and written strategies and use appropriate digital technologies for multiplication and for division where there is no remainder (ACMNA076)	Potentially supportive of multiplicative thinking if strategies based on shift of thinking from size of group (count of equal groups) to number of groups (factor idea), use of distributive law , etc. Risk: Strategies based on repeated addition, count all groups
	Investigate equivalent fractions used in contexts (ACMNA077)	Highly multiplicative if explored via equipartitioning strategies (e.g., halving, thirding and fifthing) and linked to *region* idea (e.g., thirds by fourths are twelfths). Risk: treated as rule-based procedure
	Count by quarters halves and thirds, including with mixed numerals. Locate and represent these fractions on a number line (ACMNA078)	Locating fractions on an open number line invites the use of equipartitioning strategies based on benchmarks (e.g., halving, thirding and fifthing , etc.) and links to *fractions as number* idea. Risk: treated additively
	Recognise that the place value system can be extended to tenths and hundredths. Make connections between fractions and decimal notation (ACMNA079)	Potential to relate place-value system equipartitioning and *for each* idea (i.e., for each one there are 10 tenths), see base 10 system as multiplicative. Risk: introduced before students understand whole number as multiplicative system
	Recall multiplication facts up to 10 × 10 and related division facts (ACMNA075)	Implies memorisation of facts, unclear as to how these are represented (see above). Risk: limited to *equal groups*, count of groups (i.e., 'traditional tables' representation)
	Explore and describe number patterns resulting from performing multiplication (ACMNA081)	Potential to shift thinking from count of *equal groups* to *factor* or *scalar* idea to support more efficient mental strategies (e.g., 4 of anything is double double). Risk: treated as repeated addition
	No reference to arrays, regions, Cartesian product, partition or quotition division No reference to benchmark percents (50%, 25%, 10% , etc.)	
5	Identify and describe factors and multiples of whole numbers and use them to solve problems (ACMNA098)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: considered in isolation from representations of multiplication
	Solve problems involving multiplication of large numbers by one- or two-digit numbers using efficient mental, written strategies and appropriate digital technologies (ACMNA100)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: strategies based on/limited to repeated addition, count all groups, rote learnt procedures
	Solve problems involving division by a one digit number, including those that result in a remainder (ACMNA101)	Highly supportive of multiplicative thinking if based on sharing or 'what do I have to multiply by' (i.e., *factor* idea)
	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)	Highly supportive of multiplicative thinking if based on equipartitioning strategies (e.g., halving, thirding or fifthing) and linked to *fraction as number* idea. Risk: treated as a iterative counting exercise
	Recognise that the place value system can be extended beyond hundredths (ACMNA104)	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *for each* idea (e.g., for each tenth there are 10 hundredths, for each hundredth there are 10 thousandths and so on)—this involves recognising recursive, exponential nature of

		the base 10 numeration system. Risk: limited to surface features
	Compare, order and represent decimals (ACMNA105)	Representing/locating decimals on a number line highly supportive of multiplicative thinking if on equipartitioning strategies (e.g., tenthing) and *for each* idea. Risk: this becomes rule based
	Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)	Potentially supportive of multiplicative thinking where numbers renamed to support more efficient calculation. Risk: this becomes rote procedure
	No link between hundredths and percentages	
6	Identify and describe properties of prime, composite, square and triangular numbers (ACMNA122)	Highly supportive of multiplicative thinking if based on *array*, *region* or *area* representations of multiplication and shift of thinking described above. Risk: taught in isolation from representations
	Compare fractions with related denominators and locate and represent them on a number line	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *fraction as number* idea
	Find a simple fraction of a quantity where the result is a whole number, with and without digital technologies (ACMNA127)	Highly supportive of multiplicative thinking if related to *partition* division, *fractions as operators* and/or *think of multiplication* strategy
	Multiply decimals by whole numbers and perform divisions by non-zero whole numbers where the results are terminating decimals, with and without digital technologies (ACMNA129)	Highly supportive of multiplicative thinking if based on *area* or *factor* representations of multiplication and partition division strategies (i.e., sharing and/or think of multiplication). Risk: procedures devoid of meaning, inability to check reasonableness of outcome
	Multiply and divide decimals by powers of 10	Potentially supportive of multiplicative thinking if explored in relation to structure of the base 10 system of numeration. Risk: Meaningless procedures such as 'adding 0", moving decimal point
	Make connections between equivalent fractions, decimals and percentages (ACMNA131)	Highly supportive of multiplicative thinking if based on equipartitioning strategies, *fraction as number i*dea—First mention of percentages. Risk: Meaningless rule-based procedures
	Investigate and calculate percentage discounts of 10%, 25% and 50% on sale items, with and without digital technologies (ACMNA132)	Supportive of multiplicative relationships if linked to equipartitioning strategies (e.g., halving, fifthing), *for each* and *fraction as operator* ideas and multiplication by decimal fractions. Risk: Meaningless rule-based procedures, inability to check reasonableness of results
7	Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)	Highly supportive of multiplicative thinking if linked to *for each* and *factor.factor.product* ideas
	Investigate and use square roots of perfect square numbers (ACMNA150)	Highly supportive of multiplicative thinking if linked to *factor* idea and *think of multiplication* strategy
	Apply the associative, commutative and distributive laws to aid mental and written computation	First mention of these properties yet used in mental computation much earlier and 2 digit by 2 digit multiplication in Year 6. Supportive of multiplicative thinking where factors used
	Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line (ACMNA152)	Highly supportive of multiplicative thinking if based on equipartitioning strategies and *fraction as number* idea. Risk: Taught in isolation, meaningless rule-based procedures
	Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154)	Supportive of multiplicative thinking if based on equipartitioning representations, *fraction as operator*. Risk: Taught in isolation, meaningless rule-based procedures
	Express one quantity as a fraction of another, with and without the use of digital technologies	Highly supportive of multiplicative thinking if linked to *fraction as quotient* idea. Risk: Taught in isolation, meaningless rule-based procedures
	Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157)	Supportive of multiplicative thinking if linked to *fraction as quotient* idea. Risk: Taught in isolation, meaningless rule-based procedures

	Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies. (ACMNA158)	Highly supportive of multiplicative thinking if linked to *fraction as operator* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Recognise and solve problems involving simple ratios (ACMNA173)	Highly supportive of multiplicative thinking if linked to *fraction as ratio* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Investigate and calculate 'best buys', with and without digital technologies (ACMNA174)	Highly supportive of multiplicative thinking if seen as application of proportional reasoning, related to *fraction as quotient* idea
8	Use index notation with numbers to establish the index laws with positive integral indices and the zero index (ACMNA182)	Requires multiplicative thinking and recognition of f*actor* idea. Risk: laws treated in isolation from underpinning properties
	Solve problems involving the use of percentages, including percentage increases and decreases, with and without digital technologies (ACMNA187)	Highly supportive of multiplicative thinking if linked to *fraction as operator* interpretation. Risk: Taught in isolation, meaningless rule-based procedures
	Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)	Highly supportive of multiplicative thinking if linked to *fraction as ratio*. Risk: Taught in isolation, meaningless rule-based procedures
	Solve problems involving profit and loss, with and without digital technologies (ACMNA189)	This is an application of ACMNA187
	Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)	Supportive of multiplicative thinking where linked to *factor.factor.product* idea and partitioning (both additive and multiplicative)
	Factorise algebraic expressions by identifying numerical factors (ACMNA191)	Supportive of multiplicative thinking where linked to *factor.factor.product* idea
9	Solve problems involving direct proportion. Explore the relationship between graphs and equations corresponding to simple rate problems (ACMNA208)	Requires multiplicative thinking to be achieved with understanding

RESEARCH PAPERS

INCREASING THE LD50 OF MATHEMATICS: RE-ENGAGING STUDENTS IN MATHEMATICS LEARNING

KIM BESWICK
University of Tasmania
kim.beswick@utas.edu.au

TRACEY MUIR
University of Tasmania
tracey.muir@utas.edu.au

KATARA JADE
University of Tasmania
katara.jade@utas.edu.au

LINDA FARRINGTON
Department of Education, Tasmania
linda.farrington@education.tas.gov.au

ROSEMARY CALLINGHAM
University of Tasmania
rosemary.callingham@utas.edu.au

Although most students agree that mathematics is important and useful, many capable students choose not to pursue study of Mathematics when they have the choice. In this paper we describe a project-based learning approach to mathematics adopted in a Year 9–12 school in which none of the teachers were mathematics specialists. Data about students' views of mathematics and their perceptions of mathematics learning approaches in their current school are presented. The results highlight the negativity towards mathematics that these students felt and the huge task faced by their teachers in attempting re-engage them with learning in the area.

Engagement with mathematics is one of eight dimensions of attitude to the subject identifiable in the literature on students' attitudes to mathematics (Beswick, Watson & Brown, 2006). Each of the eight dimensions are conceptualised as dichotomies: engagement or avoidance; liking or disliking; regarding mathematics as important or unimportant; useful or useless; easy or difficult; viewing oneself as good or bad at the subject (Ma & Kishor, 1997); feeling confident or anxious (Ernest, 1988); and believing mathematics to be interesting or uninteresting (McLeod, 1992). Disengagement with mathematics is thus part of a negative attitude to mathematics.

This paper reports on the attitudes to mathematics of Year 9–12 students enrolled in a school that had adopted a project-based learning approach across the entire curriculum. The students' perceptions of mathematics learning at the school are also reported and the implications of the apparently toxic effect that years spent in mathematics classes appears to have on many students' attitudes to the subject are discussed.

Why attitudes matter

Positive attitudes to mathematics have been associated with improved achievement in the subject (Ashcraft & Kirk, 2001) but negative attitudes to mathematics are prevalent among secondary school students. For example, students interviewed by Boaler (1994) described the mathematics they experienced in school as a boring, meaningless, individual activity. In the most recent Trends in International Mathematics and Science Study (TIMSS) Australian Year 8 students who reported liking learning mathematics achieved, on average, higher scores than their peers who did not express a liking for the subject (Thomson, Hillman & Wernert, 2012). It is of concern, therefore, that the percentage of Australian students who indicated that they liked learning mathematics was 16% compared with the international average of 26%. At the other end of the spectrum, 45% of Australian Year 8s reported not liking mathematics learning compared to the 31% international average. These figures are consistent with the widespread concern among teachers in the middle years with student disengagement (Luke et al., 2003; Sullivan, Tobias & McDonough, 2006).

In addition, there is evidence that Australian students' attitudes to mathematics decline with school Year level (Beswick et al., 2006). Beswick et al. (2006) found, for the 650 Year 5–8 students in their study, significant negative correlations between year level and overall attitude as well as the particular dimensions: interesting/uninteresting; engage/avoid; and like/dislike. The tendency for students to disengage with mathematics through the middle years of schooling is likely to contribute to the ongoing decline in numbers of students undertaking senior secondary and tertiary studies in mathematics and mathematics-related fields (Australian Mathematical Sciences Institute, 2013; McPhan, Morony, Pegg, Cooksey & Lynch, 2008).

Why do students disengage with mathematics?

Many factors have been identified as contributors to student disengagement in the middle years of schooling and associated negative attitudes to learning and particularly to mathematics. They include inappropriate curricula, insufficiently challenging tasks, ineffective teaching, poor physical design of learning spaces, and societal changes related to technological and social developments (Luke et al., 2003). Parents are also influential with high parental expectations for academic achievement associated with greater student engagement (Chen & Gregory, 2010).

There is evidence that teachers' beliefs about their students' capacities to learn and their own efficacy in helping students to learn also have an impact on their students' attitudes. For example, Midgley, Feldlaufer and Eccles (1988) reported that teachers of lower secondary school mathematics had lower expectations of their students' learning and of their own efficacy in facilitating learning than primary teachers. They suggested this as an explanation for the decline in mathematics achievement across the middle years. However, from their large scale study involving Canadian secondary students from disadvantaged communities, Archambault, Janosz and Chouinard (2012) concluded that engagement and achievement are more likely to be related to students' prior experiences of learning mathematics, and particularly success and difficulties that this has entailed, than to their teachers' beliefs. They specifically referred to the impacts of ability grouping in this regard. Their findings suggested that the attitude and beliefs of an individual teacher over a single school year are unlikely significantly to influence their students' engagement although they did appear to influence achievement. Long histories of lack of success with mathematics learning, therefore, appear to lead to entrenched disengagement that persists even if achievement can be boosted by an enthusiastic and encouraging teacher.

A survey of junior secondary Australian mathematics teachers found that 27% had studied no more than first year university mathematics and one third had undertaken no study of mathematics teaching methods (Harris & Jensz, 2006). Under-qualified teachers are unlikely to present mathematics in inspiring and engaging ways even if they manage to convey the content in a mathematically correct manner.

Mathematics classroom cultures in which effort and achievement are not validated by the peer group—where adolescents' needs for autonomy, identity and social success can be achieved by non-compliance, disengagement and lack of effort—have been identified as a powerful possible explanation for disengagement of middle school students (Sullivan et al., 2006). The fundamental nature of the needs that deliberate disengagement from learning can meet in these circumstances explains the difficulty teachers express in relation to motivating students to engage with learning (Sullivan et al., 2006). These authors suggested that the classroom culture may in fact be the most potent influence on student engagement.

Re-engaging students with mathematics

Attempts to re-engage students with mathematics or to prevent disengagement have tried to address each of the factors implicated in causing it (Tadich, Deed, Campbell & Prian, 2007). They have variously focussed on teachers and teaching, curricula design, and the learners themselves. Often curriculum re-design is accompanied by efforts to transform teaching approaches (e.g., Department of Education, Tasmania, 2002; Education Queensland, 2002). Approaches adopted in part with the aim of improving the engagement of middle school students include integrated and cross-disciplinary approaches (Wallace, Sheffield, Rennie & Venville, 2007). Participants in Wallace et al.'s study reported the use of teaching practices in these contexts that included multi-age student groupings, single teachers teaching across multiple learning areas and/or teaching the same students for several years, and flexible timetabling arrangements. The use of mathematics problems embedded in meaningful contexts has also been promoted as a way of improving engagement with mathematics (Jurdak, 2006). Establishing the effectiveness of these innovations in improving student engagement has been difficult as a result of the multiplicity of variables inherent in classrooms.

Approaches focussing on students have included influencing students' views about schooling and their own capacities to learn, and helping them to see the value of school achievement (Tadich et al., 2007). Others have recognised the importance of helping students to acquire skills that facilitate engagement with, and achievement in, mathematics. These include metacognitive strategies (Stillman, 2001), and study skills (Munns & Martin, 2005, cited in Tadich et al., 2007).

Project-based learning

Project-based learning involves students exploring problems or questions of interest to them. These questions tend to more closely resemble the kinds of problems encountered outside of school than do typical school tasks. The approach is characterised by student autonomy and choice (Lam, Cheng & Ma, 2009). Improved student engagement is among the benefits associated with project-based learning (Lam et al., 2009) along with opportunities to develop capacities for creative thinking, innovation (Lee & Breitenberg, 2010) and independent work (Doppelt, 2009).

Project-based learning principles are used in Australian schools associated with Big Picture Education Australia (BPEA). In some of these schools, Big Picture runs as a stream for selected students alongside a traditional curriculum structure for other students. In other cases, Big Picture principles have been adopted to a greater or lesser extent for all students in the school. Students in Big Picture schools or programs work towards five broad learning goals (BPEA, 2011) that include Quantitative Reasoning (QR).

Quantitative reasoning

In spite of the relevance of mathematical content to QR, it is important to note that QR cannot be equated to the traditional school subject of mathematics (Down & Hogan, 2010). Nevertheless, the links are sufficient for the words to be used essentially interchangeably in the context of this study. Thornton and Hogan (2003) linked the term QR to quantitative literacy as used by Steen (e.g., 2004) and to the notion of numeracy that has been defined by the Australian Association of Mathematics Teachers (1997, p. 15) in terms of being numerate—that is, able to “use mathematics effectively to meet the general demands of life at home, in paid work, and for participation in community and civic life”. QR is one of three of BPEA’s five learning goals that use the term “reasoning” defined as “the process of forming conclusions, judgments, or inferences from facts or premises” (Down & Hogan, 2010, p. 63). This definition of reasoning is consistent with the description of the Proficiency Strand of the same name in the *Australian Curriculum: Mathematics* (Australian Curriculum Assessment and Reporting Authority [ACARA], 2012).

The QR project

The project arose from a concern of the school principal and staff that a completely project-based learning model may not provide students with appropriate opportunities to meet all of the requirements of the *Australian Curriculum: Mathematics*. As described by Beswick, Callingham and Muir (2012), none of the teachers were specialist mathematics teachers, and although they demonstrated adequate mathematical knowledge for everyday purposes, most lacked confidence in their ability to assist students to develop their QR capacities. The project provided collaboratively designed professional learning for teachers, aimed at assisting them to identify opportunities for the development of QR in their students’ projects, and locate appropriate resources for teaching QR; building their confidence that the curriculum could be covered, at least as well in a project-based setting as in traditional contexts.

The school

The project was conducted in a Tasmanian Year 9–12 school that had been in operation for one year when the project commenced. Most Tasmanian secondary schools cater for students in Years 7–10, with colleges catering for Years 11 and 12 located in larger centres. In addition to the novelty of the year level range for which the school catered, it had project-based learning across the entire curriculum as part of an attempt to fully implement the BPEA model. Important principles underpinning Big Picture education include a focus on each learner as an individual, and recognition of the role of families and the broader community in helping each learner to pursue his/her interests and realise his/her potential. In keeping with this, teachers (known as advisors) each worked with a group of approximately 16 students to develop individual projects, and find opportunities for students to engage in the adult world through internships in workplaces. Students were also allowed a greater degree of autonomy in relation to dress and the use of their time than is typical in more traditional secondary schools. The physical spaces of the school provided a technology rich environment with spaces for large and small group work, as well as individual work, and comfortable and flexible furnishings.

The school catered for a diverse range of students. All had experienced mathematics learning (at least until Year 8) in another context. Their reasons for choosing to come to the school varied, but many had experienced difficulty in a traditional secondary school setting, with many having disengaged with learning to a greater or lesser extent.

Participants

Participants in the survey part of the study were 33 of the approximately 130 Year 9–12 students who comprised the school's enrolment. Five of these students were in Year 9, three in Year 10, thirteen in Year 11, eleven in Year 12, and one did not indicate a year level.

Instruments and procedure

The student survey comprised four sections, three of which were made up of Likert type items to which respondents indicated the extent of their agreement from Strongly Disagree to Strongly Agree on a 5-point scale. Sections 2 and 3, which examined the students' perceptions of learning QR at the school (10 Likert type items) and their attitudes to QR (16 Likert type items), are the relevant sections for this paper.

The items on QR learning were similar to the classroom environment items used by Beswick et al. (2006) but adapted to relate to pedagogies considered desirable in a project-based learning context. The 16 'attitude to QR' items were also adapted from those used by Beswick et al. (2006) by a simple substitution of 'mathematics' with 'QR'. Two of the 16 items related to each of eight dimensions of attitude to mathematics identified in the literature. The surveys were administered to individual students during the second quarter of the school year.

Results

Table 1 shows the percentage of participating students who Agreed or Strongly Agreed with each of the statements, along with the means and standard deviations of the participants in the QR project to the 16 attitude to QR items. Scores for the italicised items were reversed for the calculations of the means and standard deviations: a higher mean indicates a more positive attitude. Similar data for same items (with 'mathematics' substituted for 'QR') from Beswick et al. (2006) are also provided. The participants in Beswick et al.'s (2006) were 650 students in Years 5–8 in traditional Tasmanian primary and secondary schools.

Table 1. Means and standard deviations of students' responses to Likert type items.

Attitude Items	QR project (n=33) % A or SA	QR project (n=33) Mean	QR project (n=33) Std. Dev.	Beswick et al. (2006) (n=650) Mean	Beswick et al. (2006) (n=650) Std. Dev.
1. I find QR interesting.	18.2	2.64	1.19	3.50	1.08
2. Other learning goals are more important than QR.	30.4	2.85	1.06	3.13	0.97
3. I do as little QR as possible when I get the choice.	27.3	3.15	1.03	3.42	1.20
4. I enjoy QR lessons.	21.2	2.88	1.02	3.49	1.27
5. I find most QR problems fairly easy.	21.3	2.88	1.02	3.22	1.07
6. QR helps to develop my mind and teaches me to think.	42.5	3.33	0.85	3.89	1.02
7. QR we learn at school is important in everyday life.	63.7	3.76	0.75	4.13	1.06
8. QR makes me feel uneasy and nervous.	27.3	3.03	0.98	3.65	1.20
9. QR is dull and uninteresting.	39.4	2.81	1.31	3.56	1.25
10. I enjoy attempting to solve QR problems.	18.2	2.67	0.99	3.55	1.17
11. QR problems are nearly always too difficult.	18.2	3.01	0.79	3.59	1.02
12. I usually keep trying with a difficult problem until I have solved it.	24.3	2.97	0.98	3.66	1.09
13. I do well at QR.	21.3	2.85	1.03	3.44	1.21
14. Having good QR skills will help me get a job.	60.7	3.64	0.93	4.38	0.97
15. Most of the time I find QR problems too easy.	15.2	3.21	1.02	2.60	1.02
16. I sometimes get upset when trying to solve QR problems.	45.4	2.58	0.94	3.70	1.27

At least 60% of respondents agreed that QR was important in everyday life (Item 7) and helpful in getting a job (Item 14). More than 40% also Agreed or Strongly Agreed that QR develops the mind (Item 6) and just over 45% indicated that they sometimes became upset when trying to solve QR problems (Item 16). For every item the mean response of students in the QR project was lower than that of the participants in Beswick et al.'s study.

Table 2 shows the percentages of respondents indicating Agreement or Strong Agreement with each of the QR learning items along with their means and standard deviations. Nearly 80% Agreed or Strongly Agreed that that their teacher encouraged them to look for QR opportunities in all their work (Item 2), and two thirds Agreed or Strongly Agreed that they were encouraged to present QR in a variety of ways (Item 3) and helped them to identify errors in their work (Item 8). According to the students, the least common of the QR learning experience was being asked to write a QR report (Item 4).

Table 2. Means and standard deviations for responses to QR learning items

When I work on QR my teacher ...	% A or SA	Mean (n=33)	Std. Dev.
1. asks me to explain my mathematical understanding.	51.6	3.42	0.90
2. encourages me to look for QR opportunities in all my work.	78.8	4.06	0.79
3. encourages me to present QR in a variety of ways.	66.7	3.76	0.79
4. usually asks me to write a QR report to show my understandings.	27.3	2.94	0.83
5. stresses the importance of proportional reasoning such as understanding ratios and percents.	45.5	3.42	0.75
6. can identify appropriate opportunities to develop my QR.	45.5	3.48	0.76
7. shows where I can find appropriate resources to help me explore QR ideas.	42.4	3.45	0.75
8. helps me to identify QR errors in my work.	66.7	3.70	0.73
9. gives me confidence that the mathematics curriculum is being covered as well as in a traditional mathematics classroom.	45.4	3.52	0.83
10. helps me to understand maths that I missed in earlier grades.	48.5	3.42	0.83

Discussion

On a positive note, the percentage of students in this study who reported liking mathematics was greater than that for Year 8 students in the 2011 TIMSS study (Thomson et al., 2012).The attitudes towards QR of the Year 9–12 students in the school in which the QR project was conducted were more negative than the Year 5–8 students described by Beswick et al. (2006) and Watson et al. (2007). Beswick et al. noted a decline with year level in attitudes to mathematics of the students in their study; it could be that the more negative attitudes of the participants in this study are a reflection of further progression of this trend. Alternatively (or additionally) the characteristics of this particular cohort that included, for many, factors that motivated them to change schools when such a transition was not necessary, may also have contributed to greater disengagement with, and more negative attitudes to, mathematics (QR).

The principles that underpinned the way in which the school operated incorporated many of the elements identified in the literature as likely to influence students' engagement and attitudes positively. These included the involvement of parents and other significant adults in all aspects of each student's learning, the focus on meeting the needs of individual learners and designing curriculum content to match their interests, and the emphasis on building positive advisor–student and peer relationships through small groupings and individual attention. One would expect that the needs for autonomy, identity and social success highlighted by Sullivan et al. (2006) would have been met in this school environment to a greater degree than in most others.

Nevertheless, none of these students had been at the school for more than one complete year and so had experienced mathematics learning in traditional settings for from 9 to 12 years prior to coming to experiencing project-based learning. The results of this study are consistent with Archambault et al.'s (2012) conclusion that a single year of enthusiastic and encouraging teaching is unlikely to be sufficient to reverse disengagement that has resulted from years of lack of success—evidenced in this study by the fact that only about one fifth of students agreed that they did well at QR (see Table 1, Item 13).

The fact that none of the teachers in the school were mathematics specialists is a cause for concern as is the fact that many, perhaps most, of the teachers these students had encountered in mathematics classes in prior years were also likely to have lacked specialised expertise (Harris & Jensz, 2006). It must be noted though that these teachers were doing quite a bit right in terms of helping their students to learn QR (see Table 2). Given the extent of the students' negativity towards mathematics it is not clear that teachers with greater expertise would be more effective in turning their attitudes around.

The data also show that it is possible to convince students of the importance and usefulness of mathematics, both for everyday life and future careers, but that this is not necessarily in anyway related to other more emotive aspects of attitude to the subject, including their engagement with it. Students are able simultaneously to acknowledge the usefulness and importance of mathematics, and to avoid it.

Conclusion

The QR project was a small study and the number of students who participated in the survey was small. Nevertheless, the results suggest that for some students, arguably too many, years spent in mathematics classes have a toxic effect on their attitude to the subject. Furthermore, the impacts of this experience are difficult to reverse and are likely to take sustained experiences that counter those of the past and build learners' sense of themselves as competent mathematics learners and users. Attention to students' attitudes, and the ways in which typical mathematics teaching contributes to them, from the earliest years is warranted.

Efforts aimed at promoting the many applications of mathematics in current society and its crucial role in many jobs (as detailed, for example, by Deloitte (UK) for the Engineering and Physical Science Research Council, 2012) seem unlikely to be effective in addressing the decline in enrolments in advanced senior secondary mathematics and mathematically based university courses.

This paper presents quite a depressing picture of disengagement with mathematics and the vastness of the task of reversing the problem once it occurs. Its value is in drawing attention to the extent of the problem for some students and the need to rethink mathematics teaching in quite fundamental ways. The literature suggests that the practices employed in this school and the principles that underpin them are worth pursuing, but patience will be needed in drawing conclusions about their effectiveness in practice in relation to students' attitudes to mathematics.

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"IS THIS A FIVE MINUTE ARGUMENT OR THE FULL HALF HOUR?"[2] ENRICHING CLASSROOMS WITH A CULTURE OF REASONING

HELEN CHICK
University of Tasmania
helen.chick@utas.edu.au

Reasoning is one of the proficiency strands highlighted in the *Australian Curriculum: Mathematics*. This is because it is the heart of mathematical work. However, because the content strands receive the bulk of the explicit description in the document, it can be difficult to know how to ensure that reasoning plays a key role in the classroom. This paper presents strategies to enhance the nature and extent of reasoning activity taking place in classrooms, and discusses how to build a culture of justification, use questioning techniques to encourage reasoning, and develop tasks that require higher order thinking.

Introduction

An anecdote from the distant past

My mind is taken back to 1978. I am in Year 9, in a maths classroom in a standard 1950s-built Tasmanian high school. It is the advanced maths class, where we get extra maths as an elective, on top of regular maths classes. There are about 28 students in the class, only four of them girls (typical for the time), and we are being taught by Mr Smith who is the senior master for maths. We have been doing Pythagoras' theorem, and, in this time where calculators are only just being introduced into schools, all of the answers to the problems are beautiful whole numbers. Thus we meet the famous (3,4,5) triangle, and the (6,8,10) triangle (and can see the connection between the two, although we explore that no further); and we also meet some of the other Pythagorean triples like (5,12,13), (7,24,25), and (9,40,41).

At this point, someone notices something about the numbers in our triples: the first is odd, and the final pair are consecutive. Mr Smith encourages us to test this and we find (11,60,61), giving our newfangled calculators a workout in the process. Someone else notices that in the sequence 12, 24, 40, 60—formed by the second numbers in our sequence of triples—the difference between each pair of consecutive numbers goes up by 4 as 12, 16, 20. This leads us to conjecture that (13,84,85) is another of these special triples. We reach again for our calculators (a thrill to use but it still feels like cheating), and the new example is confirmed.

2 From *Monty Python's Flying Circus*, episode 29.

We try to articulate what we have discovered, and thus "9CHAM's Theorem[3]" is born. It asserts that for every odd number there is a Pythagorean triple—not that we know this name at this time—where the remaining two numbers are consecutive. Mr Smith is as excited about this as we are; it is new to him as well. There is a small amount of frustration, however, because although we are confident about the result and can see the patterns, we cannot quite articulate it mathematically. I think, too, that a few of us recognise that, in fact, we cannot be absolutely sure that this result always holds, partly because we have not been able to nail it down properly, and partly because we think it needs a proof like the ones we had been doing in geometry with a QED thingy at the end.

Commentary from the present

Well, my memory is hazy, and I confess I cannot be sure that the events above occurred in exactly the way I have described, but most of it is roughly accurate, and I *do* remember being excited during the discovery of 9CHAM's Theorem. There was a buzz in the classroom as we spotted patterns, and conjectured, and tested, and tried to explain what was happening. I do not know if *everyone* in the class felt it, or followed the arguments or all of the discussion, but it seemed as if they did. Mr Smith allowed us to conduct the investigation—indeed, he investigated with us (no, I do not think he just pretended for our sakes), and he encouraged the conjecturing, testing, and articulating that led us to the final result. Although this was the most striking example, there was generally always a positive culture of discussion and reasoning in that classroom.

Two years later my Year 11 teacher showed me the proof of 9CHAM's Theorem. Yes, it really is true (it is also relatively well-known, though not to my Year 9 class and Mr Smith). As it happens, there is even more to it. It was nice to have the result properly articulated and proven, but I do recall a vague disappointment that I had not been able to prove it for myself. (To be fair, I do not think I had really tried to prove the result for myself in the intervening years.) Partly this was because the Year 11 teacher's proof was actually of an even more general result, and I could not fully link it to the specific cases that we had encountered in 9CHAM's Theorem, which made my Year 11 teacher's proof not exactly the one I thought I was looking for. In hindsight, though, I am wondering how much I *could* have done, with the right guidance. I am not sure I would have been able to determine the necessary algebraic formulation of the patterns, but I *do* think I could have completed, for myself, the algebraic manipulation that proves the result … and I also think I could have been scaffolded to find the algebraic formulation of the 9CHAM's Theorem special cases. These experiences raise important questions about what reasoning is possible in classrooms and how to get it to happen.

Reasoning and the curriculum

It is now 2013. I am no longer a high school student; my job now is to prepare prospective teachers to teach high school mathematics. What should they be teaching? What should be going on in *their* Year 9 classrooms?

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3 CHAM = Clarence High Advanced Mathematics, and, yes, I know now that at the time it was only a conjecture … but you should tell that to everyone who ever talked about Fermat's Last "Theorem" before 1994.

The last 35 years have seen the demise of geometric proof from the curriculum, which has been bemoaned by those who see it as epitomizing a central tenet of the discipline of mathematics: namely deductive reasoning. On the other hand, for others the loss is unlamented, since many students struggled with it, it appears to have little value to real world applications, and its removal allowed space in the curriculum to emphasise other aspects of mathematical activity and content. Whatever your view of the place of geometric proof in the curriculum, however, I do not believe that the reduced emphasis on it was *intended* to imply a reduction in reasoning in the curriculum. Nevertheless, this may be what happened. Stacey (2003), in examining the results of the TIMSS 1999 video study, discussed evidence for a “shallow teaching syndrome” in Australian Year 8 mathematics classrooms. She pointed out the excessive use of repetition in the sequences of problems assigned to students, the use of problems of low complexity, and the absence of mathematical reasoning in the activities of the classroom, and suggested that these may play a role in the differences in outcomes from international testing between Australian students and their international counterparts.

Reasoning is certainly specified in curriculum documents as being an essential component of mathematics education. Like various curricula before it (e.g., Victorian Curriculum and Assessment Authority, 2008, with its “Working Mathematically” strand), the recently released *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment and Reporting Authority, 2013) places emphasis on reasoning. It is one of four proficiency strands, together with fluency, understanding, and problem solving. In the curriculum document, these four strands are treated differently from the content strands number and algebra, measurement and geometry, and statistics and probability. In some respects this is not surprising, since content and proficiency are quite different from each other as aspects of mathematical knowledge. However, if teachers are looking to the curriculum document as a guide for what they are to teach, then there is potential for them to underestimate the importance of reasoning. Although I am not claiming that volume of text in a curriculum document is indicative of the weight that is intended to be given to a curriculum strand, it is rather telling that the vast majority of the *Australian Curriculum: Mathematics* is taken up with the content strands. In part this is because the content strands have tended to be encyclopaedic, spelling out all the topics to be covered. In contrast, the potential pervasiveness of reasoning within mathematics means that the curriculum indicates the scope of reasoning only by giving a few examples of where it might arise, rather than trying to specify all the places where it could be incorporated. This pervasiveness—and the limited collection of supplied examples—puts the onus on the teacher to identify places where reasoning can be highlighted. Furthermore, the fact that some aspects of mathematics can be (and sometimes are) “taught” in classrooms in the absence or limited presence of reasoning means that it is might be useful to look at strategies that can build a culture of reasoning in the classroom.

Strategies for fostering reasoning

My suggestions in what follows have been inspired by a number of sources, including the work of Watson and Mason on questions and prompts for the mathematics classrooms (1998) and constructing examples (2005); the work of Stein, Grover, and Henningsen on task selection and implementation (1996; and also Henningsen & Stein, 1997); the list of habits of mind posited by Cuoco, Goldenberg, and Mark (1996); and Swan's work on strategies for improving learning (2005); together with other long-forgotten sources over the years, including teachers and mentors. The suggestions are intended to provide strategies, approaches and ideas for how to use tasks and questions to foster a stronger culture of reasoning in the classroom.

Tasks

Optimising routine tasks

Routine 'exercises' are an important part of mathematical learning, not least because they build fluency and understanding. You can maximise their effectiveness for reasoning by discussing why the steps follow in sequence, and how starting off with the conditions in the problem statement leads to the conclusion or answer in a logical way based on mathematical properties.

Selecting tasks that foster reasoning

Look for tasks that require multiple steps to solve, allow multiple entry points, provide opportunities to make links to other results, can be extended (see further comments later), and require or allow the opportunity for conjecturing. Do not underestimate your students' capacity to engage in "hard" problems, in a supportive environment.

Maximise the reasoning afforded by a task

Sometimes it can be difficult to identify and develop all the reasoning opportunities inherent in a particular task. A classic example is the use of pattern-spotting activities (e.g., how many matches are there in the tenth diagram in the sequence |_|, |_|_|, |_|_|_|, |_|_|_|_|, ...?). Many students will be able to identify the pattern, but more work and reasoning is required to describe the pattern using a mathematical formulation, and to justify the validity of the formulation. This principle means that it is important for the teacher to have tried the task, and to have thought in advance about what learning opportunities it offers and how to bring these out in the classroom.

Adapting tasks

Look for ways to open up a task to make it amenable to greater opportunities for reasoning. Routine exercises often can be turned into richer tasks, with a little thought. Try 'reversing' the task, by taking the answer and posing new, related tasks with that answer (e.g., What other numbers only have 3 factors? Can you find other equations of lines that pass through (2, 3)? What other parallelograms could you construct if (1, 2) and (3, 7) are vertices on one side?). Try asking "what if?" questions (e.g., If that angle changed from 30° to 60° what would happen to the remaining angles in the triangle? What would happen to the volume if the dimensions of a prism were doubled?)

Reasoning in real-world tasks

Steen (1999) points out that formal deductive reasoning is needed only rarely for real world problems; but of course this is not the only type of reasoning that can and should be fostered in classrooms. Working with real world problems—with their ambiguity and the need for modelling and approximation—provides opportunities for reasoning while devising models that ensure a good match between the model and the real-world situation, examining the impact of variations in the model's assumptions, and checking the plausibility of numerical solutions.

Questions

There are many kinds of teacher and student questions that can foster reasoning. The distinction between a task and a question is sometimes unclear, and so there is some overlap among the categories of questions suggested in what follows and with some of the principles suggested earlier for developing and using tasks.

- Questions that ask for justification (e.g., Why can you perform that step? Can you explain why the mean changes by so much when you remove that outlier?)
- Questions that allow reflection on or development of the general principles that are evident in a specific problem. Asking for "another example" is a powerful way of doing this (see Watson & Mason, 2005) (e.g., Can you give me another example of a line parallel to $y = 3x$? Can you find an example of a number which is larger than its square? And another? And another?)
- Questions that encourage students to conjecture (e.g., Can you describe the pattern? What would happen if the coefficient of x^2 changed?)
- Questions that encourage students to test and question assumptions (e.g., Can you find an example that *doesn't* work? What if it *wasn't* a right-angled triangle?)
- Questions that encourage students to explore and experiment (e.g., How rare is it to get 5 heads (or tails) in a row when tossing a coin ten times? Do all quadrilaterals tessellate?). Comments from the discussion on pattern-spotting are relevant here: such experiments are usually only part of the story, and generally they should be a precursor to a more rigorous and reasoned analysis.

Classroom culture

It can take time to develop the culture of the classroom so that reasoning is encouraged and valued. Teachers will need to model reasoning and to ask questions requiring it. The classroom needs to be a supportive environment in which students can make conjectures and put forward their reasoning without risking their self-worth. At the same time, it is important for students to know that the ideas are testable and that questioning, validating, discussing, justifying, and disproving are all essential parts of the reasoning process.

Teachers can also encourage students to look for generalisations and to encourage a sense of wonder. This can be habituated by encouraging students to reflect on the tasks they have completed and ask themselves, "What happens if...?" or "Is there a sensible 'next question' I could ask about this situation?" A teacher's own enthusiasm for reasoning processes can also foster an environment in which reasoning is viewed as engaging and stimulating work, if not actually enjoyed and valued.

Other issues

The role of content knowledge and pedagogical content knowledge

Although these strategies may help to foster a culture of reasoning in the classroom, key roles are played by content knowledge and pedagogical content knowledge. A teacher needs mathematical knowledge to determine the potential of a task and the possible directions it might lead. Moreover, the choice of task alone is not enough. Stacey (2003) in reviewing the work of Stein and Henningsen and colleagues, points out that "although good tasks might seem to be the causal mechanism, the teacher influences the choice, timing, and detail of their implementation in classrooms. A lesson may be more or less successful in sustaining high level thinking, depending on the actions and pedagogical decisions of the teacher within the classroom" (p. 121). Sullivan, Clarke, and Clarke (2009) also discuss the challenges of turning a task into classroom activity that provides genuine learning opportunities. There are many mathematical and pedagogical decisions that a teacher needs to make in advance *and* in the hurly-burly of actual teaching in order to maximize reasoning opportunities. One particular challenge for the teacher is to interpret, validate, and counter students' reasoning as required; and also to engage in his or her own mathematical reasoning during the lesson, by constructing examples and counter-examples that may further develop learning.

Classroom management

As suggested earlier, building a culture of reasoning in the classroom requires that the teacher build an atmosphere of respect for students and for the idea of determining the usefulness or validity of ideas. This requires management of the discourse that take place, so that it is respectful of others, and opinions are backed by reasons or justification. Teachers should ensure that thinking time is allowed, and that students are given an opportunity to contribute regardless of the speed at which their idea was determined (it may be worth having a frank discussion with students about the merits and fairness of the strategies you use to select students to give their responses). Monitor students as they work and ask for their conjectures and reasons. This monitoring can also allow teachers to be strategic in choosing contributions from students. Use mini whiteboards as a scratchpad for working, or a place to record conjectures, or to vote for a conjecture (by having students "vote" they are forced to commit to one hypothesis over another; however, students must also learn that "popular opinion" is not actually an appropriate reasoning strategy).

An example

The following set of questions provides an example of a series of tasks and questions that will provide students with the opportunity to explore and reason, in this case with geometry. The sequence might be conducted with any grade level from 5 to 12. Naturally, you might expect different levels of rigour and/or formality in the reasoning from different groups of students, but, regardless of the level, the tasks allow scope for rich discussions, exploration, conjecturing, testing, and justifying.

The sequence was inspired from the single question "How many right angles can a pentagon have?" from Watson and Mason (1998).

Right angles in polygons

- How many right angles can a quadrilateral have? [Explore this. Have you covered all possibilities?]
- Can you have a quadrilateral with exactly two right angles, in which the angles are adjacent to each other? Why?
- Can you have a quadrilateral with exactly two right angles, in which the angles are opposite each other? Why?
- Can you have a quadrilateral with exactly three right angles? Why?

- Can you have a quadrilateral with one right angle and an obtuse internal angle?
- Can you have a quadrilateral with one right angle adjacent to an obtuse internal angle?
- How many right angles can a pentagon have? [Have you covered all possibilities? Did you only think of the regular pentagon at first?]
- Can you have a pentagon with exactly four right angles? Why?
- How many right angles can a hexagon have? [Have you covered all possibilities? Did you only think of convex hexagons at first?]
- Is it possible to get four right angles as the internal angles of a hexagon? Why?
- Is it possible to get more than four right angles as the internal angles of a hexagon? Why?
- Why does the hexagon *have* to be convex if you have four or more right angles?
- What's the next question that we could consider?

Conclusions

Building a culture of reasoning is not easy. It demands significant content and pedagogical content knowledge as well as classroom management skills. Yet there are rewards. Research suggests that student outcomes are better, which is already a favourable result; but the anecdotal evidence also suggests reason-filled classrooms have a vibrant atmosphere with engaged students in which the discipline of mathematics is not only explored and learned, but is actively experienced by students who are actually working mathematically.

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CAN PROFESSIONAL LEARNING IMPROVE TEACHERS' CONFIDENCE, ATTITUDES AND BELIEFS ABOUT ALGEBRA?

LORRAINE DAY
University of Notre Dame Australia
lorraine.day@nd.edu.au

DEREK HURRELL
University of Notre Dame Australia
derek.hurrell@nd.edu.au

With the increased magnification of Algebra in the Australian Curriculum, many teachers, especially those in the early, primary and middle years cohorts, feel inadequate with the content to be taught. A professional learning (PL) model to address this was designed to enhance teachers' content and pedagogical knowledge, provide sufficient time and resources, address identified needs while modelling high quality teaching. The research component of the program aimed to discover whether the teachers' confidence, attitudes and beliefs about Algebra improved as a result of the experience and if so, whether similarly structured PL might be employed for a more generalised population.

Introduction

Norton and Windsor (2008) state that confidence and competence in Algebra is an important filter for more advanced courses in mathematical thinking and problem solving. Edwards (2000) calls Algebra a 'gatekeeper' to other academic fields and that participation in it increases vocational possibilities. There does not seem to be contradictory literature in relation to the importance of Algebra, and yet prior to the introduction of the nationwide *Australian Curriculum: Mathematics* (ACM) (ACARA, 2011), Algebra was a strand of mathematics which did not enjoy prominence in many Western Australian primary schools.

In the previously mandated Western Australian Curriculum Framework (Curriculum Council, 1998), Algebra was separated from the Number strand and placed under a heading of Pre-Algebra and Algebra. Anecdotal comments from teachers suggested that this gave them licence to avoid Algebra as being something in the domain of upper primary or lower secondary school and rather to concentrate on the arithmetic aspects of the Number strand. This avoidance occurred despite researchers (Chick & Harris, 2007; Kieran, 2006) stressing the importance of getting primary school students to analyse relationships, generalise, predict and notice structure, all of which are foundational for the conventional algebra encountered later.

With the introduction of the Australian Curriculum: Mathematics the topic of Algebra has received some greater prominence with the placing it alongside the Number strand to highlight the very strong links between the two. The re-emergence of prominence has prompted some teachers to re-evaluate their capacity to teach this subject. This is problematic as most primary teachers have little expertise in Algebra (Chick & Harris, 2007), and little confidence (Norton & Windsor, 2008). It is also generally recognised that the traditional approaches to teaching algebra have met with limited success (Norton & Irvin, 2007). Yet, as Norton and Irvin report, there are solutions to the issue of limited success. These are:

- Making explicit algebraic thinking inherent in arithmetic in children's earlier learning;
- Explicit teaching of nuances and processes of algebra in an algebraic and symbolic setting, especially in transformational activities;
- Using multiple representations including the use of technology; and
- Recognising the importance of embedding algebra into contextual themes (p. 552).

Further, with the continued growth of Information and Communication Technologies (ICT), some of the tedious arithmetic calculations can be avoided. Ploger, Klinger and Rooney (1997) assert that when technology is used as a tool to help prepare students for Algebra, they can generate the effects of algebraic transformations on familiar numbers. The technology relieves them of the boredom of the crunching of numbers and leaves them with space to analyse the effects.

This literature, and further literature regarding Mathematics Knowledge for Teaching (MKT) (Ball, Thames & Phelps, 2008; Hill, Schilling & Ball, 2004), Pedagogical Content Knowledge (PCK) (Ball, Thames & Phelps, 2008; Park & Oliver, 2008; Shulman, 1986) and effective Professional Learning (PL) (Clarke, 2003; Cohen & Hill, 2000; Supovitz, Mayer & Kahle, 2000; Supovitz & Turner, 2000) was used to underpin a series of PL sessions and a restricted analysis of the benefits of that PL.

The aim of this project was to investigate whether PL could improve teachers' confidence, attitudes and beliefs for the ACM sub-strand of Algebra. Specifically, PL designed around subject matter knowledge and pedagogical knowledge was investigated.

Design of the professional learning program

The participants in the PL program ranged from early childhood teachers to secondary teachers of mathematics. Some were the mathematics co-ordinators for their schools and others were full-time classroom teachers. All participants felt they needed more grounding in algebraic content and how to teach it in such a way that students would not find it daunting. The PL described in this paper was carried out over a full year, with five half-day contact sessions and support by the PL facilitators available via email in between times. There was also an opportunity for the participants to attend one or two days with another expert presenter who tailored his program to complement what was being covered in the PL.

In designing the PL, several big ideas were considered. Firstly, what were considered to be the 'big ideas' of primary algebra were identified as Pattern, Equivalence and Function. Secondly, the structure to be used throughout the PL was decided upon. Drawing upon the research of Clarke (2003), Guskey (2003) and InPraxis (2006) it was decided that the model would aim to enhance both content and pedagogical knowledge within a collaborative and professional learning community where inquiry was encouraged. Sufficient time and resources would be provided to the teachers who could design lessons and trial within their classrooms the ideas gleaned, reflect on that process and report back to the group. It was also believed that it was imperative that high quality teaching practice was modelled by the PL leaders at all times, so that the teachers could see what the lessons could actually 'look like' in their classrooms.

As there was a perceived lack of confidence in teaching Algebra, it was viewed as important to cover the content knowledge in the least threatening manner possible. The idea agreed upon was to set up a professional learning community where the participants worked through the learning activities as their students would. In this way the content was presented from an early childhood perspective through the primary years and to the middle years, so the developmental aspects were made clear. The pedagogical approach in each case was one which was appropriate for students, following a working mathematically approach, embedding the ACM Proficiency Strands, and working from the concrete through the representational to the abstract.

Data collection

The participants completed pre- and post-program questionnaires about their confidence and beliefs about Algebra. The teachers' confidence questionnaire was based on five point Likert-type scale questionnaires designed by Hackling and Prain (2005) and Riggs and Enoch (1990). The respondents were asked to indicate their confidence concerning nine statements regarding the teaching of Algebra. The teachers' beliefs questionnaire was adapted from a five point Likert-type questionnaire constructed by White, Way, Perry and Southwell (2006). The teachers were asked to indicate their beliefs about ten statements regarding their effectiveness as teachers of Algebra. The statements were concerned with constructivist beliefs about mathematics and beliefs about the meaningfulness of mathematics and, in particular, Algebra.

All participants in the PL program were invited to volunteer to be interviewed approximately four months after the conclusion of the program. The three case studies described below self selected for the semi-structured interviews, which were recorded and transcribed by the researchers.

Alice is a junior primary teacher who teaches a large class of Year Two students, without the support of an education assistant. She came into the PL program with very little confidence in her ability to incorporate algebraic reasoning in her mathematics program. As a participant in the program, Alice was enthusiastic, keen to trial the new ideas in her classroom and very willing to share her classroom experiences with all of the other participants in the group.

Beth works in a primary setting as a numeracy support teacher, predominantly supporting students in Years Four to Six. Initially her role involved withdrawing children to work on modified programs and recently it has been expanded to include supporting small groups of students within their lessons. This year Beth has begun mentoring other teachers by modelling mathematics lessons and demonstrating how she would teach lessons in a concrete manner and then develop the 'big ideas' and concepts within the lesson. Beth engaged with the PL program, enjoyed the readings provided as a research basis, and demonstrated a willingness to trial new ideas in her school and shared the results with other participants.

Claire is a secondary teacher of mathematics who does most of her teaching in the senior secondary WACE Stage 3 courses. She also has a middle school class of Year Nine students. Claire's contributions were more considered than the other two teachers, although it was obvious from her comments that she had trialled with her

Year 9 students many of the ideas presented. She tended to share her experiences with small groups of participants rather than the group as a whole, although she occasionally relayed a story to all about a particularly successful lesson or experience.

Results and discussion

The aim of this project was to investigate whether PL could improve teachers' confidence, attitudes and beliefs for the ACM sub-strand of Algebra. Specifically, PL designed around subject matter knowledge and pedagogical knowledge was investigated. The significance of this small study lay in the increased magnification of the topic of algebra in the primary school classroom within the ACM which has prompted some teachers to re-evaluate their capacity to successfully teach this important sub-strand. The small sample size meant that the aim of this study was to raise the issues as possibilities for further research rather than to provide any definitive answers.

In this section an attempt has been made to synthesise the responses provided by three of the participants in the PL program to questions asked in the pre- and post questionnaires and in a semi-structured interview. The questions posed were to gain further insight into whether the aims of the PL were achieved.

All three respondents indicated that they felt that the mixture of well-considered and well-delivered subject content, and sound pedagogical practices to deliver that content, contributed greatly to their increased levels of confidence by the end of the PL experience. This may be a recognition of the value of PCK (Shulman, 1986) in the craft of teaching.

Both Alice and Beth indicated that their confidence in developing the literacy skills needed for learning Algebra had improved and all three respondents believe their confidence in managing discussions and interpretations of algebra had increased during the PL program. In fact all three respondents alluded to confidence being gained through having the 'correct' language to articulate their general understandings of Algebra, which may be indicative of the power of having appropriate discourse.

All three respondents claimed that their content knowledge in algebra had improved since the start of the PL. Alice indicated that her perception of her own content knowledge jumped markedly when she realised what content was required to teach primary school algebra. She indicated that much she was exposed to in the PL affirmed the knowledge she already had regarding the content and the pedagogy but did strengthen both of these elements. Perhaps the biggest surprise was from Claire who was of all the respondents the person whose role was the most subject specific regarding mathematics, when she indicated that her content knowledge had improved. The growth perceived by Claire, was in combining quality pedagogy to her content knowledge (perhaps more indicative of improved pedagogical practices) and an increased understanding of what and how to teach students who were encountering difficulties.

Claire's increased understanding of how to assist students who were encountering difficulties was facilitated by attendance at the PL with teachers of students from Foundation classes to Year 12. All three of the teachers indicated that having the range of year levels represented meant that they were able to discuss how the concepts, skills and knowledge required for Algebra were developed across the range of schooling. Alice indicated that she gained greatly from knowing what happened with the foundations she built, Beth in seeing what teaching and learning had occurred before the students reached her, and Claire, the types of experiences that the students might have encountered prior to reaching secondary school. They all spoke of the richness that this heterogeneous group afforded in terms of ideas, strategies, pedagogy and even responsibility.

Pedagogical issues were very prominent in the conversations with all three respondents. Each indicated that their confidence in managing hands-on group activities had improved during the PL as had their confidence in engaging their students' interest in Algebra. Alice and Claire indicated that, as a result of a change in their pedagogical practices, they purchased manipulative materials to support their teaching and learning of Algebra. Beth did not indicate the purchase of new materials but did signify that she had developed a greater understanding of what materials could be used to support the learning. The selection of suitable manipulative materials showed a developing understanding of the nexus between pedagogical practices and a deepening of content knowledge.

All three respondents indicated that at the conclusion of the PL that they believed they were more effective at finding better ways to teach algebra. They felt that they now had a much better idea of the steps that were necessary to teach algebraic concepts and to monitor algebraic investigations. Both Alice and Beth mentioned that they would be adopting a concrete-representational-abstract teaching and learning structure to their lessons as a result of this PL experience. All three teachers indicated an increased confidence in using a constructivist model to plan mathematics units of work during the course of the PL.

The ACM supports the learning of Algebra in providing content descriptions for all students from Foundation to Year 10. This is a mandated document for all Australian schools, although at the time of writing this article the transition to this document from the Western Australian Curriculum Framework (Curriculum Council, 1998) was not complete in all schools. What was not supported in any of the conversations was a deep understanding of this document. Alice's lack of prior understanding with regards to the knowledge she already possessed in relation to primary school algebra; Beth's perception that algebra was different from other mathematical strands and in particular not realising the direct links with the Number strand plus her inability to articulate a whole school scope and sequence; and Claire's contention that she found the references to the ACM most illuminating; prompts the contention that the ACM is not as yet properly utilised or understood.

In articulating PCK in his seminal work, Shulman (1986) stated that there were three domains constituting PCK; content, pedagogy and context. Two of the respondents, Beth and Claire, talked about the importance of contextualising the algebra to facilitate learning. Claire in particular decided that this was to be a focus of her teaching.

Conclusion

What became apparent from a synthesis of both questionnaires and the semi-structured interview was that the objectives of this PL were perceived to have been met by the case study respondents. These respondents reported that their content knowledge and pedagogical knowledge both increased, as did their positive disposition towards Algebra, as illustrated through their perceptions of their beliefs and attitudes. These results suggest that a larger study could be initiated to see if PL designed around subject matter knowledge and pedagogical knowledge in Algebra could improve the content and pedagogical knowledge of teachers and therefore perhaps their pedagogical content knowledge.

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MEASUREMENT OF STEEPNESS: GRASPING THE SLIPPERY SLOPE

BRUCE DUNCAN

University of Tasmania

bruce.duncan@utas.edu.au

The measurement of steepness draws content from both the number and algebra strand, and the measurement and geometry strand of *The Australian Curriculum: Mathematics*. When students are asked to use their own methods to measure the steepness of the ground, the task also incorporates understanding, reasoning, fluency, and problem solving. Activities involving the measurement of steepness provide valuable opportunities for students in middle school mathematics classes to learn about slope as a ratio. This concept is important for both further studies in algebra and understanding measurements used in the building and transport industries.

Introduction

Which of these paths is steeper? Or, which of these paths looks steeper?

Figure 1. Two photographs of paths.

The mathematical knowledge that is needed to make accurate measurements of steepness is covered in high school mathematics classrooms but the question remains; is covering that knowledge in school enough to provide students with the means to perform that task once they leave school? How often, as teachers, do we assume that, if a student can carry out a mathematical procedure in the classroom they should also be able to carry out that procedure outside the classroom? There is a gap between the understanding that students need to complete tasks on paper and the understanding they need to solve problems in the real world, outside the classroom. The need to address this gap has led to the inclusion of the proficiency strands in the *Australian Curriculum: Mathematics*. This presentation describes findings from research that has focussed on the measurement of steepness and how it can be used in a learning sequence to help students consolidate their understanding of the key concepts, and their proficiency in working with this measurement.

The *Australian Curriculum: Mathematics* is structured to develop the links between interrelated and interdependent concepts and systems. As children progress through the years of schooling, they are expected to build on their prior understanding, develop flexible understanding of concepts, and be able to apply those concepts “beyond the mathematics classroom” (ACARA, 2013, p. 1). Research into the measurement of steepness (Stump 2001, Lobato & Thanheiser, 2002) shows that it requires not only understanding of a number of concepts, but also the ability to select and apply them appropriately. This understanding is described here, along with findings from a recent study by the author in which pre-service teachers were given the task of measuring the steepness of a number of paths, including the one in Figure 1 (yes, they are both pictures of the same path). These findings have implications for teachers and their provision of learning activities in mathematics classrooms.

Measurement

High school students are expected to understand how to perform common measurements accurately. According to the *Australian Curriculum: Mathematics*, by the time they reach Year 7, students are expected to solve authentic problems involving measurement, use formal metric units for length, mass, and capacity, and understand the relationship between volume and capacity (ACARA, 2013). Alongside the specific aspects of each method of measurement, students are expected to develop understanding of the general concepts of measurement that apply to all methods.

The learning of measurement concepts has been described by Wilson and Rowland (1993) as occurring in five stages that begin with *isolating the attribute,* or understanding what is to be measured. Stage 2 involves the ability to *make comparisons* between different instances of the attribute. Comparisons are judgements made without units or formal methods but they still require the application of mathematical understanding in order to be accurate and appropriate. Direct comparisons, such as comparing peoples’ height by standing back to back, are more easily made than comparisons that can’t be made directly; is my son taller than I was at his age? Usually, indirect comparison requires the introduction of a unit, but units can be informal or formal. Stage 3 is the use of *informal units* (the classroom is six paces wide), and in stage 4, students use *formal units* correctly. Stage 5 involves the use of *formulae to aid measurement*. Wilson and Rowland note that an emphasis on formal units and formulae that comes before the earlier concepts have been consolidated, may result in a lack of understanding of the attribute and an over dependence on procedures for solving problems.

Steepness

Steepness can be thought of as a measurement of the ‘slantiness’ of a line. That line may be a two-dimensional representation on paper or a computer screen, or it may exist between two points in three-dimensional space, such as a ramp, a roof, or a hill. In the study referred to here, steepness was explained to participants as "the extent to which the ground varies from being flat." This description implies a relationship to the horizontal, but this relationship can be perceived, and represented, in different ways (Figure 2). Two formal methods for measuring steepness are introduced to school students, the angle method and the ratio method.

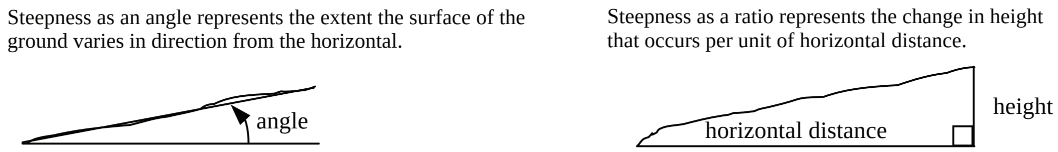

Figure 2. Two methods for measuring steepness.

Measurement with angles begins in Year 2 and it has been observed that most students by the age of 9 years are able to conceptualise slope as an angle (Mitchelmore and White, 2000). Angle measures the amount of turn between two lines. When reduced to a 2-D representation, the steepness of a hill can be perceived as the amount of turn between the surface of the ground and the horizontal. Angle can be measured using a number of units but the most common method used throughout primary and high school mathematics is degrees. Initially, children are taught to recognise angle as a fraction of a turn and then to label a quarter turn as a right angle. By Year 4, students are comparing angles with right angles and, in the following year, using the formal method to measure angles in degrees and construct them with a protractor (ACARA, 2013).

The measurement of steepness as a ratio is more complex than the angle method but it is also more widely used in contexts outside the high school mathematics classroom. The ratio method involves the perception of steepness as a change in height per unit of horizontal distance and is commonly expressed as "rise over run." Ratio, as an example of proportional reasoning, requires multiplicative thinking (Siemon, 2006) and becomes a feature of the curriculum in Year 7. Steepness is synonymous with slope or gradient of a line, a foundational concept in algebra that is first mentioned in Year 9. Steepness as a ratio is also the method used most commonly in the building and transport industries (see Figure 3). The ability to perceive steepness as a ratio, therefore, is not only of benefit to students when they meet the concept of slope in algebra but also if they enter the building or transport industries after leaving school and use the measurement of steepness as a ratio to make judgements in their work.

The unit notations that are used with ratios are diverse and problematic for middle school students. In fact, the concept of ratio as a measure is considered to be a special case that warrants explicit attention in the classroom (Simon & Blume, 1994). A ratio describes the relationship between two values with the same units. Steepness is the relationship between height and horizontal distance, which are both lengths. In this case, the notation does not require a unit to be stated. The relationship between rise and run is a proportion that is constant, no matter what unit is used to measure each length, as long as the same unit is used. As a proportion, ratio can be written as a fraction, a decimal, or a percentage. Other notations are also common, such as a colon, and the words *to*, *in*, *per*, and *for every*. Notations that can be expressed in a number of different ways can be confusing for students and adults alike.

Figure 3. The ratio method is used in the transport industry.

Mathematisation of steepness

The term 'mathematisation' or mathematical modelling (Verschaffel, 2002), is used here to describe the process by which a person selects and applies mathematical concepts and procedures appropriately to solve a problem. The concepts and procedures outlined above represent some of the understanding that needs to be drawn on when students mathematise steepness. It requires the appropriate assessment of the problem to determine what mathematical concepts and procedures should be used, and how they should be used. This process is not required of students in mathematics classrooms who are told what procedure to use to solve a number of similar problems. In this scenario, learning involves the familiarisation of these procedures but this is widely recognised as being inadequate as the connections between related concepts are not made explicit (McIntosh 2002). The Australian Curriculum: Mathematics encompasses mathematisation in the *proficiency strands*. The emphasis is on students making connections between concepts, choosing appropriate procedures, and transferring what they know to unfamiliar situations (ACARA, 2013).

The inclusion of mathematisation into school curricula is not a new idea. The National Council of Teachers of Mathematics produced a series of reform documents beginning in 1985 that emphasised that students should not only "know" mathematics but should also be able to "do" mathematics. Simon and Blume (1994) studied pre-service teachers' attempts to mathematise steepness and identified a number of obstacles to the process. In a teaching experiment designed to foster views of learning and teaching in mathematics that were consistent with the recent reforms, learning activities were designed around a social constructivist model. Pre-service teachers were given the problem of devising a system for quantifying the steepness of a series of artificial ski-slopes, and required to solve this problem through group discussion. They were told that each slope had a known height, length of base, and width of base, and asked to explain how they would determine the relative steepness using this information. The mathematisation that was required involved the selection of the appropriate quantities (height and length of base) and to combine these quantities in an appropriate ratio (rise over run). Simon and Blume concluded that an understanding of ratio-as-measure (such as steepness) involves both an understanding of proportional reasoning and an understanding of mathematisation.

The research study

A study by the author examined pre-service teachers' attempts to compare and quantify the steepness of four sloping paths. Their capacity to accurately measure the steepness of these paths was assessed against the stages of learning measurement described by Wilson and Rowland (1993). The activity took place in a convenient outside location. Participants were each given a clipboard with the response sheet attached and a pen. They did not have access to any measuring equipment. Participants were first asked to quantify the steepness of four slopes represented as two-dimensional diagrams on paper (see Figure 4). They were told to choose any method they felt comfortable using, formal or informal. Next they were asked to walk around a circuit of paths and, at four locations (including the path in Figure 1), to assess and represent the steepness. The first representation they were to provide was a drawn line against a horizontal reference; the second was a number. A final question tested participants' ability to simply compare steepness by asking them to order the paths from flattest to steepest. The slopes of the paths differed by approximately 5°.

Figure 4. Participants were asked to quantify the representations of slopes using a method of their own choosing.

A striking difference was observed between participants' ability to accurately compare the steepness of the four paths and their ability to accurately represent the steepness as a diagram or as a number. All participants were able to make more than half the comparisons accurately, and most participants (84%) were able to successfully order all four paths according to steepness. From this result, it can be concluded that all participants understood what was meant by "steepness" and that most were able to perceive differences of steepness of 5°. In comparison, only 12% of participants were able to represent and quantify steepness of each path to within 5°. The understanding that exists between a person's capacity to judge a difference and their capacity to quantify that difference is the aim of learning in measurement.

The series of tasks described above required progressively more mathematisation, from the quantification of a diagram, to the drawn line representation, and then, the quantification of the actual paths. It was observed that participants' error increased with this increase in mathematisation demands. Figure 5 shows the results from the survey illustrating the relationship between mathematisation demands and error. Each diagram shows the steepness to be measured and the participants' attempts to represent and quantify that steepness.

Quantification of a diagram

Drawn line representation

Quantification of an actual path

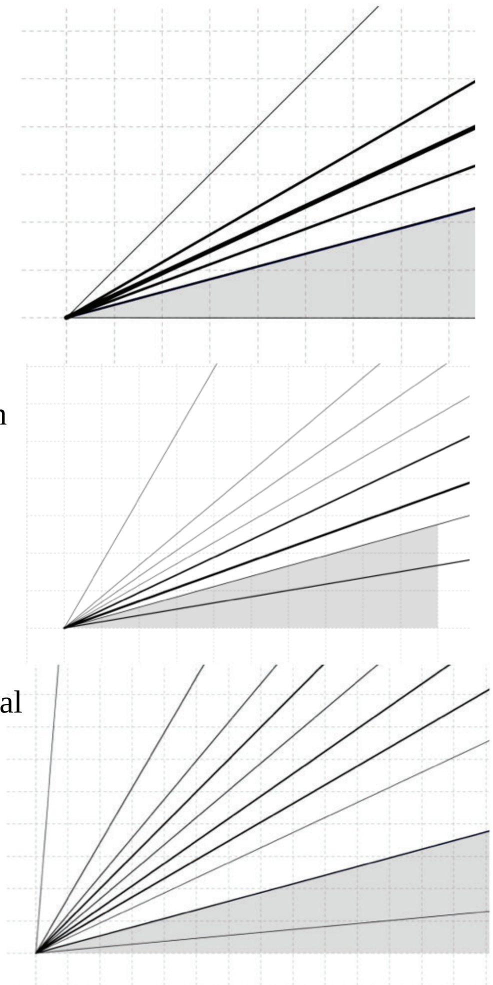

Figure 5. Results from the survey activity. The shaded area represents the steepness to be measured, each line represents a participants' attempt to represent it accurately.

The aspects of mathematisation required to complete each task are summarised in Table 1. For the task of measuring steepness accurately, each of these aspects need to be identified and successfully applied. The research concluded that the errors made by participants in these tasks result from the lack of understanding of one or more of the appropriate concepts.

Table 1. The aspects of mathematisation required to measure steepness accurately

Task	Appropriate concepts
Quantification of a diagram	selection of appropriate method (angle or ratio) estimation of size of angle or, estimation of quantities in ratio and, construction of a ratio
Representation of the steepness of a path as a drawn line	imagination of horizontal reference selection of an appropriate point of view from which to judge steepness creation of an accurate, two-dimensional representation of space
Quantification of an actual path	as in "quantification of a diagram" but applied to either the drawn line representation or the actual path

Implications for teachers

Teachers need to deal with mathematisation, or the proficiency strands, explicitly in the classroom. Designing learning activities that require students to make decisions, provides them with an illustration of the need to make decisions as well as the context in which they can explore and test their own decision making skills. The activity described here is a simple one that can be conducted in most school environments. To follow this activity with an exploration of the results gained and a discussion of the sources of error can provide rich learning opportunities for students. To then follow this discussion with a plan for accurately measuring steepness, designed and tested by students, can help consolidate these understandings. These activities also provide opportunities for assessment from both the teacher's and the student's point of view.

The task of measuring steepness involves the application of a number of mathematical concepts. Each of these concepts can be explored and tested via this activity. In doing so, the students' understanding of each concept can be developed towards the broad and flexible understanding that is described in the *Australian Curriculum: Mathematics*. Through experiencing these concepts in a range of contexts, students are more likely to develop conceptual understanding of measurement, proportional reasoning, slope as a ratio and mathematisation.

As teachers, we cannot assume that students' success in paper-based tasks is indicative of their potential to succeed in problem solving outside the classroom, particularly when the tasks used to assess students' understanding do not require them to select the concepts and procedures that should be used. Practice in mathematisation enhances conceptual understanding and students' proficiency in applying mathematical concepts outside the classroom.

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WHEN DOES ½ = ⅓?
MODELLING WITH WET FRACTIONS

NOLEINE FITZALLEN
University of Tasmania
noleine.fitzallen@utas.edu.au

Many fraction activities rely on the use of area models for developing partitioning skills. These models, however, are limited in their ability to assist students to visualise a fraction of an object when the whole changes. This article describes a fraction modelling activity that requires the transfer of water from one container to another. The activity provides the opportunity for students to explore the part–whole relationship when the whole changes and respond to and reason about the question: When does ½ = ⅓?

Although the use of concrete materials for the teaching and learning of fractions is strongly advocated, many teachers in the middle years do not use concrete materials for fraction development (Van de Walle, Karp & Bay-Williams, 2010). The emphasis is often placed on determining the fraction of shaded areas of geometric shapes and multiple procedural computations of fractions of groups. Little attention is given to the conceptual development of fraction understanding (Mills, 2011).

Hands-on activities that are developed for students in the middle years often use the context of pizza, cookies, and food as real-life contexts and models (e.g., Bush, Karp, Popelka & Miller Bennet, 2012; Cengiz & Rathouz, 2011; Wilson, Edgington, Nguyen, Pescocolido & Confrey, 2011). Such activities put an emphasis of the use of the circle area model for the development of understanding of part–whole relationships, but do not address other representations such as fractions as a measure, ratio, and operator. Dominating students' experiences with the circle area model limits students' ability to transfer their knowledge of fractions to different models and contexts (Clements & McMillen, 1996). It is, therefore, important to use a variety of fraction models in order to support students to make the connections among the different fraction representations.

Mathematical models

The choice of what model to use to foster particular mathematical understanding needs to be based on the model's ability to provide links between the features of the model and the target mathematics knowledge. Stacey and colleagues (2001) describe this as *epistemic fidelity*. Another factor that influences the usability of a particular model is the *process of engagement* students undertake with the model and is dependent on the specific socio-cultural practices of the students and established classroom practices. A third factor raised by Stacey et al. is *accessibility*. Accessibility is optimal when students "*see through it* [the model] to the underlying principles and relations, without being confused by features of the model itself" (Stacey, Helme, Archer & Condon, 2001, p. 200).

The three factors described by Stacey et al. (2001)—epistemic fidelity, process of engagement, and accessibility—determine the effectiveness of concrete materials used as models. Collectively the factors contribute to the *transparency* of the model. Transparency is achieved when the inherent features of the model, and the way in which the model can be manipulated within particular classroom practices, supports effectively students' development of mathematical knowledge (Meira, 1998). Meira also stresses that concrete materials provide a focus for discussing mathematical ideas. In some cases the concrete materials provide vital links between the mathematics and its application in real-life contexts—an element absent in many mathematical activities.

Fraction models

Typically, three fraction models are used in the middle years of schooling—area, length, and set models. Area models help students visualise parts of the whole, length or linear models show that there are always other fractions found between two fractions, and set models show that the whole is a set of objects and subsets of the whole make up fractional parts. The three different models impart different meaning and provide different opportunities to learn. Activities designed with these models for students in the middle years mirror the way in which they are used in the primary classroom. Therefore, they have nothing more to offer the students as they progress into secondary education. The repeated use of the same models and activities in the middle years, particularly "determine the fraction of the shaded area," does not acknowledge the need to extend students' problem solving and reasoning skills as advocated by the *Australian Curriculum*: *Mathematics* (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2013), nor does it acknowledge the need to provide older students with meaningful activities that make connections to other mathematical ideas and concepts.

Alternative fraction model

In this section a fraction activity is described that introduces a different fraction model—a liquid volume model. The activity uses water in containers to explore what happens when a quantity of water is transferred into a different container, thereby, giving students the opportunity to explore what happens when the whole changes in a part–whole relationship. A collection of different sized containers is required for the activity. Odd shaped containers make it more interesting and more challenging when visualising the fraction filled. Using a smaller container for the second part of the activity than in the first part will result in fractions greater than 1. Examples of the type of containers that could be used are shown in Figure 1 and the activity is described in Figure 2.

Figure 1. Collection of containers.

When is ½ = ⅓?

Set the scene

You have a container with water in it to water some plants. The container is not full. Your container starts to leak very slowly so you have to transfer the water into another container. First, mark the level of the water in the container. Now transfer the water into a different sized container.

Estimate

Estimate the fraction of the container taken up by the water for each container.

Measure

Measure the volume of the water.

Record and comment

	Container 1	Container 2
Volume of container		
Estimation of fraction of container filled with water		
Measure of water		
Calculated fraction of water in the containers		
Express the calculated fraction as a decimal		
Percentage of container filled with water		
How close was your estimate to the actual fraction for each container?		
Which container was easiest to estimate the fraction filled?		
Comment on the question: When is ½ = ⅓?		

Figure 2. Initial investigation.

After students have conducted the initial investigation they can use the fractional quantities and the measurements made to answer questions that will assist them to develop fluency in fraction calculations. For example:

- What volume of water is required in the second container to have an equivalent fraction to that in the first container?
- How much water does each plant get if you give six plants an equal share of the water in your container?
- What fraction of the container would each of the six plants get?
- Before watering the plants you drank one fifth of the water. You then used two thirds of the water left to water three plants. What fraction of the container was used to water the plants?

Classroom discussions that occur during and after the activity can include the relationship between fractions, decimals, and percentages as well as responses to the overarching activity question. There is also the opportunity to discuss the need to convert the units of measure used. Calculations involving a 2 litre container may involve converting the containers volume of 2 litres to 200o millilitres.

Utilisation of this activity within a sequence of learning activities designed to enhance students understanding of fraction concepts will provide the opportunity to address multiple mathematics learning outcomes in Year 7 of the mathematics curriculum.

- Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153)
- Express one quantity as a fraction of another, with and without the use of digital technologies (ACMNA155)
- Connect fractions, decimals and percentages and carry out simple conversions (ACMNA157)
- Find percentages of quantities and express one quantity as a percentage of another, with and without digital technologies. (ACMNA158) (ACARA, 2013).

Conclusion

This activity takes advantage of the inherent nature of water to develop students understanding of fractions. Because water is a liquid, it can take the shape of the container in which it is stored without changing its volume. Therefore transferring water from one container to another allows immediate visualisation of the original quantity as a fraction within the new container. This property increases the epistemic fidelity of the fraction model underpinning the activity. The use of containers used every day by students and the familiarity students have with standard-sized drink containers increases the accessibility of the activity. There is, however, the need to explore the use of this activity further. As Meira (1998) suggests, students from different socio-cultural backgrounds may engage with this activity in unexpected ways and it is important to determine if the liquid volume model provides the transparency needed to make it an effective learning model for fraction development.

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IMPLEMENTING JAPANESE LESSON STUDY: AN EXAMPLE OF TEACHER–RESEARCHER COLLABORATION

SUSIE GROVES
Deakin University
susie.groves@deakin.edu.au

BRIAN DOIG
Deakin University
badoig@deakin.edu.au

WANTY WIDJAJA
Deakin University
w.widjaja@deakin.edu.au

DAVID GARNER
Creekside K–9 College
garner.david.s@edumail.vic.gov.au

KATHRYN PALMER
Melton Network
palmer.kathryn.k@edumail.vic.gov.au

There is growing worldwide interest in Japanese Lesson Study as a model for professional learning, with large-scale adaptations of lesson study taking place in many countries. This paper describes how teachers and researchers collaborated in a lesson study project[4] carried out in three Victorian schools. It describes Japanese Lesson Study and the typical structured problem solving research lesson that forms the basis for lesson study; and discusses how the collaborative planning process and the resulting research lessons, together with the post-lesson discussions, provided teachers and researchers with the opportunity to collaborate in the research process.

Japanese Lesson Study

Japanese Lesson Study is a professional learning activity with origins that can be traced back for almost a century. Unlike many Western initiatives, richly funded and mandated, lesson study in Japan is neither funded nor mandatory. Essentially school-based and organized by teachers themselves, it pervades primary school education—and to a lesser extent secondary school education—across the country, with teachers researching their own practice in school-based communities of inquiry.

Lesson study first came to worldwide attention as a vehicle for professional learning through Yoshida's (1999) doctoral dissertation and Stigler and Hiebert's (1999) accounts of Japanese structured problem solving lessons based on the *Third International Mathematics and Science Study* (TIMSS) video study. Since then, there has been phenomenal growth of lesson study as a vehicle for professional learning in countries such as the USA, UK, Malaysia, Indonesia and Australia.

Japanese Lesson Study has four components:

- formulation of over-arching school goals related to students' learning and long-term development;
- group planning of a *research lesson* addressing these goals;

4 The *Implementing structured problem solving mathematics lessons through Lesson Study* project was funded by the Centre for Research in Educational Futures and Innovation, Deakin University. The project team consisted of Susie Groves, Brian Doig, Colleen Vale and Wanty Widjaja.

- one team member teaching the research lesson while the planning group, and others, observe in order to gather evidence of student learning; and
- the post-lesson discussion where the planning group and other observers (usually including an "outside expert") discuss and reflect on the evidence gathered during the lesson, using it to improve the lesson, the unit, and instruction more generally (Perry & Lewis, 2008, p. 366).

In Japan, the *research lesson* in mathematics is based on "structured problem solving", a major instructional approach designed to create interest in mathematics and stimulate creative mathematical activity (Takahashi, 2006). Typically such lessons have four stages: posing the problem; students solving problems individually, in pairs or small groups; whole-class discussion; and summing up (Shimizu, 1999). These lessons have a single focus and address a single problem designed to "achieve a single objective in a topic" (Takahashi, 2006, p. 4).

Critical in the process of planning a research lesson is the selection of the problem or task for the problem solving activity through *kyozaikenkyu*, which is an intensive and complex investigation of a large range of instructional materials, including textbooks, curriculum materials, lesson plans and reports from other lesson studies, coupled with a study of students' prior understandings (Watanabe, Takahashi & Yoshida, 2008). While teachers cannot engage every day in such deep *kyozaikenkyu,* conducting it for the purpose of a research lesson leads to a deeper understanding of the curriculum and the mathematical content and goals underpinning it, as well as the importance of matching problems to both the mathematical goals of the lesson and students' knowledge (see also Doig, Groves & Fujii, 2011).

Public observation and debriefing of research lessons is a key feature of Japanese Lesson Study. Typically a research lesson will be observed by all members of the lesson planning team, the school principal, the other teachers at the school (or the other teachers in the same subject area at secondary schools), and an "outside expert" who acts as the final commentator at the post-lesson discussion. Depending on the scale of the research lesson, there may be many additional outside observers—50 to 100 observers would not be unusual. Observers focus on student learning and are expected to base their comments in the post-lesson discussion on evidence they have collected during the lesson. The purpose is to promote thoughtful, data-focussed discussion of the lesson.

Teachers act as researchers in all phases of the Japanese Lesson Study process, researching the curriculum, teaching resources, known student misconceptions, and formulating their own research questions to be addressed through the research lesson and subsequent post-lesson discussion.

Our lesson study project

The *Implementing structured problem solving mathematics lessons through Lesson Study* project worked with two Year 3 or 4 teachers from each of three schools from a Melbourne school network to explore ways in which key elements of Japanese Lesson Study could be embedded into Australian mathematics teaching and professional learning. Teachers were supported not only by members of the Deakin research team, but also by a key leading teacher at each school (e.g., a curriculum specialist or numeracy coach) as well as the network numeracy coach—a total of ten participants.

Participants took part in an initial whole-day professional learning session on lesson study in June, and completed one lesson study cycle during each of Terms 3 and 4 of 2012. Each lesson study cycle involved two cross-school teams of three teachers and two leading teachers or coaches planning a research lesson on the same topic during four two-hour planning sessions. Each team was supported by two of the university researchers. One member of each team taught the research lesson in front of observers, with both teams participating in the post-lesson discussions. Key staff at each school, together with all interested teachers who could be released from their classes at the time of the research lessons, as well as other professionals such as numeracy coaches and leadership teams from other network schools, and mathematics educators, were invited to observe the lessons and take part in the post-lesson discussions. Approximately 30 people observed the fourth research lesson in December 2012. Due to the perceived success of the project, the project has continued into the first half of 2013, with two days of teacher release for each participating teacher being funded by the Melton Network. Two research lessons are now being planned for the second week of Term 2.

In this paper, members of the Deakin University research team discuss how the collaborative planning process and the resulting research lessons, together with the post-lesson discussions, provided teachers and Deakin researchers with the opportunity to collaborate in the research process, while one of the school numeracy coaches and the network numeracy coach provide their perspectives on the project.

The collaborative planning process

Detailed and careful planning is central to the Japanese Lesson Study process. Planning for lesson study in a Japanese school involves setting overarching goals, as well as goals for the unit of work in which the research lesson is embedded, and goals for the research lesson itself. Teachers need to identify key mathematical ideas to be explored in the lesson and anticipate students' mathematical solutions. In keeping with the spirit of Japanese Lesson Study, which sets out to engage teachers as "investigators of their own classroom practices" and "researchers of teaching and learning in the classroom" (Takahashi & Yoshida, 2004, p. 438), teachers and coaches took full responsibility for the planning of the research lessons. The Deakin research team facilitated the planning process by sourcing potential mathematical tasks to be explored, modelling a problem solving lesson using a problem similar to the one to be used in the first research lesson, and providing resources such as articles on lesson study and sample lesson plans.

During the first planning meeting in each cycle, teachers and numeracy coaches in the project engaged in solving the mathematical problem proposed for the research lesson and participated in a discussion of their solutions. Having first-hand experience in solving the mathematical problem and discussing the attributes of various solutions was instrumental in helping teachers anticipate the learning potential for students and possible misconceptions students might have when working on the problem. Furthermore, engaging in solving the mathematical problems provided teachers with opportunities to deepen their mathematical content knowledge.

Anticipating students' solutions is a key element of the lesson planning process in Japanese Lesson Study (Shimizu, 2009). It gives teachers a clear idea of what to look for when they observe students' work, thus enabling them to orchestrate a productive whole-class discussion that carefully sequences students' solutions. The main teaching and learning takes place during this whole-class discussion, which is designed to help students learn something "new" and advance their mathematical thinking. (Shimizu, 2009; Takahashi, 2006; Watanabe, Takahashi & Yoshida, 2008).

Anticipating students' mathematical solutions was a new element in the planning process for all teachers and coaches. Similarly, orchestrating an extended whole-class discussion was not a common practice in their mathematics lessons. Initially teachers expressed concern about allocating 20 minutes for a whole-class discussion and predicted that this would be challenging for their students. In order to allow teachers to become more familiar with such a lesson structure and to build their confidence in implementing such lessons, the research team encouraged teachers to work closely with their school numeracy coach in trialling a similar problem solving task in their classrooms. Teachers in both planning teams agreed to trial another problem with their class and record students' responses. As a result, teachers became more comfortable with conducting extended whole-class discussions, with one teacher commenting that she had been "quite wrong" when she had previously predicted that her class would not be able to come up with many different solutions or be able to spend extended time sharing these. This was a major breakthrough for this teacher. Other teachers came to similar conclusions after trialling the research lessons in different classes prior to the research lesson day. Sharing the insights gained from trialling these problem solving lessons in the planning meetings was instrumental in advancing the planning process. Through this trialling process, teachers were encouraged to examine in detail various elements of the research lessons, such as the exact phrasing of the task, ways to elicit students' mathematical thinking through questioning, and planning the sequence of students' solutions to enable a progression of ideas.

At the beginning, there might have been an expectation that the researchers would lead the way in planning the research lesson. However, members of the planning teams shared responsibilities to identify links between lesson goals and curriculum documents. Collective effort by every member was evident through the sharing of resources. The numeracy coaches played a salient role in supporting teachers to conduct the trial lessons by arranging a release time for teachers to observe each other's trial lessons, analysing students' work and helping teachers to plan questions to elicit students' thinking. The fact that members of the research team stepped back and let the teachers and coaches take control of the planning process was initially challenging for some teachers. However this thinking had shifted by the end, after teachers had observed the benefits of developing their own clear ideas about different elements of the lesson through the process of articulating their thoughts and ideas, guided by questions from members of the research team. There was a strong sense of mutual trust among members of the planning teams, driven by the intention to work on common goals to generate knowledge by examining classroom practice with questioning attitudes, an indication that the planning teams were working as communities of inquiry (Groves, Doig & Splitter, 2000; Jaworski, 2008).

In-depth planning of a research lesson requires a large time commitment. While teachers and coaches saw the real benefits of in-depth planning in deepening teachers' knowledge of mathematics and in the changes to their lessons, ways to address the common concern about the amount of time and continued support from the school community required remain to be explored.

Teachers as researchers

On the surface, Japanese Lesson Study would not appear to be related to teachers acting as researchers. However, examining one's practice is a core aim of the research lesson. The purpose of the research lesson in Japanese Lesson Study is not to provide "a demonstration that showcases a particular teacher or approach" (Watanabe, 2002, p. 37), but rather to provide a proving ground or test-bed for an experiment in teaching and learning. While this may seem a grandiose claim for a single lesson, albeit well-designed and taught, Lewis and Tsuchida (1998) report that "Japanese teachers repeatedly pointed to the impact of 'research lessons' … as central to individual, schoolwide and even national improvement of teaching" (p. 12).

How does this work? In a Japanese Lesson Study cycle, teachers in the planning group choose goals and design a lesson to achieve these goals. The goals may be to improve student attitudes to mathematics, to develop new skills, or to try an alternative approach to a curriculum topic. In most cases, the goals include one that is directed towards developing student understanding. For example, in our project, the task in one research lesson was to find the number of dots in a 23 x 3 array, without counting the dots individually. One of the two planning teams listed the following as their two goals for their research lesson: "to encourage students to use more effective multiplicative thinking strategies (including the use of arrays and partitioning); and to ensure students' mathematical explanations match their use of the diagram".

These goals reflect the planning group's own goals or research questions, one of which was "to build the content knowledge of teachers as well as their capacity to ask more precise questions about the student responses". In their lesson plan, this group included a section on how these lesson goals related to their own lesson study goals, stating that:

> In this lesson we are looking at how the teacher poses the problem in order to elicit student thinking about multiplicative strategies. The teacher questioning and discussion should progress student thinking at their point of need and the collaborative planning for this lesson should result in improved teacher practice and student learning.

Although the teachers' research questions are phrased as goals, it is clear what the teachers planning the research lesson wish to investigate.

Once the planning group has agreed on the goals, the lesson plan starts to take shape. A critical feature of the planning is to anticipate likely student solutions. Without a tradition of such lessons to fall back on, teachers in the planning groups trialled the task in their own classrooms, in order to identify likely solution strategies. Researching likely solutions to a problem is a feature of planning for a research lesson, revealing to the inquiring teacher not only many aspects of how children interpret tasks, but also the range of strategies that students employ in solving the problem. In the problem involving finding the total number of dots in the 23 by 3 array, teachers' research in their own classrooms found the following strategies used by the Year 3 and Year 4 students: counting all the dots; using repeated addition; skip counting by 3s; writing the number sentence $23 \times 3 = 69$; and using the vertical multiplication algorithm. While some teachers were surprised with the range of strategies found, others were surprised at the achievement of some of their thought-to-be less capable students. As teachers gained more interesting insights into their students' thinking, this also honed the questions to be used within the lesson itself, as teachers discovered the effect of using different wordings of the task on student responses. This emphasis on deciding on an exact wording to a task in order to stimulate desired responses from students took on a life of its own and became a major influence in creating later lessons.

Finally, the observers invited to the research lesson (a hundred extra eyes) were asked by the planning team to look for evidence that would support the achievement of their goals for the research lesson, thus helping the teachers gauge the effectiveness of their endeavour. For example, the planning team referred to earlier, stated:

> We would like the observers to focus on one or two students to collect data on the strategies used in the lesson. Specifically we would like to know if the strategy used by the students matches their recorded method using the diagram and if the student is chosen to share, how well does the student articulate the strategy used and recorded method?

Over the complete lesson study cycle, teachers were continually investigating "What would happen if we...?" and worked on answering their own questions. In a presentation at the 2012 Mathematics Association of Victoria annual conference, two points were highlighted that under-scored the heightened interest in researching practice by the lesson study project teachers, namely the benefits to teachers and students coming from: planning in teams with clear lesson goals; and trialling lessons before conducting them.

In this project, it was apparent to both the teachers and the university academics, that the teachers were researchers in the project just as much as were the academics.

Creekside College: A need for lesson study

As a numeracy coach in a school of over 1400 students, leading the development and evolution of a problem solving culture in mathematics looms as a challenging task. For teachers to teach *through* problem solving, rather than the more commonplace "teach a problem solving strategy a week" approach, it is vital to build a collaborative, learning community model for planning mathematics units and lessons. Teams of teachers need to work as professional learning communities, where their mathematical knowledge for teaching is developed collaboratively and in an ongoing way, enabling them to teach within a problem solving paradigm of mathematics teaching and learning. If building teachers' mathematical knowledge for teaching is the priority, then Japanese Lesson Study offers a model within which this can take place. Lewis, Perry and Murata (2006) outline the conjecture that more than simply planning a lesson, Lesson Study strengthens three pathways to instructional improvement (see Table 1).

Table 1. Lesson study strengthens teachers' mathematical knowledge for teaching (Lewis et al., 2006, p. 5)

Teachers' knowledge	**Teachers' commitment and community**	**Learning resources**
Knowledge of subject matter	Motivation to improve	Lesson plans that reveal and promote student thinking
Knowledge of instruction	Connection to colleagues who can provide help	
Capacity to observe students	Sense of accountability to valued practice community	Tools that support collegial learning during lesson study
Connection of daily practice to long-term goals		

Although the success of Japanese Lesson Study as a model for improving instruction and teacher content knowledge in Japan has been well-researched and documented, the ability of non-Japanese schools and systems to adopt it as successfully must be considered. Lewis, Perry, Hurd and O'Connell (2006) conducted research into the effectiveness of North American schools and districts in utilising and adapting a lesson study approach to improve teacher instruction and student achievement. They found a distinct improvement in student achievement data in mathematics with the inception of their lesson study approach. Teachers also commented on the enhanced collaboration and development of collective efficacy in the culture of the school. While the whole process is built strongly around the established lesson study processes of Japan, the schools in the United States were continuously mindful of making it work in the USA, not simply replicating the exact program as observed. This not only allowed the schools to develop a model that worked for them, but also allowed the schools, teachers and professionals involved to take ownership of the lesson study process. It is these two key pieces of research that have lead me to believe that incorporating aspects of lesson study, if not entire lesson study cycles, into the established planning and teaching practices of Creekside College teachers would be a key strategy in the improvement of mathematics teaching and learning at our school. The project with Deakin University therefore provided the perfect catalyst for change.

A school-based coach's experiences of the lesson study project

The opportunity to take part in the lesson study project provided a rich experience with myriad benefits, challenges and future implications for both my coaching practice and the teaching and learning practice of the teachers involved in lesson study. Successes of the project included, but were not limited to:

- collaborative planning within a team;
- exploration of developmental continua throughout the planning meetings;
- increased mathematical knowledge for teaching, reported by all teachers at the conclusion of each lesson study cycle;
- the opportunity to work with 'more experienced others' throughout the planning process;
- planning, teaching and reflecting on a problem solving approach to mathematics;

- modifications to the established lesson structure to incorporate more teaching taking place through reflection and sharing;
- consideration and planning for anticipated student responses to the problem throughout the planning process;
- building the confidence of the classroom teachers involved in the project;
- careful, more deliberate task selection, design and modification to meet the learning goals of the lesson and unit;
- the rigorous nature of the planning documentation;
- the honest and open nature and culture of the post-lesson discussion, enabled by the thorough discussion of the lesson, lesson plan, teaching and learning; and
- multiple cycles allowing all involved to hone skills and reflect on learning through the new implementation.

Future implications for mathematics teaching and learning at Creekside College as a result of the lesson study project included, but were not limited to:

- extending share time to around fifteen to twenty minutes in most numeracy lessons;
- student solutions being deliberately selected and ordered across a continuum of learning rather than just having a student read our their own work;
- use of moderation of problem solving task as a pre-assessment for units;
- running lesson study teams throughout the year;
- eventually having each team of teachers running a lesson study cycle;
- importance of teachers planning in a way that builds their knowledge of misconceptions and how they teach through these; and
- in my role as coach, leading the development of teachers' task design and questioning skills.

It is important to conclude with a reflection on why this project was so important and what it means for the future. I feel vindicated in my belief that if we can develop a planning model where teachers can build their knowledge for teaching, then we can improve teachers' practice and, most importantly, improve student learning. The ability of this project to bring together mathematics researchers, numeracy leaders and classroom teachers was a vital component in "launching" lesson study. Merely reading about it and then trying to implement it within schools would not do the process justice. Having researchers who have been involved in lesson study—on multiple occasions, in multiple schools, across a number of years and countries—allowed us to run an authentic lesson study, the benefits of which are countless. A project, which allowed teachers to engage in their own research hand in hand with more experienced others, provided ongoing opportunities for self-reflection and the ability to engage in a genuine professional learning community. I see my role as one where I "*teach* a man to fish" rather than *give* him a fish. Without this view, I believe teachers will never gain the knowledge and confidence to teach high quality mathematics programs and engage learners as problem solvers. Leading and empowering others in collaborative learning communities is essential if long-term, sustainable change is going to occur. Lesson study provides one such paradigm, and this project has been the catalyst for establishing lesson study cultures in our Australian schools.

Perhaps the most significant 'product' of the lesson study project for Creekside College is the implementation of our first lesson study cycles within the school in 2013.

A group of six teachers from across different year levels will be engaging in a full lesson study cycle each term throughout the year. It is hoped that this will become part of the culture of not only mathematics teaching and learning practice, but also an in-built component to quality teaching and learning practice across all curriculum areas into the future.

Lesson study in the Melton Network

As a numeracy coach to over 20 schools across the Melton Network in the Western suburbs of Melbourne, I have been trying for many years to implement the concepts underpinning Japanese Lesson Study. Last year I got the opportunity to act as a project facilitator in an authentic lesson study project. This involved inviting three schools in my network to become involved in Deakin University's project. Considerations included finding three schools in close proximity to each other to overcome travelling issues; teachers were able to move between schools during their lunch break. The first step was to convince the school-based numeracy coach and the leadership team at each school that this was a worthwhile project. As the facilitator of the school-based numeracy coaches' professional learning in my network, I had previously discussed the merits of a lesson study approach to develop teacher content knowledge. So with Deakin University support of funding and personnel this was an easy task and all schools approached were extremely eager to be involved. All school numeracy coaches and leadership teams within the network were invited to attend each research lesson. While not all attended, those who did were excellent advocates for the process and soon there was a need to give all principals within the network some professional learning around the lesson study process. As a result the Melton Network of schools agreed to support the original schools in continuing a final lesson study for Term 1 in 2013 so that all six classroom teachers in the project could have the opportunity to conduct a research lesson.

As a result of the professional discussions and participation in the lesson study project, schools involved in the project have:

- created greater levels of collegiality between teachers and schools involved in the project;
- helped to build a common professional language and common understanding of high quality pedagogy;
- provided opportunities for teachers to share high quality teaching practice, thereby providing a forum to share ideas, success and challenges;
- had a reason to learn together as a result of participating in a practical project that will help improve student learning;
- had to carefully prioritise the most important themes to tackle in the research lesson;
- shared collective responsibility for producing more effective learning for all students;
- used and built on what they know;
- created and implemented plans for achieving their project aims—they think big, but start small and manageable;
- identified the professional learning strategies that most help them learn; and

- combined outside-provided support (research findings, Network Numeracy Coach, external consultants—Deakin University) and work-embedded support (lesson observations, team-teaching, coaching).

While there have been numerous benefits from involvement in this project for both the network and the schools involved, the next challenge is to sustain this work. As my role as network coach is funded through National Partnerships funding, it is unlikely it will continue after this year. Many schools are placing their full time school based numeracy coaches back into a full time classroom role and therefore won't have the time to support the intensive planning needed to develop research lessons. External funding from both Deakin University and, this year, from Melton Network has definitely been a huge reason for the success of this project. However, I am confident that all schools involved in the project will try to modify and implement many of the aspects of lesson study they have experienced through their involvement in this project.

Conclusion

In Japan, lesson study is the main form of systematic professional learning undertaken by teachers. Outside Japan, lesson study is sometimes understood superficially as an activity aimed at perfecting individual lessons. However, it should rather be seen as an activity that allows teachers to collaborate with one another to research their own practice. For example, Lewis and Tsuchida (1998) quote a teacher as saying:

> Research lessons help you see your teaching from various points of view... A lesson is like a swiftly flowing river; when you're teaching you must make judgments instantly. When you do a research lesson, your colleagues write down your words and the students' words. Your real profile as a teacher is revealed to you for the first time (p. 15).

Lesson study in Japan usually involves the participation of outside experts—typically educational consultants, district personnel, or university academics. While these outside experts may only participate in the post-lesson discussions, their contributions help teachers reflect on their practice and often inject new knowledge about relevant research findings. Findings from our project suggest that lesson study in Australia can also provide the opportunity for genuine teacher–researcher collaboration.

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I WANT TO DO ENGINEERING AT UNI: SHOULD I STUDY ONE MATHS SUBJECT OR TWO IN YEARS 11 AND 12?

MICHAEL JENNINGS
The University of Queensland
msj@maths.uq.edu.au

The proportion of students studying advanced mathematics in Years 11 and 12 in Australian schools has been declining since the mid-1990s. As such, universities are accepting students into mathematics-based degrees with weaker mathematical backgrounds than previous generations. How do students perform in university mathematics subjects if they have only studied intermediate mathematics at school? This paper investigates student performance in university mathematics subjects at The University of Queensland, based on whether students studied intermediate mathematics only or both intermediate and advanced mathematics at school.

Introduction

There are substantial and ongoing concerns in the Australian and international tertiary education sectors about students' transition from secondary to tertiary mathematics. These include falling participation rates in advanced mathematics in secondary school, declining enrolments in university mathematics, less stringent university entry requirements, and increasing failure rates in first-year university.

Students who performed very well in secondary school often struggle at university with topics such as differentiation from first principles and integration by substitution, topics they studied in secondary school. Even after two semesters of university mathematics, students still have major difficulties with these topics, and failure rates of 30–35% are not uncommon across some first-year mathematics courses. Do students who only studied one mathematics subject at high school struggle more than those who did two? Do students who perform well at secondary school somehow lose their knowledge between the end of secondary school and the start of university some four months later, or did they not really have a good understanding at school yet somehow managed to get good grades?

Enrolment numbers in secondary school advanced mathematics

The number of students studying higher level mathematics in Australian secondary schools declined in the mid–late 1990s and early part of this century. Although the raw numbers of students studying mathematics in the last two years of secondary school is increasing (due to more students completing Year 12), the proportion of students taking *advanced* mathematics dropped from 14.1% of the Year 12 student population in 1995 to 10.1% in 2010 (Barrington, 2011). The decrease was more marked in Queensland (Mathematics C—13% in 1995, to a low of 7.9% in 2002, and 8.7% in 2012), the state of interest in this paper.

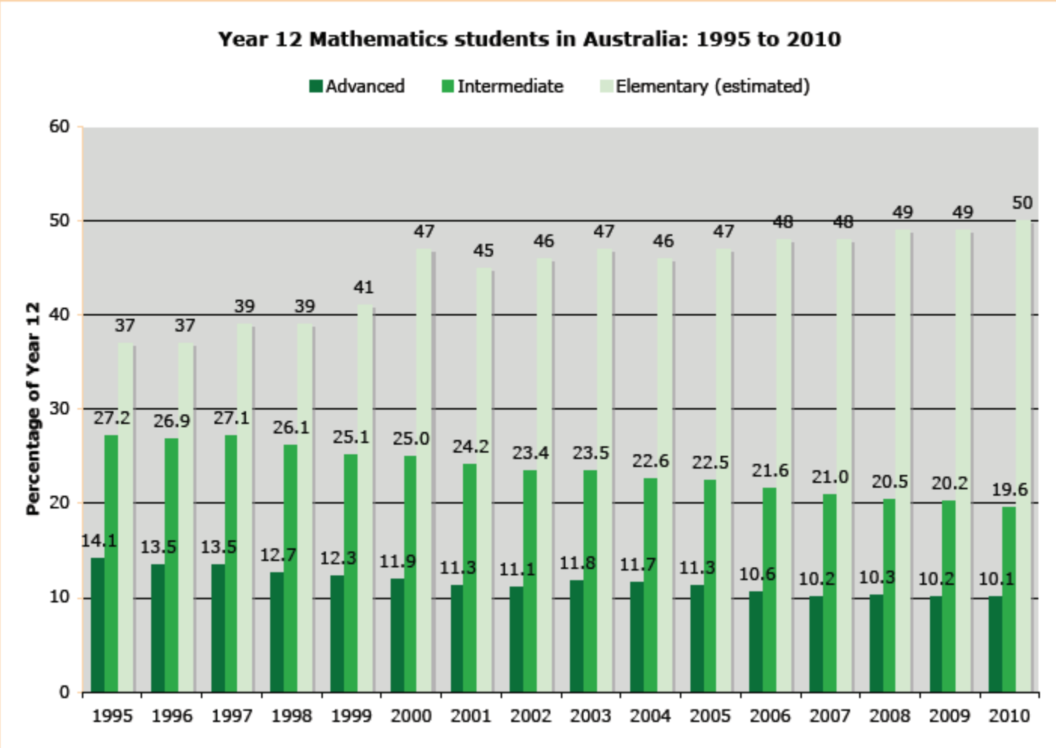

Figure 1. Year 12 mathematics numbers 1995–2010 (Barrington, 2011).

Several years ago universities throughout Queensland introduced a bonus point system to attract more students to study advanced mathematics (and languages) at school. Raw numbers of enrolments have slightly increased yet the overall percentage is still only 10.1% nationally (8–8.7% in Queensland). The start of the decline in advanced mathematics enrolments coincided with universities removing prerequisites from their engineering courses, yet there are other reasons as well.

The *Maths? Why Not?* project (McPhann, Morony, Pegg, Cooksey & Lynch, 2008) found several such reasons why students are less willing to take advanced mathematics at school. The project investigated mathematics teachers', school counsellors', and (to a lesser extent) students' beliefs as to why students were not choosing to study higher level mathematics in their last two years of secondary school. The reasons stated as to why students were not choosing to study higher level mathematics in their last two years of secondary school can be grouped into five categories:

- school influences;
- university influences;
- sources of advice influences;
- individual influences; and
- other influences.

While the *Maths? Why Not?* report was a first in Australia-wide data collection on reasons for choosing senior secondary mathematics, its findings are somewhat limited. As noted, students were not the main group asked why they were not choosing to study higher level mathematics, mathematics teachers and school counsellors were. The author is currently undertaking a Queensland state-wide research project on the transition from secondary school to tertiary mathematics by focussing on the *students* as they progress from school to university.

Secondary-tertiary transition in mathematics: The University of Queensland context

The University of Queensland (UQ) is Queensland's oldest university and is a member of the G08 and Universitas21. It has a history of attracting the highest achieving students in Queensland and northern New South Wales and has not had a strong focus on supporting the transition from secondary to tertiary learning environments. UQ had (and still has) higher entry score requirements than other universities for many programmes and had strict prerequisites that had to be studied at school before enrolment.

Since the move from elite to mass education, many prerequisites for degrees have been removed, with students able to study these former prerequisites once they are at university. These take the form of bridging courses that typically cover secondary school content of science and mathematics courses, yet they only run for one semester compared to two years at secondary school. Consequently, in these bridging courses it is impossible to teach the same amount of content, and, importantly, students do not have as long a time period to understand and consolidate the material, and develop automaticity and fluency.

The removal of prerequisites had an immense effect on the nature of engineering cohorts in particular. Until the mid-1990s students had to have studied at secondary school both intermediate *and* advanced mathematics, plus chemistry *and* physics, in order to enrol in engineering. Since the mid-1990s, students have only needed intermediate mathematics plus chemistry *or* physics to enrol. (UQ was the *first* Queensland university to drop the advanced mathematics prerequisite for engineering. All other universities soon followed suit.) As a consequence, only 60 to 70% of recent first-year engineering students have studied both intermediate and advanced mathematics at school (The University of Queensland, 2007–2013). This has left 30–40% of students entering engineering without two years of further integration, matrices, vectors, sequences, series, and complex numbers, all important topics for engineering.

First-year mathematics courses at UQ

Students who have not studied advanced mathematics at high school study Mathematical Foundations, a course which revises some of the intermediate mathematics content (namely trigonometry, functions, differentiation, and integration) then looks at matrices, vectors, sequences, series, and complex numbers. This semester-long course has 500–600 students enrolled in first semester: approximately 65% engineering students and 25% science students. Upon successful completion students then study Calculus and Linear Algebra 1 the following semester, perhaps at the same time as Multivariate Calculus. Students who have studied advanced mathematics at high school go straight into Calculus and Linear Algebra 1 in first semester.

Comparisons of the mathematical abilities and understanding of these two groups of students (high school intermediate mathematics only and high school intermediate and advanced mathematics) are performed at the beginning of first semester via diagnostic testing. Diagnostic testing, one of the suite of innovations introduced to address transition issues and the increased diversity of backgrounds, knowledge, and abilities, was reintroduced in 2007 after a 20 year break. The tests over the last seven years have revealed that many first-year students appear to remember, or understand, little of their Year 11 and 12 mathematics in comparison to topics they had studied in primary and early secondary school. Questions on calculus, an area only studied in Years 11 and 12, had the lowest success rate. Students who had studied intermediate and advanced mathematics subjects at high school performed better on all questions than those who had just studied intermediate mathematics, and students performed considerably better in topics to which they had more exposure (see, for example, Jennings, 2008, 2009, 2011; Kavanagh et al., 2009). However, the results suggest that for both groups, students' understanding of the topics most recently studied, in this case, differentiation and integration, appear not to have been strongly consolidated, with students not having developed automaticity and fluency.

One question in the diagnostic test that showed considerable difference between the two groups was that involving a definite integral. While most students could integrate a polynomial, only 30% of the intermediate mathematics only cohort could evaluate an elementary definite integral (the integral from x = 0 to 2 of 2x+3, with respect to x), whereas 75% of the intermediate and advanced mathematics cohort were successful. This difference could be explained in part by the extra unit of integration (30 hours worth) that students studying advanced mathematics at high school do.

Integration is approximately one-sixth (35 hours) of the Queensland Years 11 and 12 intermediate mathematics syllabus and includes the following topics:

- Definition of the definite integral and its relation to the area under a curve;
- The value of the limit of a sum as a definite integral;
- Definition of the indefinite integral;
- Rules of integration: $\int a\,f(x)\,dx$, $\int [f(x) \pm g(x)]\,dx$, $\int f(ax+b)\,\frac{1}{ax+b}\,dx$;
- Indefinite integrals of simple polynomial functions, simple exponential functions, sin $(ax + b)$, cos $(ax + b)$ and $1 \div (ax+b)$ $\frac{1}{ax+b}$;
- Use of integration to find area;
- Practical applications of the integral;
- Trapezoidal rule for the approximation of a value of a definite integral numerically.

Advanced mathematics includes an extra 30 hours of integration, including:

- Integrals of the form: $\int \frac{f'(x)}{f(x)}dx$ $\int [f'(x) \div f(x)]\,dx$, $\int f(g(x)).g'(x)\,dx$;
- Simple integration by parts;

- Development and use of Simpson's rule;
- Life-related applications of simple, linear, first-order differential equations with constant coefficients;
- Solution of simple, linear, first-order differential equations with constant coefficients.

Mathematical integration is a fundamental skill that all engineering students will be required to use at various stages and levels throughout their undergraduate and professional lives. Given that 70% of the advanced mathematics bridging subject cohort are engineering students, the lack of facility with integration at the beginning of university is a cause for concern. The overlap between high school advanced mathematics and what is taught in the advanced mathematics bridging course is not a perfect match; only integration by substitution is taught in the bridging course. Integration by parts and first-order differential equations are taught at various times in the other three compulsory mathematics courses for engineers. Simpson's rule is never taught.

Over the last five years students who have studied the advanced mathematics bridging course have had considerable difficulty with integration by substitution questions on end of semester examinations. Even after another semester of study (Calculus & Linear Algebra 1) students' results on integration questions (including integration by substitution, parts, and partial fractions) are disappointing, with average marks in all questions less than 50%. This is of grave concern to both mathematics and engineering staff. Jennings et al. (2012) are currently working on a project to improve students' ability to understand and apply mathematical integration to a variety of problems across the entire length of the engineering degree.

By contrast, students going straight into Calculus & Linear Algebra 1 after doing both intermediate and advanced mathematics at school perform better in not only integration questions, but also differentiation. Another interesting point to note is that a student's final grade in the advanced mathematics bridging course is a very good predictor of success in higher mathematics courses. Data over the last seven years reveal that a student who passes with the lowest possible pass (a grade of 4 out of 7) will generally find it difficult to pass Calculus & Linear Algebra 1 in the next semester. Until recently, engineering students who studied only advanced mathematics at school would study the bridging course in first semester, then both Calculus & Linear Algebra 1 *and* Multivariate Calculus in second semester. A clear majority of students with a low pass in the bridging course who attempted both courses in the same semester failed *both*. As a result, Multivariate Calculus is now also offered over summer semester, and students with a low pass in the bridging course are advised to study Calculus & Linear Algebra 1 in second semester then Multivariate Calculus over summer. Students with a credit, distinction, or high distinction (5, 6, 7 out of 7) in the bridging course perform well in both Calculus & Linear Algebra 1 and Multivariate Calculus in second semester.

Conclusion

The title of this paper is 'I want to do engineering at uni; should I study one maths subject or two in Years 11 and 12?' The short answer is 'two'! It is perhaps not surprising that students who study both intermediate and advanced mathematics at high school perform better in university mathematics subjects. The extra subject means two extra years of mathematics and two more years of thinking mathematically. It has to count for something. While bridging courses are offered at university, their short nature does not allow students to catch up on the work they would have done in the same subject at high school. In addition, students who just pass the bridging course generally struggle in subsequent mathematics courses.

With only 8–10% of students across Australia studying advanced mathematics at school, some would say the future is grim if these numbers do not increase. "Australia will be unable to produce the next generation of students with an understanding of fundamental mathematical concepts, problem-solving abilities and training in modern developments to meet projected needs and remain globally competitive" (Mathematics and Statistics: Critical Skills for Australia's Future, The National Strategic Review of Mathematical Sciences Research in Australia, 2006, p. 9).

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MATHEMATICS ACHIEVEMENT OF GRADE 8 STUDENTS FROM ASIA-PACIFIC IN TIMSS 2011

BERINDERJEET KAUR

National Institute of Education, Singapore

berinderjeet.kaur@nie.edu.sg

TIMSS (Trends in International Mathematics and Science Study) 2011 is the fifth in a series of international mathematics and science assessments conducted every four years. TIMSS is designed to provide trends in fourth and eighth grade mathematics and science achievement in an international context. This paper reviews the mathematics achievement of eighth-graders from ten Asia-Pacific countries (Australia, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, New Zealand, Singapore and Thailand) that participated in TIMSS 2011. The achievement data show that Korea, Singapore, Chinese Taipei, Hong Kong and Japan ranked as the top five countries respectively. The paper also examines Number test items, for the topic fractions and decimals, that Australian students found relatively difficult and speculates some instructional issues that may explain the poor achievement.

Introduction

TIMSS (Trends in International Mathematics and Science Study) 2011 is the fifth in a series of international mathematics and science assessments conducted every four years. TIMSS is designed to provide trends in fourth- and eighth-grade mathematics and science achievement in an international context. In TIMSS 2011, 45 countries participated at the eighth grade level. The Asia-Pacific countries that participated at the eighth grade were Australia, Chinese Taipei, Hong Kong, Indonesia, Japan, Korea, Malaysia, New Zealand, Singapore and Thailand. Data was collected from participating students, their teachers and school leaders with the help of assessment tasks and background questionnaires. The TIMSS 2011 International Results in Mathematics (Mullis, Martin, Foy & Arora, 2012) is a comprehensive report of all the data collected and analysed for mathematics assessment of grades four and eight students. This paper draws on the data from the report and discusses the achievement of grade eight students from Asia Pacific countries. It also examines Number test items, for the topic fractions and decimals, that Australian students found relatively difficult and speculates some instructional issues that may explain the poor achievement.

Student participants and tests

Representative samples of eighth graders participated in the study. They were in their eighth year of formal schooling with average ages ranging from 14.0 to 14.5 years. The TIMSS 2011 tests comprised of both mathematics and science items. Fourteen different booklets containing a selection of the 215 mathematics and 217 science items were administered to the students. Each student completed the test in one booklet. Testing time was 90 minutes. The 217 mathematics items (118 multiple choice and 99 constructed response type) were classified by content domain and cognitive domain. The four content domains were Number, Algebra, Geometry, and Data and Chance, while the three cognitive domains were Knowing, Applying and Reasoning (Mullis, Martin, Ruddock, Sullivan & Preuschoff, 2009).

Mathematics achievement

Table 1 shows the ranking and average scale scores of the Asia-Pacific countries that participated in TIMSS 2011 and TIMSS 2007. The five East Asian countries were in the top five ranks for both TIMSS 2011 and TIMSS 2007. Korea, Singapore, Chinsese Taipei, Hong Kong and Australia improved their average scale scores in 2011 compared to 2007, and Australia made a significant upward move to the 12th place with an average scale score higher than the international average of 500.

Table 1. Rank and average scale scores of Asia-Pacific countries.

	TIMSS 2011		TIMSS 2007	
Country	**Rank**	**Average Scale Score**	**Rank**	**Average Scale Score**
Korea, Rep of	1	613 (2.9)	2	597 (2.7)*
Singapore	2	611 (3.8)	3	593 (3.8)*
Chinese Taipei	3	609 (3.2)	1	598 (4.5)*
Hong Kong, SAR	4	586 (3.8)	4	572 (5.8)
Japan	5	570 (2.6)	5	570 (2.4)
Australia	12	505 (5.1)	14	496 (3.9)
International Avg	-	500		
New Zealand	16	488 (5.5)	-	-
Malaysia	26	440 (5.4)	20	474 (5.0)
Thailand	28	427 (4.3)	29	441 (5.0)
Indonesia	38	386 (4.3)	36	397 (3.8)

Standard errors are shown with ( ).
* No significant difference between average scale scores

International benchmarks of mathematics achievement

The international benchmarks presented as part of the TIMSS 2011 data (Mullis, Martin, Foy & Arora, 2012) help to provide participating countries with a distribution of the performance of their eighth-graders in an international setting. For a country the proportions of students reaching these benchmarks perhaps describe certain strengths and weaknesses of mathematics education programs of the country. The benchmarks delineate performance at four points of the performance scale. Characteristics of students at each of these four points are elaborated in the next section.

Table 2 shows the percentage of students from the Asia-Pacific countries reaching TIMSS 2011 international benchmarks of mathematics achievement. It is worthy to note that almost half of the students from Chinese Taipei, Singapore and Korea were at the Advanced benchmark. Furthermore in all the five Asia-Pacific countries that were ranked as the top five, more than 70% of their students were at the High benchmark level, with the exception of Japan (61%) and almost 90% of their students were at the Intermediate benchmark level. In contrast, particularly for Australia, only 9% of their students were at the Advanced Benchmark level and 29% at the High benchmark level. Nevertheless the proportions of students from Australia at the advanced and high benchmarks have improved compared to TIMSS 2007. In TIMSS 2007, only 6 % were at the advanced level and 24 % were at the high level. For New Zealand, only 5% were at the advanced level and 16% of the students were below the low level. For Malaysia, Thailand and Indonesia, only 2% or less of the students were at the advanced level and 35%, 38% and 57% of students respectively were below the low level.

Table 2. Percentages of students reaching TIMSS 2011 international benchmarks of mathematics achievement.

Country	Advanced benchmark (625)	High benchmark (550)	Intermediate benchmark (475)	Low benchmark (400)
Chinese Taipei	49 (1.5)	73 (1.0)	88 (0.7)	96 (0.4)
Singapore	48 (2.0)	78 (1.8)	92 (1.1)	99 (0.3)
Korea, Rep of	47 (1.6)	77 (0.9)	93 (0.6)	99 (0.2)
Hong Kong SAR	34 (2.0)	71 (1.7)	89 (0.7)	97 (0.3)
Japan	27 (1.3)	61 (1.3)	87 (0.7)	97 (0.3)
Australia	9 (1.7)	29 (2.6)	63 (2.4)	89 (1.1)
New Zealand	5 (0.8)	24 (2.6)	57 (2.8)	84 (1.6)
Malaysia	2 (0.4)	12 (1.5)	36 (2.4)	65 (2.5)
Thailand	2 (0.4)	8 (1.3)	28 (1.9)	62 (2.1)
Indonesia	0 (0.1)	2 (0.5)	15 (1.2)	43 (2.1)
International Median	2	17	46	75

Standard errors are shown with ( )

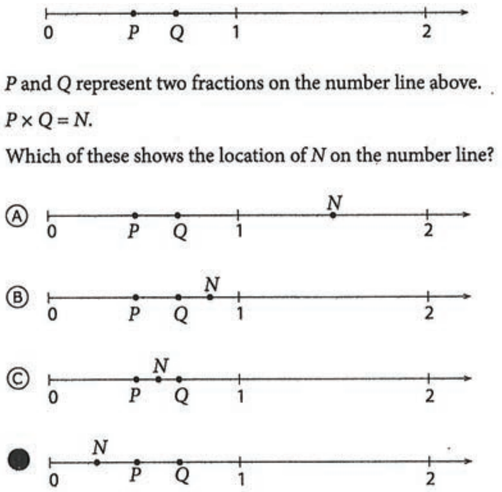
What can students at each of these benchmarks do?

Advanced International benchmark

At the advanced international benchmark students can

- reason with information, draw conclusions, make generalizations, and solve linear equations;
- solve a variety of fraction, proportion, and percent problems and justify their conclusions;
- express generalisations algebraically and model situations;
- solve a variety of problems involving equations, formulas, and functions;
- reason with geometric figures to solve problems; and
- reason with data from several sources or unfamiliar representations to solve multi-step problems.

Figure 1 shows an item that students reaching the advanced benchmark were likely to answer correctly.

Content Domain: Number Cognitive Domain: Reasoning Description: Given two points on a number line representing unspecified fractions, identifies the point that represents their product	Country	Percent correct
	Chinese Taipei	53 (2.0)
	Hong Kong SAR	47 (2.5)
	Singapore	45 (2.0)
	Korea, Rep of	44 (2.0)
	Japan	43 (2.1)
	Australia	23 (2.1)
	New Zealand	19 (2.3)
	Malaysia	18 (1.4)
	Thailand	12 (1.5)
	Indonesia	10 (1.7)
	International Avg	23 (0.3)

Standard errors are shown with ( ).

Figure 1. An advanced international benchmark item.

High international benchmark

At the high international benchmark students can

- apply their understanding and knowledge in a variety of relatively complex situations;
- use information from several sources to solve problems involving different types of numbers and operations;
- relate fractions, decimals, and percents to each other;
- basic procedural knowledge related to algebraic expressions;
- use properties of lines, angles, triangles, rectangles, and rectangular prisms to solve problems; and
- analyse data in a variety of graphs.

Figure 2 shows an item that students reaching the high benchmark were likely to answer correctly.

Intermediate international benchmark

At the intermediate international benchmark students can

- apply basic mathematical knowledge in a variety of situations;
- solve problems involving decimals, fractions, proportions, and percentages;
- understand simple algebraic relationships;
- relate a two-dimensional drawing to a three-dimensional object;
- read, interpret, and construct graphs and tables; and
- recognise basic notions of likelihood.

Figure 3 shows an item that students reaching the intermediate benchmark were likely to answer correctly.

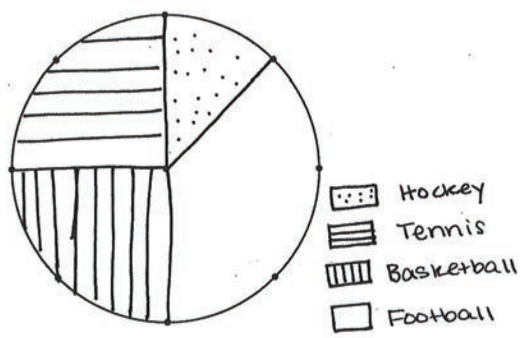
Content Domain: Data and Chance
Cognitive Domain: Applying
Description: Constructs and labels a pie chart representing a given situation

480 students were asked to name their favorite sport. The results are shown in this table.

Sport	Number of Students
Hockey	60
Football	180
Tennis	120
Basketball	120

Use the information in the table to complete and label this pie chart.

Popularity of Sports



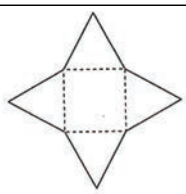
The answer shown illustrates the type of student response that was given 2 of 2 points

Country	Percent correct
Singapore	85 (1.5)
Korea, Rep of	85 (1.4)
Chinese Taipei	80 (1.7)
Hong Kong SAR	76 (1.8)
Japan	75 (1.7)
Australia	67 (2.3)
New Zealand	59 (2.5)
Malaysia	50 (2.2)
Thailand	45 (2.3)
Indonesia	28 (2.2)
International Avg	47 (0.3)

Standard errors are shown with ( ).

Figure 2. A high international benchmark item.

Content Domain: Geometry
Cognitive Domain: Knowing
Description: Given a net of a three-dimensional object, completes a two-dimensional drawing of it form a specific viewpoint



The shape shown above is cut out of cardboard. The triangle flaps are then folded up along the dotted lines until they touch the edges of the flaps next to them.

Complete the diagram below to show what the shape would look like when viewed from directly above.

The answer shown illustrates the type of student response that was given 1 of 1 points

Country	Percent correct
Japan	89 (1.2)
Australia	87 (1.2)
Korea, Rep of	85 (1.3)
New Zealand	84 (1.7)
Singapore	83 (1.4)
Hong Kong SAR	77 (2.0)
Chinese Taipei	74 (1.7)
Malaysia	53 (1.8)
Thailand	51 (2.4)
Indonesia	27 (2.2)
International Avg	58 (0.3)

Standard errors are shown with ( ).

Figure 3. An intermediate international benchmark item.

Low international benchmark

At the low international benchmark students have some knowledge of whole numbers and decimals, operations, and basic graphs. Figure 4 shows an item that students reaching the low benchmark were likely to answer correctly.

<table>
<tr><td>Content Domain: Algebra
Cognitive Domain: Knowing
Description: Evaluates a simple algebraic expression</td><td>Country</td><td>Percent correct</td></tr>
<tr><td rowspan="11">$y = \frac{a+b}{c}$
$a = 8$, $b = 6$, and $c = 2$
What is the value of y?
● 7
Ⓑ 10
Ⓒ 11
Ⓓ 14</td><td>Korea, Rep of</td><td>92 (1.0)</td></tr>
<tr><td>Chinese Taipei</td><td>91 (1.0)</td></tr>
<tr><td>Singapore</td><td>91 91.1)</td></tr>
<tr><td>Japan</td><td>86 91.5)</td></tr>
<tr><td>Hong Kong SAR</td><td>83 (1.8)</td></tr>
<tr><td>Australia</td><td>71 (2.6)</td></tr>
<tr><td>Indonesia</td><td>65 (2.4)</td></tr>
<tr><td>New Zealand</td><td>61 (2.6)</td></tr>
<tr><td>Thailand</td><td>56 (2.2)</td></tr>
<tr><td>Malaysia</td><td>47 (2.1)</td></tr>
<tr><td>International Avg</td><td>71 (0.3)</td></tr>
</table>

Standard errors are shown with ( ).

Figure 4. A low international benchmark item.

Performance of Australian students on some Number items

In this section we examine the performance of Australian students on the 13 Number items for the topic: Fractions and Decimals from the pool of TIMSS 2011 released items (see http://timssandpirls.bc.edu for complete set of released mathematics items for grade 8). Table 3, shows the percent correct for each of the items. The data for Singapore students and the international average score is also shown in the table. This allows us to add a perspective of the achievement of fellow students from Singapore on the items and the opportunity to learn the relevant content at the respective year levels.

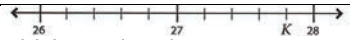
Thompson, Kaur, Koyama and Bleiler (2013) noted that in comparative studies achievement scores must be examined alongside with the opportunity to learn for students in their respective school systems. In a country like Australia that have state based schooling systems with their respective curriculum guidelines for mathematics it is almost certain that not all Year 8 students would have had the opportunity to learn the topic fractions and decimals to the same depth. In contrast, the Skills, Properties, Uses and Representations (Thompson & Kaur, 2011) of aspects of fractions and decimals tested by the 13 items were in the intended mathematics curriculum for Singapore students spread across Years 3 to 5. As shown in Table 3, for most of the items the achievement of students from Singapore was certainly at mastery level. Thompson et al (2013) noted that Singapore and Japan provide opportunities for students to learn number concepts earlier in the primary curriculum compared to the US and that students in Singapore were more likely to master concepts during the grade at which they were first introduced, at least through grade 3, than was true for students in Japan and US.

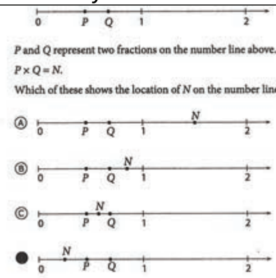
Barry Kissane and Marian Kemp from Western Australia (WA) were both approached by the author for some insights into the possible reasons about the achievement of Australian students. In their email communication (Kissane & Kemp, personal communication, February 7, 2013) they speculated the following issues that may be associated with the achievement of students in Australia:

- Profiles of Year 8 students across Australia vary as in some states it is the first year of secondary school while in others it is the second;
- Variations in state based intended curriculum (certainly in 2011);
- The intended curriculum serves only as a guide. Therefore the implemented curriculum is subject to variation;
- Lack of exposure to the test format of TIMSS;
- Generally the curriculum focuses on understanding numbers, their representations and nature of operations before focussing on calculation.
- As the items are focused on calculation of one form or another and equivalence, students in Year 8 may not be 'ready' to do such items.
  - For example, calculating with fractions (item 3.7) would rarely be something that students did in Year 8, we suspect, and perhaps more likely in Year 9. The language of 'a correct method' may also be problematic. We suspect that many WA Year 8 students would not regard any of the choices offered as a correct method, as they would not have been taught the standard algorithm that is involved in the question. They would certainly not have spent much time practicing the use of the algorithm, either, even if they had seen it.
  - In Item 3.12 knowing how to calculate percentages of quantities, knowing that division is not commutative is more likely to be familiar to Year 9 students than to Year 8.
  - Item 3.8 involves solving a proportion, and we would be surprised if Year 8 students were experienced at such things, which are very hard. They may do it in Year 9 or even Year 10 level.
  - Item 3.13 is described as Reasoning, with which we agree, although it does depend on thinking about calculations with fractions and decimals (as well as a level of comfort with the algebraic representation of $P \times Q = N$. It would surprise us if Year 8 students were familiar with all the necessary things to get this right. Year 9 is a more likely year for that.
  - Item 3.10 is also more likely to be Year 9 work, involving calculation with fractions.
  - Item 3.11 involves conversion from fractions to decimals and rounding to a specified number of decimal places. We would place this as more typical of Year 9 than Year 8 also.
  - The only other item we would place beyond Year 8 is Item 3.6, which would seem to fit at the end of the Calculate section, involving estimation. It's the idea of a 'best' estimate that suggests to us that it is more characteristic of Year 9 than Year 8.
- Year 8 students in WA are expected to do Items 3.1, 3.2, 3.3, 3.4, 3.5 and 3.9. In these items their achievement ranged from 67% - 82% compared to 92% - 96% for students in Singapore. The significant difference in the achievements, aside from cultural and motivational factors, may be a consequence of time on task. Assuming that the time in school devoted to maths is the same in the two

countries our hunch is that Singapore students may do more maths practice outside school than typical Year 8 students in Australia do. While most Year 8 students would get homework, it would generally be a small amount (and often regulated by the school), and frequently not done by lots of students. The idea of practising something to reach mastery levels would not be typical Year 8 behaviour (at least for the bottom half of Year 8 students), we think.

Table 3. TIMSS 2011 Grade 8 mathematics released items for the topic Fractions & Decimals.

Item no TIMSS ID Type Cognitive Domain	Description of Item	Percent correct		
		AU	SG	Int Avg
3.1 M052216 MC Knowing	Which number is equal to $^3/_5$? A 0.8 B 0.6 C 0.53 D 0.35 Answer key: B	70 (2.7)	96 (0.7)	68 (0.3)
3.2 Mo52231 CR Knowing	42.65 + 5.748 = Answer: _______________ Correct response 48.398	82 (2.0)	94 (0.8)	72 (0.3)
3.3 M042032 MC Knowing	Which fraction is equivalent to 0.125 ? A 125/100 B 125/1000 C 125/10 000 D 125/100 000 Answer key: B	67 (1.8)	93 (0.9)	70 (0.3)
3.4 M042024 MC Knowing	 Which number does K represent on this number line? A 27.4 B 27.8 C 27.9 D 28.2 Answer key: 27.8	73 (2.3)	94 (1.0)	54 (0.3)
3.5 M032094 MC Knowing	$^4/_{100}$ + $^3/_{1000}$ = ______ A 0.043 B 0.1043 C 0.403 D 0.43 Answer key: A	68 (1.8)	92 (1.1)	62 (0.3)
3.6 M032166 MC Knowing	Which of these is the BEST estimate of (7.21 × 3.86)/10.09 ? A (7×3)/10 B (7×4)/10 C (7×3)/11 D (7×4)/11 Answer key: B	66 (2.5)	92 (1.1)	57 (0.3)
3.7 M052228 MC Applying	Which shows a correct method for finding $^1/_3 - ^1/_4$? A (1–1)/(4–3) B 1/(4–3) C (3–4)/(3×4) D (4–3)/(3×4) Answer key: D	34 (2.7)	83 (1.5)	37 (0.3)
3.8 M042031 MC Applying	The fractions $^4/_{14}$ and $^\square/_{21}$ are equivalent. What is the value of $\square$? A 6 B 7 C 11 D 14 Answer key: A	45 (2.2)	83 (1.6)	50 (0.3)
3.9 M042041 MC Applying	A workman cut off $^1/_5$ of a pipe. The piece he cut off was 3 metres long. How many metres long was the original pipe? A 8 m B 12 m C 15 m D 18 m Answer key: C	79 (1.7)	92 (1.1)	70 (0.3)
3.10 M032064 CR Applying	Ann and Jenny divide 560 zeds between them. If Jenny gets 3/8 of the money, how many zeds will Ann get? Answer: ____________ Correct response: 350	34 (2.2)	76 (1.6)	27 (0.3)

3.11 M032725 CR Knowing	Write $3\,{}^{5}/_{6}$ in decimal form, rounded to 2 decimal places. Answer: _______ Correct response: 3.83	31 (2.4)	73 (1.5)	25 (0.3)
3.12 M052214 MC Knowing	Which of these number sentences is true? A ${}^{3}/_{10}$ of 50 = 50% of 3 B 3% of 50 = 6% of 100 C 50 ÷ 30 = 30 ÷ 50 D ${}^{3}/_{10} \times 50 = {}^{5}/_{10} \times 30$ Answer key: D	36 (2.1)	67 (1.6)	41 (0.3)
3.13 M032662 MC Reasoning	 Answer key: D	23 (2.1)	45 (2.0)	23 (0.3)

Legend: AU – Australia; SG – Singapore; Int Avg – International Average;
MC – multiple choice item; CR – constructed response item
Standard errors are shown with ( ).

Concluding remark

It is evident that curriculum alignment across the states as well as push towards mastery of concepts and skills taught at respective grade levels are necessary for improvement of achievement scores of Australian eighth graders in future TIMSS cycles of testing.

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DO YOU SEE WHAT I SEE?

ROSE KNIGHT
Australian Catholic University
rose.knight@acu.edu.au

ANN DOWNTON
Australian Catholic University
ann.downton@acu.edu.au

Spatial visualisation is an important aspect of everyday life. It is also an important tool for success in mathematics and science studies. Research has shown spatial visualisation is challenging to teach and difficult to learn. Experiences with geometrical figures and solids in the primary years supports the development of this important skill. This paper, reports on a one week intervention in a Year 5/6 class that focused on students' spatial visualisation skills.

We think spatially in many everyday situations such as interpreting a map, putting flat pack furniture together, interpreting a diagram and packing a suitcase. Shapes and their manipulation are so much a part of our daily lives that we hardly notice the actions involved (Ehrlich, Goldin-Meadow & Levine, 2006). Information is often presented in a visual format that requires interpretation and processing to make sense of it, but many people do this so naturally that rarely do they stop to consider the ways in which they are thinking (Presmeg, 1986; van den Heuvel-Panhuizen & Buys, 2005).

In mathematics, this type of thinking takes on a deeper significance. As indicated by Battista (2007), visualisation is an important tool for success in developing geometric concepts which are associated with success in mathematics and science. These disciplines form the basis for further study or employment in the fields of mathematics, science, technology, engineering, medicine and arts (Newcombe, 2010). Experiences with figures in the primary years of schooling may assist in abstract reasoning about reflection in secondary school mathematics (van den Heuvel-Panhuizen & Buys, 2005).

Lohman (1979) suggests that spatial visualisation is the ability to construct, retain, retrieve and manipulate visual images, and underpins much of the geometry in secondary school. Although spatial visualisation is not explicitly stated in the Australian Curriculum, Mathematics, (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2012), it is explored under the Geometry and Measurement content strand. In USA, the National Council of Teachers of Mathematics (NCTM) Standards (2000) states "students should develop visualisation skills through hands on experiences with a variety of objects. Later they should become comfortable in drawing perspective views, counting component parts and describing attributes that cannot be seen but can be inferred" (p. 43).

Providing opportunity for students to experience hands on learning in the context of spatial visualisation can be challenging for teachers, as many teacher resources use two dimensional (2-D) representations of the real world that need to be reinterpreted as three dimensional (3-D) objects. Using images in place of objects can be difficult, yet it is necessary that students learn to interpret the 2-D representations of their world. Students need to be able to make sense of the information from 2-D images as well as representing this information about the real world with 2-D drawings.

Spatial visualisation is challenging to teach and difficult to learn. When working with teachers on a project to improve students' capacity to use 3-D models and represent them as 2-D representations, Nivens, Peters and Nivens (2012) found that many of them were apprehensive about teaching the topic as they "lacked the confidence in their geometric abilities and their knowledge of the vocabulary" (p. 346). If teachers are not confident with thinking spatially and/or cannot recognise when they are thinking spatially, then they may have some difficulty in adequately teaching their students about this type of thinking.

Researchers have nominated strategies which may improve the way students develop spatial skills. These include:

- having teachers and parents understand what spatial visualisation is and the kinds of pedagogical activities and materials to support its development (Newcombe, 2010);
- encouraging, supporting and modelling engagement in age-appropriate tasks of a playful nature. This can be through books, language, providing opportunities to imagine, jigsaw puzzles, and making sketches of 2-D and 3-D shapes;
- using technologies that are useful in developing spatial ability such as taking photos with a camera, to look at different points of view. Computer software is an option, although many teachers will not be as familiar with relevant programs, as they were not exposed to them when at school. These programs include drawing programs such as Geogebra and LOGO (Battista, 2007) as well as computer games like Tetris (Newcombe, 2010);
- providing concrete experiences that allow children to build, draw and read drawings (Ben-Chaim, Lappan & Houang, 1988).

This paper describes a 60-minute lesson that was part of a one-week classroom intervention in a Grade 5/6 class (10 to 12 year olds). The focus of the study was to gauge the impact of such an intervention on students' spatial visualisation skills relating to 2-D representations of 3-D objects. An overview of the lesson sequence provides some background to the task selection, the content, materials used, and where the lesson described in this paper was situated.

Overview of the lessons

The lessons discussed in this paper were adapted from tasks on the *Maths 300* website (http://www.maths300.esa.edu.au) and *Mathematics in Context—Side Seeing* (National Center for Research in Mathematical Sciences Education & Freudenthal Institute, 1997). These lessons were chosen because of their problem solving nature, and their clear connections to real life. The students were provided with an opportunity to play, collect data and use problem solving strategies and basic skills, to prove or disprove any theories which may arise (Lovitt, 2000).

The following aspects of the representation and the concrete objects were considered when planning the lesson sequence.

1. Picture to object view—from a picture create an object using manipulatives.
2. Object to picture—from an object create a picture.
3. Picture to picture—which of the following shapes would fit in to complete the large object.
4. Word to picture—listening to instructions and creating a model from it.
5. Words to object (Bishop, 1977).

The materials used in the lessons were chosen because of their flexibility and usefulness. As the use of manipulatives, such as the whiteboard, in the lesson proved to be a novelty for the students, 'play' time was incorporated into the start of subsequent lessons, before introducing a 'tuning in' activity. In the first lesson, students were asked to look at everyday items taken from home (drink bottle, milk bottle, football, etc.) and draw them from different perspectives. Cubes were used in the other lessons as they are the basic units for the 3-D dimensional objects. All the lessons allowed the students to build, draw and evaluate 3-D objects. They also provided the opportunity for students to manipulate materials.

The first two lessons introduced the students to drawing objects from different points of view, i.e., taking an everyday 3-D object, and representing it as a 2-D picture.

- Lesson One: Point of View, in which the students were introduced to the concept that looking at objects from different perspectives produces different results. Also included in this lesson was the challenge of creating a sketch of a 3-D object in 2-D. Students were also challenged to imagine what they would see if they were sitting at the other side of the room, and to draw this image.
- Lesson Two: Cube Nets, had the students creating a cube from GeoShapes and determining which nets will create a cube. This lesson was to highlight the fact that all 3-D objects have a 2-D representation. It also provided the opportunity to view data not as numbers, but as the nets they had created. From this, the challenge was to see if there were any patterns evident from the data.

In subsequent lessons, students were provided with the opportunity to manipulate cubes and objects in order to see them from different points of view: front, left, right, back and top. They also drew the objects and buildings in two different ways, orthogonal (front, side and top views), which is like a plan, as well as isometric (using triangular dot paper to draw cube buildings side on). Using the isometric dot paper required the student to turn the cube so that it was viewed from a corner. Reading the "plans" and creating the building was another aspect of the lessons. An outline of each of these three lessons follows:

- Lesson Three: Blocks and Buildings helped the students to develop their ability to communicate and reason about 2-D representations of 3-D objects, and highlighted the importance of language. It was important for students to realise that not all elements of a building are in view and that some blocks are hidden, depending upon the viewer's position.
- Lesson Four: Block Views followed on from the previous lesson where students realised that in some 3-D drawings it is inconclusive as to how many blocks might have been used to make a building, as some could be obscured, and therefore top views could be different. Block views highlighted that the top view alone does not

always give the information about the number of stacked blocks in each section of the building; several versions of the building are therefore possible.

- Lesson Five: Four Cube Houses, in which the students were architects designing houses, each using four cubes, for a housing estate. The first investigation was to find the 15 different constructions (given particular constraints) that could be made from four cubes, and then the students constructed plans of all the houses for council, using isometric drawings.

The remainder of this paper describes the first lesson, Point of View. The lesson began with the students seeing three everyday items, a mug, drink bottle and milk container placed on a table in the centre of the room. The table could be seen by all students from their seats. We were aware that the students had not had any experience in the mathematics classroom in relation to perspective drawings. Students were asked to draw what they could see of the objects in the centre of the room (Figure 1). The use of the mini whiteboard was intentional as it allowed the students to ‘have a go’, and should they not be satisfied with the result their initial sketch it easily be erased. The ‘have a go’ attitude is one we were hoping to encourage throughout the series of lessons as this would be when we believed that real learning would take place.

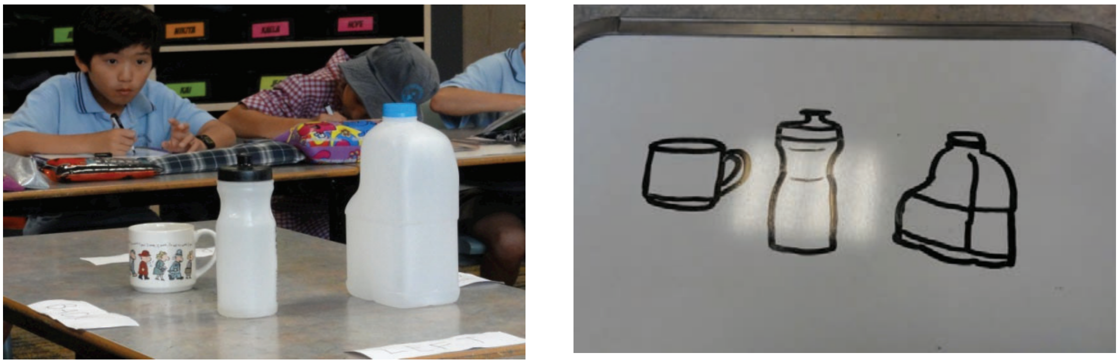

Figure 1. The objects on the table and an example of one student’s representation of these.

Once students had completed their sketch, we asked three volunteers from different positions in the room (who had done a reasonable job) to bring their sketches to the front for all to see (Figure 2). These students described their sketches to the class and then we posed the following question, “We are all looking at the same objects, so why are our sketches different?”

The language the students used to describe the positions of the objects on a table was clear as they were able to correctly use positional language such as middle, left, right, front and back. We were also able to determine that one view had to be identified as the front view. Signs were placed on the table to indicate each of the views. The top view was described as a ‘bird’s eye’ view. The students were also able to confidently discuss why it depended on where you were standing to get a particular view.

Figure 2. Students share their sketches with the class.

A different set of objects was then placed on the table. Students were asked to divide their white boards in half by drawing a line with their marker. On one side of the board they were to sketch what they could see from their seats. The other half of the board was for a sketch of what they thought they may see if they were sitting on the opposite side of the room (Figure 3).

Figure 3. A student's sketch of what she saw from her seat, on the left and what she imagined, on the right.

On completion of their sketches the students placed their boards on their seats and moved around the room to look at each other's sketches. During our walk, one girl let out a shriek, which grabbed the attention of all in the room. With everyone looking at her she embarrassingly said that she did not even know there were three objects on the table because from her seat she could only see two. This was a perfect lead into a real teaching moment, that often in sketches information may be missing and a form of abstraction is required (Battista, 2007).

In the class discussion that ensued students mentioned imagining, however the terms visualise or visualisation were not mentioned. They noticed that the sketches were similar to the objects, but that some of the information was missing. From this we were able to highlight that the 3-D objects may be curved, but the curves cannot always be easily shown in a 2-D representation, if they are facing you and we have to use strategies to find a solution. Some of the strategies they suggested included imagining themselves at the other side of the room viewing the objects, or physically place themselves in the position to see the object from another viewpoint.

To conclude the lesson, students were asked to draw a top view of the five items on the table. They were then given five photos of the five objects from different viewpoints around the table (taken prior to the lesson). Each photo had a letter in the corner as a reference for the students to use. The students' task was to indicate on their drawing on their whiteboard where the photographer was standing when taking each photo. The purpose of this task was to highlight further that what we see is very much dependent on where we are standing. Often we are not able to see some aspect of the object and must infer what an image may be. Imagining a different perspective is still very difficult for 11 year olds as they still have egocentric responses (Rigal, 1996).

At the conclusion of the lesson the students, were posed the following question "What mathematics have we attended to in our lesson today?" One student replied, "None, we were doing art". Another agreed with this statement, as we were drawing and had not used numbers.

Insights from this lesson

As the students developed their understandings, they knew what to do, and they also improved their skills to mentally visualise and manipulate these sketches. The opportunity to physically manipulate objects and make connections aided this development. Exposing the students to an activity and manipulatives alone, will foster some improvement, however questioning students and asking them to explain their mathematical thinking is vital. The discussion relating to what the objects might look like from different viewpoints revealed the different strategies that drew on their use of spatial visualisation. Rigal (1996) mentioned they may also imagine the rotation of the objects, but this was not evident in this lesson. Providing students with more opportunities to engage in experiences such as this may have revealed the use of this strategy and opportunities for students to make connections between 3-D objects and their 2-D representations.

It was interesting to note that throughout the intervention, those students who were considered "good" at mathematics often found some of the drawing very challenging and were not comfortable in situations where their sketches were on display. On the other hand, some students who were considered "weaker" at a number of components of mathematics were very engaged and interested and proved generally successful in tasks. Wheatley and Wheatley (1979) suggested that this may be dependent on which side of the brain the information is processed.

For some students, making a start on this task proved quite a challenge. Prompts, which have been described as "strategies that can be directed at students when they need to be more supported, rather than have the students listen to additional explanations" (Sullivan, Mousley & Zevenbergen, 2006), were used throughout the lesson. An example of such a prompt was the use of the screen of the digital camera. Taking the picture at eye level and reviewing it was converting the 3-D object to a 2-D representation (Figure 4). The simple prompt was able to focus the students' attention as to what they were required to draw.

Figure 4. Using the camera as a prompt enabled some students to make links and correctly understand the task.

The lesson described in this paper highlights ways in which students can engage in hands on learning experiences relating to spatial visualisation. Providing manipulatives, which are everyday items in a student's environment, and experiences that enable them to represent 3-D objects as 2-D representations may assist them to interpret the 2-D representations of 3-D objects. As teachers we need to have an appreciation of the complexity associated with spatial visualisation in order to guide our students. Although only one lesson was described in this paper it is evident that a five day, short sharp, focused, intervention can be a useful vehicle for increasing students' spatial visualisation skills, in particular 2-D representations of 3-D objects.

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BUILDING A SOLID FOUNDATION FROM WHICH TO LAUNCH OUR FUTURE MATHEMATICIANS

ROBIN NAGY
Cranbrook School, Sydney, NSW
rnagy@cranbrook.nsw.edu.au

It is essential to retain a focus on building students' mathematical reasoning and comprehension rather than merely developing superficial understanding through procedural learning. All too often this approach 'takes a back seat' because of examination and assessment pressure, where the importance of 'How?' supersedes that of 'Why?' It is not *what* we teach that is important so much as *how* we teach it. This session explores conceptual methods in the teaching of Secondary mathematics. It will appeal to both new and seasoned teachers, providing food for thought and suggesting practical approaches to teaching mathematics for understanding rather than regurgitation.

Introduction

Many teachers of mathematics find the time pressures and constraints of examination and assessment driving them towards teaching by rote learning instead of developing in their students a deep conceptual understanding of the material being covered. When students embark on university courses, their subsequent ability to cope with new material and novel problems and applications is hampered by their lack of solid mathematical foundations. Teachers need to be encouraged to present mathematics in a variety of ways which enhance the systemic understanding of concepts and the development of a systematic methodology.

Political and social pressures of examination-based assessment and achievement standards have inevitably dictated what and how we teach. The teaching of mathematics has, in many secondary school classrooms, become so dominated by assessment that 'the tail is wagging the dog'. A preoccupation exists to equip students with the skills necessary to 'pass the test' and this in turn, prescribes a procedural approach to teaching mathematics. This 'How' based style of teaching leads to concepts not being properly taught and understood, due to perceived time pressures teachers to get their students 'up to speed' on examination-style questions. Instead of examinations existing to assess mathematical knowledge and reasoning, they are seen as the *raison d'être* of the course and students are taught on a 'need to know' basis with exploration beyond the constraints of exam questions actively discouraged in many mathematics departments. Unfortunately, teaching 'to the test' is often an effective method of achieving good marks and it is possible to achieve creditable performances in mathematics examinations without really understanding any of the underlying concepts. By contrast, 'enrichment' is often seen as an intangible add-on for the brightest classes, an alternative to 'acceleration' and one which does not bring with it any concrete benefits.

In my opinion good teaching and good examination results are not mutually exclusive; indeed there is a strong positive correlation between the two. Mathematics teachers have an obligation to ensure that the 'Why' is taught together with the 'How' and that students' examination performance is indicative of their general comprehension of the subject. Learning is seldom a linear process and in order to develop mathematical deductive reasoning, students will necessarily need to struggle to develop their own understanding and reasoning processes, with plenty of bumps and hiccups along the way. In the words of Lao Tsu (Giles, 1905, p. 45) "Failure is the foundation of success, and the means by which it is achieved."[5] This can be disconcerting for teachers and students alike and both must be prepared for a turbulent journey. Much of what is presented here could be labelled 'enrichment', which has often come to mean 'more than just teaching what they need to know (to answer questions)'. My firm contention is that the methodology of all teaching should endeavour to include such 'enrichment'.

Example topic 1: Pythagoras' Theorem

I have chosen to look at Pythagoras' Theorem by way of an example topic, to demonstrate the two different approaches to teaching mathematics, the procedural or 'How' and the conceptual or 'Why' approach.

A 'How' approach would involve teaching the formula $c^2 = a^2 + b^2$ for a right-angled triangle, explaining how to identify the hypotenuse and showcasing examples of typical questions which occur: finding the hypotenuse, finding one of the other sides, applying to 'real world' questions. This could be achieved in a few lessons with little or no conceptual enlightenment attained in the areas of mathematical proof or method, but rather a superficial understanding of how to answer questions based on a formula which we call Pythagoras' Theorem.

By contrast, a 'Why' approach might introduce the topic with a hands-on investigation such as Perigal's Dissection (Figure 1). Created in 1838 by Henry Perigal (1801–1898), a London Stockbroker and amateur mathematician, the construction consists of a right angled triangle with squares drawn on each of the sides. One of the adjacent sides is then dissected by drawing lines through its centre, parallel to the sides of the largest square and the four quadrilaterals formed can be rearranged, together with the square on the other adjacent side, to fit exactly inside the square on the hypotenuse. This is not a formal 'proof', but is a good graphical illustration and introduction to Pythagoras' Theorem; indeed Perigal (1891) postulated that Pythagoras probably discovered his theorem with a similar if not identical approach. The students can then be asked to propose a generalisation of their result using algebra. This is a constructivist approach to teaching Pythagoras' Theorem which can then be followed up by some examples demonstrating a more formal rigorous proof. It is a more powerful technique than the 'How' approach, as it encourages students to build their

5 Lao Tsu (c. 604–531 BC) was the founder of Taoism.

own knowledge and conclusions and to generalise their results. Not only will students be less likely to confuse the hypotenuse with the other two sides if they have this graphical foundation, but they will be more likely to remember and understand the theorem in the long-term.

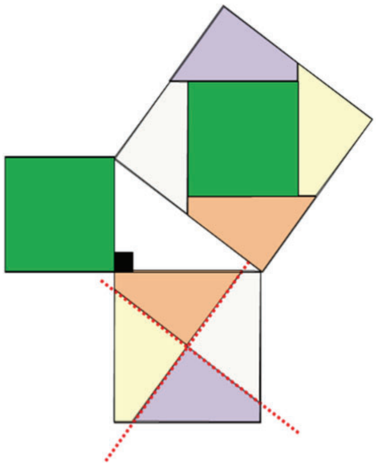

Figure 1. Perigal's dissection—a graphical illustration of Pythagoras' Theorem.

There are many proofs of Pythagoras' Theorem and students should be exposed to some of these, in order to understand the important mathematical concept of proof and its essential role in forming the structure of mathematical reasoning. Animated graphical proofs can be found on sites such as YouTube, for instance: http://www.youtube.com/watch?v=ajuUO8h0IxY and a variety of algebraic proofs are readily available from text books and online. Pythagoras' Theorem can then be applied to standard problems involving right-angled triangles with the students conceptually understanding the underlying 'truth' behind the theorem.

Example topic 2: The distance formula

Many teachers teach their students to find the distance between two points (x_1, y_1) and (x_2, y_2), using the 'distance formula': $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. But this formula could just as correctly be written: $d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$ or $d=\sqrt{(x_2-x_1)^2+(y_1-y_2)^2}$. This unwieldy formula simply represents Pythagoras' theorem where the two 'adjacent' sides are the difference in the x and y values of the coordinates. There are any number of such 'rules' to learn for a typical Senior Mathematics course and it is tempting to just tell students to 'learn' and apply the formulas by rote without attempting to explain where they come from.

A 'why' approach to teaching this topic emphasises that the distance between two points can be found by looking at the right-angled triangle formed by the difference in x and y coordinates (see Figure 3). The distance squared is the difference of the x coordinates squared plus the difference of the y coordinates squared. This provides a wonderful opportunity to introduce the symbols Δy and Δx meaning 'a change in' y and x respectively, long before calculus appears on the scene. It also provides a much more understandable formula: $d^2=\Delta x^2+\Delta y^2$ and facilitates the comprehension of the difference between, say, x values of 2 and –1 being 3 rather than 1 (these two values straddling the y-axis).

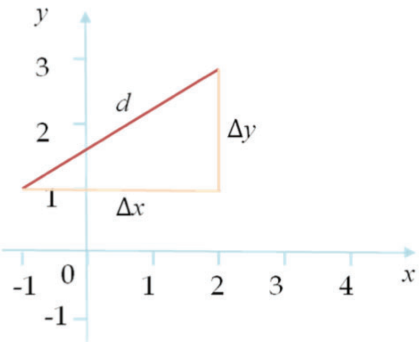

Figure 2. Considering Pythagoras' Theorem, the distance, d, between two points is the hypotenuse of a right-angled triangle, with adjacent sides formed by the difference in x and y values, Δx and Δy respectively.

Thus: $d^2 = \Delta x^2 + \Delta y^2$.

Why do we learn mathematics? Where will I use this in life?

Perhaps the most haunting question in mathematics teachers' classrooms is: "Where is this going to be useful in life?" It is a common misconception that topics within mathematics are useful in most people's lives. Educators, politicians, text book and syllabus writers frequently fall into the trap of attempting to justify the teaching of mathematics by rationalising its usefulness to the real world. However, attempting to validate the place of mathematics in our curriculum merely on the grounds that it is 'useful' does the opposite. Excusing the learning of mathematics as merely being a 'useful' skill, minimises our discipline to one of utilitarianism. No other subject experiences such a pressure to validate its place in the classroom in terms of 'usefulness' to life. When do most people 'use' Art, English literature, music theory, history, or the sciences on a daily basis?

Whilst 'numeracy'[6] may indeed be useful, the sort of mathematical topics and procedures we teach from Year 7 onwards are not 'useful' in most people's day to day life. They *can* be invaluable in specific situations and specific occupations, but even as a teacher of mathematics, I do not 'use' simultaneous equations or trigonometry very often outside my teaching.

But that does not mean that it is not important to learn mathematics as a rigorous academic discipline. Mathematics *is* important. It is an abstract system of logical, deductive reasoning and methodology, which is pure and perfect (i.e., true). This discipline *is* useful, as it allows us to engage and communicate in higher order and abstract thinking across a spectrum of subjects and life events. Mathematics is the only thing we can 'prove' to be correct (based on some fundamental axioms). For most students, learning mathematics can be considered as a mental parallel to weight training. This analogy is a very effective way of explaining to students the importance of

6 In fact the word 'numeracy' is an ill-defined but ubiquitous term whose meaning appears to be commonly understood in politics and education to suit the given situation, but it was in fact invented by a committee in 1959 (the Crowther Report on UK Education) to represent the 'mirror image of literacy'.

correct and systematic method and for addressing 'Why are we doing this?'-type questions. Of course, recent brain research highlighted in books such as *The Learning Revolution* by Dryden and Vos (1999) suggests that the process of studying (mathematics or otherwise), grows dendrites and makes connections between the neurons in the brain in an analogous way to weight training creating muscle fibre and toning the physique.

So the real benefit, to most students, of studying mathematics is that it develops their higher-order skills such as deductive reasoning and logical and organisational thinking. Once their brain has been developed in this way they *will* be able to use and further develop these 'brain muscles' in any number of useful contexts in their lives, utilising pathways and connections originally developed in the mathematics classroom.

We can say that the 'effects of studying mathematics' are extremely useful for everybody!

Maths makes you 'mentally fit'

However, like weight training, brain development with mathematics only works effectively if you are doing the 'exercises' correctly. We like to find ways of doing exercises which do not 'hurt' and are easier, but they do not necessarily yield the same results. Another analogy I frequently use is to liken my role as a mathematics teacher to that of a personal trainer. A personal trainer may 'spot'[7] a client who is bench pressing, but that client will only be improving muscle tone if he himself is doing the majority of the work in moving the weights up and down (which involves intensive effort and focus). Once the personal trainer becomes the main source of power in moving the bar, the client is simply holding onto the bar whilst the personal trainer gives his own arms a good workout, lifting it up and down. From an observer's perspective the two situations appear identical, but only the former will yield muscle development in the client. It is easy to presume that just because students are answering mathematical questions, they are successfully learning, but I would contend that it is *how* they are learning which matters most in their brain development. Thus, the *process* is as all important in developing mathematical reasoning as it is with weight training in building and toning muscle fibre. These are all good analogies which can be used with students to convince them to set their work out correctly and take the time to work at a speed where they can be assured that each stage of their working is 100 percent correct.

The method and process are the *only* things which matter in studying mathematics, not the answer—that is usually in the back of the book!

Example topic 3: Rearranging equations—a focus on method

Traditionally, rearranging equations was taught by learning the four 'rules' of taking items 'over the equals sign'. Adding became subtracting and vice versa and multiplication became division, but confusion often existed with which number was divided by which and why? Fortunately most teachers now use the analogy of a balance beam and explain that in order to maintain balance, whatever you do to one side of an equation you must do to the other. There are now many excellent animated visual

7 'Spotting' in weight training means assisting in pushing a weight. Typically in order that the client maximises their physical capability to ensure optimal muscle development.

resources, such as the 'Algebra Balance Scales' from the National Library of Virtual Manipulatives (Utah State University, 2010) which can be used to reinforce this analogous idea of an equation as a balance beam. This is a 'why' approach, but one which needs to be further enforced with a rigorous adherence to method.

I insist that my students write down at every stage what they are doing to both sides of an equation in manipulating the algebra (see Figure 3). This is the 'metacognition' of mathematics. I also insist on lining up the equals signs, so the correspondence to the fulcrum of a balance beam is maintained. I encourage students to use coloured pens and write down what they intend to do to both sides *before* writing the next line in their working. To this end, I always have a set of brightly coloured pens in my classroom and gladly give them out to students who want to use them (for the metacognition only!). It also helps if the ink smells of strawberries!

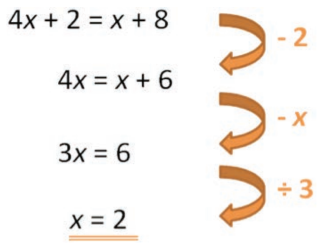

Figure 3. Setting out equations with the metacognition on the right hand side.

I emphasise to students that I am not particularly interested in the answers, which appear in the back of the book in any case—these are merely a way to check whether the working is error-free—I am only interested in the 'process instructions' which take you from line to line. I commend good working with a system of reward stamps and do not reward correct answers which are not set out in this exemplary manner. Initially students find the process monotonous, but so, I argue, is lifting weights—once they see how easily the result 'falls out', and how neat their work looks, they take it in their stride. I *always* set my work out like this on the board (at every level—including the highest level of Senior Mathematics—we must exemplify what we preach!).

Inverse (or 'undoing') operations

I believe it to be important to introduce the concept of inverse (or 'undoing') processes as early as possible in Year 7 or 8. For instance, the inverse (or undoing function) of +3 is –3 (see Figure 4).

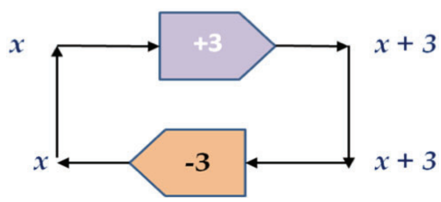

Figure 4. Function 'machines' showing the inverse processes +3 and –3.

If students are fluent with the concept of an inverse, it comes as no surprise that inverse trigonometry functions are required to 'undo' trigonometric functions. For instance, the inverse (undoing function) of sin( ) is called arcsin( ) or $\sin^{-1}$( ) (see Figure 5). I also like to use analogous ‘real-world’ functions and their inverses, such as silver plating and de-silver plating and discuss how different a function and its inverse are, typically, to each other. We also discuss ‘self-inverse’ functions such as the reciprocal function or, for instance the function $10-x$.

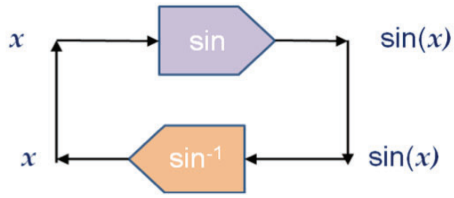

Figure 5. Function ‘machines’ showing the inverse processes sin( ) *and* $\sin^{-1}$( ).

We can see how to apply this to solve an equation $5=\dfrac{2}{\sin\theta}$ in Figure 6.

$$5=\frac{2}{\sin\theta}$$
$$\downarrow \times \sin\theta$$
$$5\sin\theta=2$$
$$\downarrow \div 5$$
$$\sin\theta=\frac{2}{5}$$
$$\downarrow \sin^{-1}(\;)$$
$$\theta=\sin^{-1}\left(\frac{2}{5}\right)$$
$$\underline{\theta\approx 23.58^{\circ}\,(4s.f.)}$$

Figure 6. Setting out correct metacognition (or ‘what are you doing at every step to both sides’) including an inverse trigonometric function to ‘undo’ the function sin( ).

Similarly, the inverse of an exponential function is called a logarithm (see Figure 7). This is how I introduce the topic of logarithms and builds a conceptual ‘why’ understanding rather than the more usual procedural definition of logarithms, adopted by most teachers and text books.

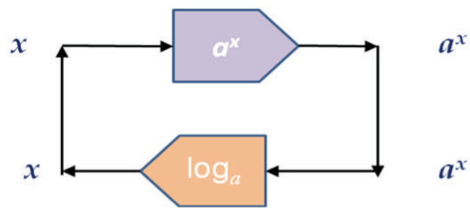

Figure 7. Function machines showing the inverse (undoing) of an exponential function a^x.

Logarithms can then be used to ‘undo’ their respective exponential functions as in Figure 8. Where the equation $2^{x+3}=91$ is solved in this manner.

$$2^{x+3}=91$$
$$\downarrow \log_2(\;)$$
$$x+3=\log_2 91$$
$$\downarrow -3$$
$$\underline{x=(\log_2 91)-3}$$

Figure 8. Using the base 2 logarithm function to ‘undo’ a base 2 exponential function 2^x.

Frustratingly, most calculators do not have a base n logarithm function, although this is beginning to change with new models[8]. Nevertheless, students can apply base 10 logarithms to generate a numerical answer to the same equation as can be seen in Figure 9, using logarithm rules. This generates a solution which is equivalent to that of Figure 8, and deduces the 'change of base' rule (we can conclude from this example that $\log_2 91 = \frac{\log_{10} 91}{\log_{10} 2}$).

$$
\begin{aligned}
2^{x+3} &= 91 && \log_{10}(\ ) \\
\log_{10} 2^{x+3} &= \log_{10} 91 && \\
(x+3)\log_{10} 2 &= \log_{10} 91 && \div\log_{10}(2) \\
x+3 &= \frac{\log_{10} 91}{\log_{10} 2} && -3 \\
x &= \frac{\log_{10} 91}{\log_{10} 2} - 3 &&
\end{aligned}
$$

Figure 9. Using the base 10 logarithm function to 'undo' a base 2 exponential function 2^x.

Conclusion

Mathematics is a science but mathematics teaching is an art; invention is the key to inspirational teaching and learning. It is important to keep fresh as a teacher of mathematics; to come up with new analogies and ways of explaining topics and never to be afraid to try something new and 'off-the-wall'. Sometimes it works and adds to your repertoire and sometimes it does not. But even if the analogy falls down or you realise you could have explained it better after you have already made an attempt, never forget that it is the *process* that is important—students will be learning from your mistakes as well as your polished set-pieces! Very often students will learn more from what goes wrong and how you (and they) work out what the problem is.

Education is all about the journey; this applies to the cyclical process of struggling, persisting and overcoming obstacles in producing new understanding and capacity for thought. It also applies equally to the outcome of education which hopefully remains with us long after we have left the classroom environment. American athlete Greg Anderson tells us to: "Focus on the journey, not the destination. Joy is found not in finishing an activity but in doing it." This is equally true of teaching and learning and is further epitomised by a quote from the tennis player Arthur Ashe: "Success is a journey, not a destination. The doing is often more important than the outcome." The word 'success' could easily be replaced with 'education'.

To use another of my analogies, there are several ways to guide a group through a forest:

- You can take them on well known tracks, enabling them to navigate the same track time and again, quickly and efficiently. This is not only boring, but virtually

8 It is alarming that some syllabuses do not allow these calculators to be used in their examinations as it does not allow meaningful assessment of the 'change of base' logarithm rule, which ironically has only come to prominence as a rule, due to the lack of a base n logarithm function on calculators; a perfect example of the 'tail wagging the dog'.

useless for future life unless they happen to be in the same forest on the same track.

- You can take them 'off-piste' using your own navigational skills. This will give them more of a sense of adventure and demonstrate that it is possible to reach the same destination in a variety of ways.
- You can teach them how to navigate themselves. You will be teaching them the skills which they can use time and again in their lives in many new and varied situations.

I enjoy inventing and taking new paths each time I teach, and believe that by doing so, I am equipping my students with the flexibility skills they need to be able to negotiate any new situation. Keeping notes for me is a sure way to become stale and I prefer to 'reinvent the wheel' with every new class, as an important strategy to keeping fresh, on my toes and exciting as a teacher. When a path leads nowhere, it is sometimes the best educational experience for your class (and perhaps for you too); in other words, when you make a mistake. You should never be frightened of this and indeed should emphasise it. You are a teacher, not a mathematics genius, but you might well be teaching one!

It is important to ask yourself 'Why?' when you teach every new concept and to avoid teaching 'recipes'. Let your students develop procedures of their own from the conceptual understanding they glean. If they are unable to do so, they may not have totally understood the concepts and could need more work on the foundations. Occasionally I feel compelled to teach 'recipes', especially where a syllabus appears to be specifically written to test recall of a formula rather than conceptual understanding of it; or when the constraints of departmental homogeneity dictate the length of time I am able to spend on a particular topic, but I always endeavour to explain to students my reasons for this and to revisit the topic at a later date if possible, for deeper conceptualisation.

The seventeenth century English statesman George Saville once remarked: "Education is what remains when we have forgotten all that we have been taught". I believe that the 'why' is the educational constituent of learning mathematics.

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CONSTRUCTING KNOWLEDGE OF THE FINITE LIMIT OF A FUNCTION: AN EXPERIMENT IN SENIOR HIGH SCHOOL MATHEMATICS

PHAM SY NAM
Phan Boi Chau Gifted High School
Nghe An Province, Vietnam
phamsynampbc@gmail.com

MAX STEPHENS
University of Melbourne
m.stephens@unimelb.edu.au

A finite limit of a function is a difficult mathematical concept, even for good students. It is a key to the study of many areas of mathematics. Textbooks typically introduce two kinds of definitions of the concept of limit: a sequence version and an epsilon-delta version. Even able students find understanding both definitions difficult. Using a constructivist approach, this study used tasks that support students in constructing the concept of the finite limit of a function. Dynamic manipulations enabled students to form and verify hypotheses, reject the wrong ones and construct knowledge about the finite limit of a function in an easier way.

Introduction

The concept of the limit of a function is difficult to teach and understand. When presenting this concept Vietnamese textbooks for mathematically gifted students start for example with the function $f(x)=\frac{2x^2-8}{x-2}$ and then consider a sequence (x_n) different from 2 and pay attention to answering the question: If $\lim x_n = 2$ then $\lim f(x_n) = ?$ Textbooks typically use a sequence version of limit to arrive at a definition of the form:

> Let the $(a;b)$ interval contain a point x_0 and if f is a definite function on the set of $(a;b)\setminus\{x_0\}$. Then the limit of function f is a real number L as x tends to x_0 (or at points x_0) if every sequence (x_n) in the set $(a;b)\setminus\{x_0\}$ (that is $x_n \in (a;b)$, and $x_n \neq x_0$ with all n) that $\lim x_n = x_0$, we have $\lim(x_n) = L$. (Cf. Definition 1, Vietnam Education Publishing House, 2010, p. 153).

This definition shows the close relationship between the concept of sequence limit and the limit of a function. This creates a number of advantages for the formation of the concept of the limit of a function from the concept of the limit of a sequence, and allows the properties and theorems relating to the limit of a sequence to be transferred to the properties and theorem of the limit of a function naturally. However, to make it easier to define the limit of a function as well as demonstrating some characteristics for function limit, Vietnamese textbooks also present a second definition (ε,δ version) of the form:

Let the interval $(a;b)$ contain points x_0 and f be a definite function on the set of $(a;b)|\{x_0\}$. We say that the limit of function f is a real number L as x tends to x_0 (or at points x_0) if for every positive number ε, there exists some positive number δ such that if $x \in (a;b) \setminus \{x_0\}$, the distance between x and x_0 is not over δ, then $f(x)$ is in distance from L not more than ε" (cf. Definition 2, Vietnam Education Publishing House, 2010, p. 154).

The challenge is how to assist students to be able to link both definitions. They can be assisted to connect Definition 2 with Definition 1 based on observation that when the value x_n is close to x_0 the value of $f(x_n)$ is close to L. This is true for any sequence $x_n \to x_0$ so it should also be true for any number close to x_0. However, it is not easy to realize this.

Finzer and Nick (1998) have argued that dynamic geometry software (DGS) and dynamic manipulations have the potential to create a new approach in teaching and learning mathematics in school. Can the use of dynamic models help students construct knowledge of the limit of a function more easily? In this article, we focus on several research questions: How do students construct knowledge of the concept of limited function through experiments on such a model? Can teaching practice based on the use of dynamic models help students to a guided discovery of knowledge of the limit of a function?

Research framework

From a constructivist perspective, such as that advocated by Confrey (1991), the following key ideas inform the research framework used this paper:

- Individuals' learning is not passive but active, i.e., individuals act on their environment to construct knowledge.
- The process of knowledge construction is developmental and evolutional; it is not static but dynamic.
- Knowledge is not an explanation of truth, but is a rationalization of individuals' experiences. Thus, individually constructed knowledge, even under the same situation, will be different from each another

Knowledge can be formed through the process of inter-influence between previous learning and related new learning. During the learning process, pupils are able to create knowledge by actively involving themselves in using the existing experience so as to solve any contradictions which may arise to achieve a common understanding with the new information.

According to Kant, "all human cognition begins with observations, proceeds from thence to conceptions, and ends with ideas" (Polya, 1965, p. 103). In this sentence the terms "observation", "conception", and "idea" were used. Polya re-expressed the sentence as: "Learning begins with action and perception, proceeds from thence to words and concepts, and should end with training certain new properties of intellectual gift" (p. 103).

In designing teaching within a broad constructivist framework, it is essential to create conditions to guide students' 'self-discovery' of knowledge of the limit of a function. Therefore, to make the building of knowledge successful and to achieve high results without taking too much time, any 'discovery' should be situated in a learning environment with pedagogical purposes carefully planned by the teacher. Applying this to learning about the limit of a function by using dynamic software, students will need help, in particular, to pay attention to the mathematical features of the images they observed.

Research design

In this research design, the following factors are key: the teacher design schedule, the teaching design idea, the design of mathematical tasks, data collection and analysis.

1. Teacher design schedule

In the design of guided learning activities to support the concept of the limit of a function, Stephens (2012) pointed to the following four elements where the teacher:

- Needs to focus on the important key conceptions of the limit of sequence, not focusing too much on general teaching strategies or overall descriptions on the limit of sequence. For example, in teaching the limit of sequence, it is important to focus on activities which make clear the process of confirming a sequence limit of 0, by considering ε as a given arbitrary small positive number, then to prove existence of positive number δ such that that $|f(x)-L|<\varepsilon,\ \forall|x-a|<\delta$.
- Needs to have a clear plan how to respond students' incorrect answers.
- Should have a longer-term plan to consistently develop students' deep understanding of the limit of function.
- Should utilise concrete examples that are familiar and easy for students to understand to help them understand the limit of function and its relationships.

2. Teaching design

As stated at the outset of this paper, the concept of the limit of a function is difficult to teach, particularly to understand the relationship between the two definitions. To meet this challenge, it was important to create activities for students to understand the concept intuitively. Several tasks and related questions were designed in order to assist them to understand correctly the key ideas and to propose their own innovative ideas.

Task 1 is designed to help students understand the concept intuitively when x tends to a then $f(x)$ tends to L. Its goal is to help them to describe clearly how $f(x)$ tends to L when x tends to a. Task 1 was carried out in various levels. Initially, students determine the distance between x and a such that $|f(x) - 8|$ is smaller than some given number.

Then they need to see that there always exists a range of values x such that $|f(x) - 8|$ is smaller than any arbitrary positive number. These ideas are prerequisite for a formal definition of the concepts we desire students to achieve: if given an arbitrary small positive number ε, it is always possible to find a positive number δ so that with $|x - 2| < \delta$ then $|f(x) - 8| < \varepsilon$ After students build concept of limit of function, Task 2 is introduced which is intended to create innovative ideas to support students' formulations of the concept.

3. Design of mathematical tasks

It is important to design mathematical tasks and activities for students where:

- tasks actively engage students in mathematical thinking;
- tasks take into account students' previous mathematical knowledge and experiences;

- a range of tools support students' understanding of the mathematical concepts involved.

To obtain data to answer the key research questions, two mathematical tasks were used with two classes at Grade 11. The inclusion of Question 1 (see Figure 1 below which incorporated DGS in the actual classroom) was designed to help students recognize intuitively, that when x gets closer to 2, but different from 2, then $f(x)$ tends to 8. However, to gain more detailed information about how the value of the function $f(x)$ changes when x tends to 2, students were asked to perform Questions 2, 3, and 4 (see below). The purpose of Question 2 is to help students recognize in detail what values of $|x - 2|$ are required for $|f(x) - 8|$ to be less than 0.1; 0.01; and so on. Question 3 is intended to give students confidence that, if given any arbitrary small positive numbers ε, it is always possible to find a number δ such that $|x - 2| < \delta$ then $|f(x) - 8| < \varepsilon$. In order to deepen their understanding of the mathematics students were asked carry out Question 4.

Task 1

In Figure 1 below is graph of function $f(x) = \dfrac{2x^2 - 8}{x - 2}$. If $x \neq 2$ then $f(x)$ is defined. Consider a sequence $x_n = 2 + \dfrac{1}{n}$, when value of n changes then x_n will change. Using a dynamic geometry model (available to the class) students could survey the model by changing the value of parameter n by dragging the parameter bar slider, which gives values of n as shown beneath the table in Figure 1, and observe changes in the value of $f(x)$ as x_n gets closer to 2. The purpose of this dynamically guided activity is to assist students to see that, as n becomes larger and larger, the value of x_n gets closer and closer to 2. They can then see how this impacts on the value of $f(x)$.

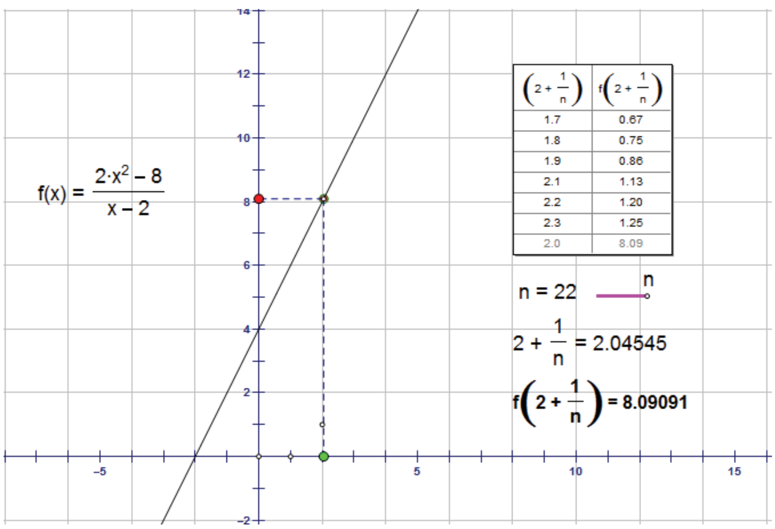

Figure 1. The limit of function model.

Question 1. Change the value of n in Figure 1 by dragging the tip of the n-parameter bar. What number does $f(x_n)$ tend to as n tends to positive infinity?

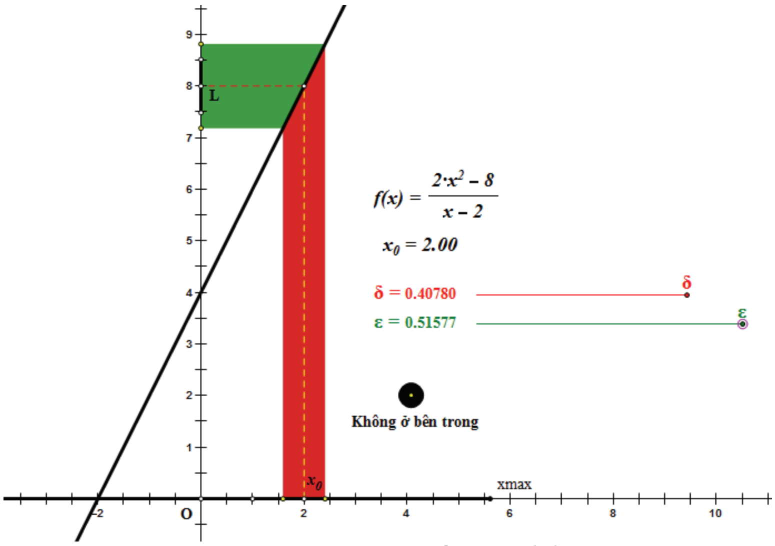

Figure 2. Dynamic δ, ε model.

Figure 2 shows a red ribbon (vertically) with its projection perpendicular on the horizontal axis generating an image of all points belonging to $\left[2-\delta, 2+\delta\right]$. It generates a segment on the vertical axis which is a perpendicular projection of the green (horizontal) ribbon. By changing the width of the vertical ribbon by dragging the tip the (red) bar which changes δ (See Figure 2, immediately below x_0), the width, ε, of the horizontal (green) ribbon changes as a result. When the height of the green ribbon is less or equal to the length of the segment, a circle in the graph will turn green with a remark "inside". Otherwise, a remark "not inside" will appear. The advantage of this dynamic model is that it allows students to see how smaller and smaller values of δ directly impact on the values of ε. As a result, students are better placed to answer the following Questions, 2 and 3, and to understand more deeply how these questions lead to Question 4 where δ and ε are introduced formally but not without prior experience.

Question 2: When $f(x)$ differs from 8 by less than 0.1; 0.01 how close does x approach 2? Use the model in Figure 2 to explore even smaller values.

with $\lvert x - 2\rvert$ < ... then $\lvert f(x) - 8\rvert$ < ... 0,1
with $\lvert x - 2\rvert$ < ... then $\lvert f(x) - 8\rvert$ < ... 0,01

Question 3: Now we play a game. In groups of two, a first student has to start with a number less than 0.01, the second student has to try to find out how much smaller $|x - 2|$ needs to be so that $f(x)$ differs from 8 by a number smaller than that given by the first student. Then change turns and fill out the results in the following table:

with $\lvert x - 2\rvert$ < ... then $\lvert f(x) - 8\rvert$ < ...
with $\lvert x - 2\rvert$ < ... then $\lvert f(x) - 8\rvert$ < ...

Question 4. Given ε is an arbitrary small positive number. Is it possible to find a number δ such with $|x - 2| < \delta$ then $|f(x) - 8| < \varepsilon$? Please give your explanation.

Task 2

Question 1. How could you demonstrate that the function *f(x)* has a limit L when x tends to a?
Question 2. To look at the image of the function graph in Figure 1, observe and make comments about the existence of the function limit when x tends to a, the monotony of the function in the interval (–∞;2)(2;+ ∞), the relation between the monotony of the function, function values in each interval and the results for the limit of the function. Please give an explanation (if possible) of the conditions for the limit of a function.

Research results

In this section, we focus on the four results: interactions among students, teacher's support to students facing difficulty, teacher's handling of students' correct results, and teacher's responses to students' incorrect results.

Interactions among students

In the process of monitoring the work between the groups, it was found that the majority of students actively worked to produce results for the whole group. Each group took turns operating with the dynamic models that were provided. Each group appointed one member to operate the model, one member to record results, and the remaining members to make observations and recommendations to resolve any difficulty. The roles were rotated among the members. When facing difficulty, some groups discussed together and solved their problems. Members from other groups were ready to support those groups that were experiencing difficulty.

Teacher's support to students facing difficulty

Sometimes students faced difficulty answering a question. For example, when operating the dynamic geometry model to answer Task 2, Question 1 in Task 2, some groups moved the N bar too fast, and so failed to see clearly the change of the corresponding value $f(x_n)$. In such cases, students were asked to move the bar so that the increase (decrease) of the value of n by each unit could be seen more clearly as a result of the changes. In Task 2, many groups initially only produced one or two results. They were encouraged to try to find other results by focusing on the images obtained. For example, let sequence $x_n, \lim x_n = 1$, find $\lim f(x_n)$. Make comments about $\lim f(x_n)$ and $f(1)$ and what result can be generalise from this?

It was easy to see that the given function $f(x)$ is increasing on $\left(-\infty;2\right)$, $\lim\limits_{x\to 2} f(x) = 8$, compare $f(x)$ with $x \in (-\infty;2)$ and 8. From that propose results for general case. We have $f(x) > 4$ with $x \in \left(0:+\infty\right)$. Compare $\lim\limits_{x\to a} f(x)$ and 4 with $a \in \left(0;+\infty\right)$. What result can be generalised from this?

Teacher's handling of students' correct results

Many good results were given by the groups. Students were asked to clarify where the results came from and prove them (if possible). Some results may still not be able to be proved, but these were valuable, because the discovery of new results did motivate students in seeking a deeper understanding.

The first task helped students recognise that, intuitively, when x approaches 2 then $f(x)$ approaches 8. When all students had completed this task, three groups were invited to present in more detail their results:

"When n increases, then $2+\frac{1}{n}$ decreases and $f\left(2+\frac{1}{n}\right)$ fades to 8".

However, to gain more detailed information about how the value of the function $f(x)$ changes when x is close to 2, students were asked to answer Question 2. The purpose of this question is to help the students to recognize in detail how as $|x - 2|$ becomes smaller, then $|f(x) - 8|$ can be made to be smaller than 0.1; 0.01; etc. All groups gave different values || satisfying the question. Summing up the results of the groups caused some students be surprised, their question was "Why there are so many different values ||?" This question provided an interesting opportunity to ask the groups to clarify the meaning of the phrase "existence of a number ...", that the students met in the form of the definition, and to help them to recognize that there are infinitely many such numbers.

Question 3 is intended to give students the belief that, if given a positive number (later denoted as ε) no matter how small, we always find some other number (later denoted as δ) such that with $|x - 2| < \delta$ then $|f(x) - 8| < \varepsilon$. All groups results appeared to be correct. But in order to demonstrate a firmer mathematical demonstration, students needed to answer Question 4. In the results for Question 4, the groups gave different values. Ten groups gave a value $\delta = \frac{\varepsilon}{2}$, and four groups gave $\delta < \frac{\varepsilon}{2}$.

In the implementation of Task 2, the variety of results can be divided into the following categories:

1. Results

Results of the method of demonstration that the function $f(x)$ having limit L when x tends to a. Ten groups thought that: "to prove the function $f(x)$ has a limit L when x tends to a, we have to prove that every sequence of numbers $(x_n), \lim x_n = a$ then $\lim f(x_n) = L$." Nine groups thought that: "Firstly consider a given arbitrary small positive number ε, then we need to prove existence of a positive number δ so that $|f(x) - L| < \varepsilon, \forall |x-a| < \delta, x \neq a$." These results helped the students acquire knowledge of results necessary for solving subsequent exercises.

2. Result of limit existence

Five groups thought that "limit of function when x tends to number a (if any) is unique". Because proving this result was beyond the students' ability, we just noted that this was the correct result. We additionally required "From the above results, give a proof of the function limit when x tends to a would not exist". Five groups applied negative propositions to give the correct result: "If two sequences $(x_n), (x_n')$ so that $\lim x_n = \lim x_n' = a$, $\lim\limits_{x\to a} f(x_n) \neq \lim\limits_{x\to a} f(x_n')$ then $\lim\limits_{x\to a} f(x)$ would not exist." The results provided a method to prove that the function limit does not exist. Nine groups gave a result based on the fact that: $\lim x_n = 2, \lim f(x_n) = 2(x_n + 2) = 8$ to argue that: "Given $f(x) = ax + b$, if $\lim x_n = x_0$ then $\lim f(x_n) = ax_0 + b$". But only five groups could explain their proof.

3. Results in the form of inequality

Five groups produced the results: "The function $f(x)$ co-varied on the interval containing (a, b), $\lim_{x\to b} f(x)=L$ then $f(x)<L\ \forall x\in(a;b)$". Four groups produced the results: "The function $f(x)$ co-varied on the interval containing (a; b), $\lim_{x\to a} f(x)=L$ then x_0, $f(x)>L\ \forall x\in(a;b)$". Three groups gave the results: "If $f(x) > m$ for every interval I containing x_0, except possible x_0, $\lim_{x\to x_0} f(x)=L$ then $L\geq m$".Only one group was able to generalise: If $f(x) > g(x)$ for every D (containing x_0) then $\lim_{x\to x_0} f(x)\geq \lim_{x\to x_0} g(x)$.

Teachers' responses to students' incorrect results

When some groups gave incorrect answers, it was helpful to discuss these incorrect answers in front of the whole class. Firstly, teachers can ask students to check if the results may be based on images in the models not yet firmly proven. Secondly, teachers can ask students to undertake further tasks or activities hoping students will recognise their mistakes. Thirdly, teachers can give a counter example and ask students to check and compare it with the answers.

When producing the results for Question 1, two groups thought that: "when n approaches positive infinity, the value $f(x_n)$ equals 8". This result can be obtained from observation that when increasing the value of n, then the red point demonstrating the change of the value $f(x_n)$ does not move. To help these students recognize their error, we posed the question, "We know that $x_n\neq 2$, whether $f(x_n)$ can be equal to 8?" With the correct calculation of $f(x_n)=2(x_n+2)$ combined with $x_n\neq 2$, the students realized their mistake.

In answering Question 4, two groups gave a result $\delta>\frac{\varepsilon}{2}$, in this case, we ask, "Is it possible to infer $|x-2|<\frac{\varepsilon}{2}<\delta$ from $|f(x)-8|=2|x-2|<\varepsilon$?" This helped students realise their mistake. In the results for the Task 2, three groups concluded that $f(x)$ can only be defined at x_0, if $\lim x_n=x_0$ then $\lim f(x_n)=f(x_0)$. This result is true only if the function is continuous, but the concept of continuous functions had not yet been learned. Two groups thought that: if $f(x)>m$ for every interval I containing x_0, except possible x_0, $\lim_{x\to x_0} f(x)=L$ then $L>m$". To help the students realise this mistake, we asked them again to observe the obtained results when doing the Task 1, considering on $(2;+\infty)$ that $f(x) > 8$,compare $\lim_{x\to 2} f(x)$ and 8.

Thus, to acquire correct knowledge students have to go through a process that inevitably involves misconceptions and difficulties. The resolution of these problems requires students to be given suggestions by the teacher, asking them to perform activities necessary check the correctness of their conceptions. In explaining the validity of the concept, the manipulation of visual models and being able to describe the results mathematically was the key to success.

Conclusion

The limit of a function is a highly abstract concept. Understanding the nature of this concept is an important prerequisite for mastering subsequent calculus concepts. During the teaching process we found that a constructivist framework, relying on and supported by dynamic models, gave students the opportunity to explore important mathematical ideas associated with idea of limit. In particular, the dynamic model allowed students to give a concrete meaning to and to explore the relationships between δ and ε, assisting students to build by themselves a proper understanding of the concept. Besides obtaining answers that teacher might expect, also it also appears that students' answers may be incorrect, or incomplete. These represent an opportunity for teachers to design and implement appropriate activities to help students get the correct understanding and to avoid these misconceptions. Dynamic models were essential to these learning activities. They provided an important bridge in the teaching and learning of abstract concepts such as the concept of a finite limit sequence.

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THINKING DEEPLY OF SIMPLE THINGS: 45 YEARS OF THE NATIONAL MATHEMATICS SUMMER SCHOOL

LEON POLADIAN
The University of Sydney
leon.poladian@sydney.edu.au

LEANNE RYLANDS
University of Western Sydney
l.rylands@uws.edu.au

The purpose of this paper is to better inform the mathematics community about the ANU–AAMT National Mathematics Summer School. This two week residential program is for the discovery and development of mathematically gifted and talented students. It takes about 64 mathematics students who have one year of secondary school left to complete and about a dozen students who have just completed secondary school from all over Australia. We present the summer school goals, how we attempt to achieve them and why we believe that we are successful.

Introduction

The National Mathematics Summer School (NMSS) was established under the joint educational sponsorship of the Australian National University (ANU) and the Australian Association of Mathematics Teachers Inc (AAMT) in 1969 by the late Professor A. L. Blakers (AM) of the University of Western Australia. The goals and organisation of the school are based on the belief that mathematics holds a central place in every civilised society, and that Australia must encourage all of its citizens to develop their mathematical talents as far as possible. The school is an investment in the youth and future of Australia that benefits the community as a whole. The goals of the school are to:

- provide opportunities for secondary school students with a real interest in (and talent for) mathematics to mix with similarly talented students from across Australia, and to develop their interest and potential in a non-competitive environment;
- provide an academic program with real depth and challenges to engage the students in the material;
- give students the opportunity to meet mathematicians, undergraduate and postgraduate students of mathematics and related disciplines, and young people working in the sciences;
- give students an appreciation of research in mathematics and science;
- provide opportunities for networking with those of similar interests and abilities, building contacts that are likely to prove useful in the future;

- draw students from a variety of backgrounds, schools and locations (including regional and rural); and
- provide students with an increased awareness of the opportunities for study and careers in mathematics and related disciplines.

The main activity of NMSS is an in-depth study of three or four different areas of mathematics. Each is very challenging and will extend every student. In addition, the program is non-competitive and very much “hands-on”. The emphasis is on doing mathematics, not just on listening to someone talking about it. Students are given extensive problem sets which encourage them to explore, conjecture, argue and reinvent. There are always enough difficult problems to challenge even the most gifted of this highly-talented group.

Whilst NMSS is unique within Australia, there are similar programs elsewhere.[9] The flagship course at NMSS is Number Theory; the curriculum and questions for this course were inspired by the legendary program[10] run by Arnold Ross at Ohio State University. Indeed Arnold Ross himself played an influential role in the teaching philosophy of NMSS and taught the Number Theory course from 1975 to 1983.

In the remainder of this paper we first discuss the NMSS participants (both staff and students) and then the teaching philosophy and outcomes. As much as possible we use the words of the participants themselves to illustrate our points.

The students and staff

Each year about 64 students who are currently in Year 11 are invited to participate in NMSS. Final selection is made by the Director based on recommendations from selectors appointed in each state and territory by the relevant mathematics teachers' association. There is a nominal quota roughly proportional to the population of each state or territory. Selection is on the basis of mathematical achievement and potential —in so far as this can be assessed—and each state is free to choose its own selection process. Existing mathematics competitions are a valuable source of information for selection, but additional tests, teacher nominations or other criteria are also used, since mathematical ability is more complicated to measure than by complete reliance on any one indicator. The Director also tries to ensure equity in the participation from different schools and regions. In addition, each year about a dozen participants are asked to return (we call them Experienced Students) both to further their own mathematical development and to assist in the educational and social program of the school.

The 64 students are divided into groups of eight for tutorials. Each group is provided with a tutor for three hours of tutorials each day and to provide mentoring during the 90 minute individual study sessions in the evenings. The tutors are almost always former NMSS students, sometimes only a few years older than the students, with a flair for teaching and inspiring.

> The tutors, too, were just kinda like grown-up versions of us. (Rachel)

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9 The AMS maintains a USA-based list http://www.ams.org/programs/students/high-school/emp-mathcamps

10 The Ross Mathematics Program (http://www.math.osu.edu/ross/)

The lecturers are chosen based on their interest and passion for teaching and acknowledgment by their peers as outstanding communicators of mathematics; many of them are former students and tutors. Even our welfare officers are expected to participate in the mathematical mentoring of the students (indeed at least four of our welfare officers have been former students). Finally, where possible, staff members and invited speakers are chosen to represent the greatest diversity of career paths (academia, government, industry, medicine, law, etc.).

> Opportunity to talk to such approachable mathematicians, equipped with the knowledge, experience and attitude to allow students to learn a great deal from them. (Daniel)

> I liked the breadth of experiences of the tutors. It gave us insight into the possible career pathways and also let us explore our interests in depth. (Samantha)

Students meet others with similar interests and abilities and so discover that they are not strange or unusual. Allowing adequate time for social interactions is thus also critical.

> We had so much more space and scope to meet and interact with other people when it wasn't all through organised activity/lectures. (Kaela)

> We can feel that we actually fit in to the community. As to me, it is a very strange but beautiful feeling, when I am not the one that is always left out anymore, it feels strange. (Yvette)

> Some of us are still friends and valuable contacts several decades later. (Leanne)

Pedagogical philosophy

The style of teaching at NMSS is a form of guided discovery learning (also called enhanced discovery learning). Topics are chosen so that students with relatively little mathematical background can explore easily and eventually wrestle with the deep ideas. This philosophy is captured in the motto "Think deeply of simple things". Number Theory is the jewel in the crown and is taught every year; other topics are delivered each year and include Topology, Projective Geometry, Chaos Theory, Sequences, Game Theory, Finite Automata, Cryptography and Knot Theory. Only three topics are presented each year with each topic at least a week long. The choice of topics depends largely on lecturer expertise. (The experienced students get four topics.)

The choice of mathematical activities at NMSS is necessarily quite different to competitive activities such as the Olympiad training schools, just as training for the 400m sprint at the Olympics in the footsteps of Cathy Freeman is very different to exploring and crossing the Great Dividing Range in the footsteps of Blaxland, Lawson and Wentworth. Our students made these comparisons.

> Olympiad maths focuses on specific types of problems and problem solving, NMSS had questions which were open-ended and just allowed us to explore (e.g., exploring primes, units, generators in different rings), which was also new and exciting. (Rachel)

> ...while it was interesting to hear about what other people were doing in the different science fields, and what their experiences were, I preferred the way it was done at NMSS, with a majority of the lectures being actual maths, with [only] a couple of guest lectures. I think I got a lot more out of NMSS because of this. (Tim)

Our courses are built around daily tutorial problem sets which form "a laboratory for mathematical ideas". The problems encourage students to explore and then to draw their own conclusions from what they find. Students first see easy questions; later, the problem sets lead to explorations of increasing depth as their ability to handle abstract ideas grows.

> I liked the problem sheets because we didn't have to answer everything; the questions were meant to help you explore ideas on your own and think about the subject as a whole instead of solving particular exercises. (Penelope)

In particular, many of the problem sets contain lists of conjectures (see Figure 1.) headed only with the instruction "Prove or disprove or salvage if possible". Not only do the students get no hints as to whether the statements are true or not, they are encouraged to think of how to salvage false statements.

PROVE OR DISPROVE OR SALVAGE IF POSSIBLE

P8. Every integer > 1 has a prime divisor.

P9. If $p_1 = 2$, $p_2 = 3, \ldots$ are primes, then $p_1 p_2 p_3 \ldots p_n + 1$ is prime.

P10. Every integer $x \in \mathbb{Z}_m$ for which $(x, m) = 1$ has an inverse x^{-1} in $\mathbb{Z}_m$.

Figure 1. Conjectures to investigate.

With such experience, students gain the confidence to apply this same approach to prove or salvage their *own* conjectures.

> ...we were encouraged to think for ourselves independently rather than asking for help and being told what to do. Ultimately, this allowed me to see what my own mathematical abilities were like and to discover what I was capable of, without the help of others. (Emma)

Guided reinvention is the principle that students should experience learning mathematics as a process similar to the process by which mathematics was invented (Freudenthal 1973, 1991; Treffers, 1987; Gravemeijer, 1994). Freudenthal stated that students should not be considered as passive recipients of ready-made mathematics, but rather that education should guide the students towards using opportunities to reinvent mathematics by doing it themselves. If students progressively mathematise their own mathematical activity, then they can reinvent mathematics under the guidance of the tutors and lecturers.

> I loved that we are solving the question, rather than mathematicians in the past. (Wonjae)

> I soon learnt to draw on the quasi-enigmatic responses from tutors and lecturers as pure motivation towards my own satisfying discoveries in maths. (Lauren)

> I would not be able to feel the joy I did in proving a.0=0 if I was shown a proof of it instead. (Tim)

Although most of our staff have no explicit training in education or exposure to pedagogical theories most of them instinctively combine modelling, coaching, scaffolding, articulation, reflection and exploration. This combination is intended to help students emulate the way experts approach mathematics and are associated with an educational theory called Cognitive Apprenticeship (Collins, Brown & Newman, 1987). The very carefully designed questions on the tutorial sets themselves create the scaffolding and encourage explorations. The lecturers and tutors provide the modelling and coaching in the way mathematicians think, and the individual study sessions concentrate, in particular, on articulation and reflection. Our tutors continually encourage students to articulate and re-articulate their ideas and reflect on any outcomes rather than quickly move on to new problems.

> I personally really like working with people trying to solve problems, whether by explaining what I have done to someone to better understand it myself, or to listen to the way others have solved it to learn off them. If I am working by myself for a long time I do tend to get distracted. (Clare)

Number Theory lends itself readily to this style of learning. Mason (2006) discusses various case studies in number theory and draws out two threads: firstly "dimensions of possible variation" (e.g., changing the numbers or conditions in an example) and secondly "range of permissible change" (e.g., does the result break when you go from primes to composites). These two common threads acting together lead to his idea of "invariance in the midst of change" which is to appreciate and express generality. See Figure 2 for an example of how we use these ideas to scaffold students' exploration of cyclic groups. Rather than random exploration (a critique often made of conventional discovery learning), very specific suggestions of what to explore are made (but this scaffolding disappears as students gain confidence and experience).

NUMERICAL PROBLEMS (Food for Thought)

* P11. Study the structure of the system (group) of units under multiplication in $\mathbb{Z}_7, \mathbb{Z}_{11}, \mathbb{Z}_{17}, \mathbb{Z}_{15}, \mathbb{Z}_{22}, \mathbb{Z}_{25}, \mathbb{Z}_{18}, \mathbb{Z}_{105}$, in that order. We are interested in knowing if there exists an element x whose powers $x, x^2, x^3, \ldots$ cover the whole set $\mathbb{Z}_n^*$. When that happens we say that the group $\mathbb{Z}_n^*$ is cyclic. Which of these groups are cyclic? Any conjectures? **(You should draw up a table of the powers of elements in each group, or a good many of them. I want you to do each example in the order specified: $7, 11, 17, 15, 22, \ldots$)**

Figure 2. A guided-exploration question.

Evidence of transformation

Renzulli and Reis (1997) consider three factors important for the development of gifted behaviour: above average ability, creativity, and task commitment. In particular, the third factor, task commitment, includes perseverance and self-confidence. Unfortunately, many of our students are accustomed in secondary school to getting every problem correct on the first (or second) attempt, and to be able to complete mathematical tasks in a single sitting. When confronted with the self-guided discovery type activities their perseverance and self-confidence may be tested for the first time. We want our students to be in what Vygotsky (1978) calls the Zone of Proximal Development (ZPD) also colloquially referred to as a 'stretch zone' that surrounds a student's 'comfort zone'. If the problems become too hard, students may find themselves in their 'panic zone'. The variety of questions and tutor guidance ensure that students move out of the comfort zone into the stretch zone without destroying their self-confidence.

> I have never admitted "I don't know" so many times before NMSS, and I suppose that's a good thing when it comes to school too. (Neha)

> The inspiration to tackle challenges, regardless of how difficult they may seem or how stupid I may feel. (Jessica)

> NMSS has also helped me reconsider my attitude towards maths—I do not have much self-confidence, but NMSS taught me not to give up on problems too quickly and not to panic if I can't immediately see a solution. (Penelope)

By the end of the two weeks, most of the students are amazed at how much they have accomplished and post-school surveys indicate that the NMSS has succeeded in raising their intellectual horizons.

> Compared to school mathematics learning, NMSS maths was far more unexpected, required far more intuitive jumps, was far harder—but was also far more interesting... maths was far more than repetitively deriving hundreds of expressions that look exactly the same (as is much of school maths, unfortunately), and [NMSS] has really given me a powerful incentive to study maths at a higher level than school. The ability to discover mathematics, create and prove your own theorems, create your own fields, rings and surfaces was a wonderful change from any other mathematics I have ever been exposed to. (Jonathan)

> I learned that math is about understanding. I have spent a lot of years in my life doing questions for the teachers to tick the boxes, and then they went on when I want to understand... I appreciate the people [who] gave me time and allowed me to ask questions in this summer school. (Yvette)

Embracing the 'after-math'

Students leave NMSS knowing that mathematics is far more than applying formulas or churning through problems and getting the answers correct. Participants see a much broader view of mathematics than at school They question, struggle and discover; learn about conjecture and proof first hand; and discover the research experience. The potential of some students is amazing and cannot be realised easily at secondary school. NMSS lasts just two weeks, but it opens up new mathematical worlds. Students develop higher-level skills and become more mathematically able.

It is imperative to take every chance available to build this potential. An excellent example of how great this potential can be is Terence Tao, one of the world's best mathematicians. He attended NMSS at the age of 10. He first competed in the International Mathematical Olympiad the same year (the youngest participant to date in either program) subsequently winning a bronze, silver and gold medal. He went on to win the Fields Medal (mathematics' highest honour, comparable to the Nobel Prize).

Two weeks of mixing with staff who are enthusiastic about mathematics, together with hearing NMSS alumni talk about their careers and the role that mathematics has played, gives some idea of the large number of careers that use mathematics[11]. Our alumni have spoken about careers in the sciences, IT, philosophy, accounting, teaching, management, finance and much more. Increasing the understanding and interest of some of our brightest students can pay huge dividends when several decades later some go on to high level and influential positions in society.

The students leave the school having had fun. In their feedback some weeks after the school, many say that their time at NMSS has been the best two weeks of their life. At a time when we hear too often about people finding mathematics hard, hating it and proudly stating that they are hopeless at mathematics, it is wonderful to see our students leave full of pride and enthusiasm for mathematics.

NMSS has even inspired others to follow our example. Two former NMSS students, Tara Murphy and James Curran, are the co-directors of the National Computer Science School. Tara and James developed the current Computer Science School from a regional school into a national residential program with a structure and pedagogy

11 http://www.amsi.org.au/careers/career-resources gives many examples of jobs requiring mathematics.

inspired by NMSS.[12] For more information about NMSS, including information on selection, contact the lead author of this paper or go to www.nmss.org.au.

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12 http://www.ncss.edu.au

LAUNCHING CONFIDENT NUMERATE LEARNERS

PETER WADE
Delany College
pwade@parra.catholic.edu.au

ANN GERVASONI
Australian Catholic University
ann.gervasoni@acu.edu.au

CATHARINE MCQUADE
Delany College
cmcquade@parra.catholic.edu.au

CATHERINE SMITH
Catholic Education Office Parramatta
catherine.smith@parra.catholic.edu.au

This paper explores how a secondary school in western Sydney used educational research as an impetus to change its mathematical education culture over a three year period. Key changes ocurred in four areas: leadership; pedagogy; structures for teaching and learning; and mathematical environments. These included increased professional conversations, adoption of a numeracy lesson structure, regular use of manipulatives and open ended tasks and a structured intervention program for mathematically vulnerable students. Critical to the development of these changes were partnerships with a university academic and the CEDP system leadership team as well as school leadership participation in professional learning.

Introduction

The *Melbourne Declaration on Educational Goals for Young Australians* (MCEETYA 2008) recognised that numeracy is an essential skill for students in becoming successful learners at school and in life beyond school, and in preparing them for their future roles as family, community and workforce members. The numeracy continuum, as described by the Board of Studies NSW Mathematics Syllabus for the Australian Curriculum (2012, p.7), outlines a progression of learning that can be used when observing students working on problems in mathematics from Kindergarten to Year 10. The ability to make informed decisions and to interpret and apply mathematics in a variety of contexts is said to be an essential component of students' preparation for life in the 21st century. So what can be done when evidence presents that students are failing to progress on this continuum despite good teaching and curriculum provision? In 2009, a system of Catholic schools in western Sydney developed a strategic approach to support its schools address this issue.

The *National Numeracy Review Report* (May 2008) provided the Catholic Education Diocese of Parramatta's (CEDP) System Learning team with research findings and recommendations that would inform the development of a new numeracy strategy for its Diocesan primary and secondary schools. The CEDP investigated various approaches and found that the *Extending Mathematical Understanding* (EMU) program (Gervasoni et al., 2012), in association with teachers using the assessment interview and framework of growth points from the *Early Numeracy*

Research Project (Clarke, Sullivan & McDonough, 2002), had been shown to improve children's learning and confidence with mathematics and enhance teachers' pedagogical content knowledge.

In 2010 the CEPD launched its new numeracy strategy with the *Numeracy Now Project* that was based on these approaches with 10 primary and 4 secondary schools. This paper examines the learning gained during this project by one of the participating secondary schools.

Context for the Numeracy Now Project at Delany College

Delany is a Year 7–12 Catholic co-educational college providing schooling for students in outer western Sydney. The College has an enrolment of 420 students who come from 38 different cultural backgrounds. The College attracts funding under the *National Smarter Schools' Partnership—Low SES* and is part of the CEDP system of schools.

In 2010 the College was invited by the CEDP to join a pilot program entitled the *Numeracy Now Project* that adopted the *Inquiry and Knowledge Building Cycle* (Timperley, 2008) to inform teacher learning. This cycle highlighted the need for engagement in systematic evidence-informed cycles of inquiry that builds relevant professional knowledge, skills and dispositions. The cycle begins by identifying the knowledge and skills students need in order to close the gaps between what they already know and can do, and what they need to know and do, to satisfy the requirements of the curriculum. As part of this project, CEDP also engaged an academic partner, Dr Ann Gervasoni from the Australian Catholic University, to assist with further developing the *Numeracy Now Project* strategy and provide professional learning for Principals, Mathematics Leaders, and Specialist Intervention Teachers.

Participation in the *Numeracy Now Project* initially involved the Principal and School Mathematics Leader participating in a 6-day professional learning course that focused on instructional leadership in mathematics; development and implementation of a school action plan that was supported by CEDP teaching educators; assessment of students using the Mathematics Assessment Interview (MAI) (Clarke et al., 2002); and provision of the Extending Mathematical Understanding Intervention Program (Gervasoni et al., 2012) for students who are mathematically vulnerable.

The professional learning program provided the College with access to research findings and professional learning about the work of highly effective mathematics teachers, instructional leaders and the characteristics of productive learning environments. As part of this process, the leadership team developed an action plan to implement and report upon during their initial year of professional learning.

The development of the team's action plan began with first assessing the Year 7 students' whole number knowledge using the MAI developed as part of two research projects, the Early Numeracy Research Project (Clarke, Sullivan & McDonough, 2002) and the Bridging the Numeracy Gap Project (Gervasoni et al., 2010). This was the first time that the interview had been systematically used in a secondary school context. The MAI data was most revealing and useful for the leadership team in focusing their action plan. The data demonstrated that many students did not have the whole number knowledge that their teachers assumed, but also highlighted exactly where the curriculum, instruction and class organisation needed to be refined to best enable all students to learn. The MAI data also highlighted that many Year 7 students were mathematically vulnerable in various whole number domains (see Table 1). Table 1 shows the percentage of Year 7 students determined to be vulnerable in each of the four domains at Delany in 2013. These results are typical of cohorts enrolled in the College as evidenced by MAI data collected over a four-year period, 2010 to 2013.

Table 1. 2013 Delany College Year 7 MAI summary data.

MAI Whole Number Domain	% Vulnerable (*n* 88)
Counting	65%
Place Value	82%
Addition & Subtraction Strategies	36%
Multiplication & Division Strategies	60%

Changes in leading mathematics learning and teaching

Key to the success of the *Numeracy Now Project* was collaboration with an academic partner, and support from CEDP in adopting a leadership triad (team) model that included the College Principal, a Teaching Educator from the CEDP and one of the College's Lead Teachers. This involvement of school leadership ensured that the project gained traction (Hargraves & Fink 2006) and was more likely to lead to sustained changes to practices that would become imbedded in the culture of the classroom and College.

From the earliest beginnings of the *Numeracy Now Project*, the Delany College Principal took a hands-on role in leading the project. Attending the EMU Leading Mathematics Learning and Teaching course, along with the Teaching Educator and Lead Teacher, was the beginning of a discourse founded in research and peppered with readings provided by the academic partner, Dr Ann Gervasoni. The four CEDP secondary schools involved in the *Numeracy Now Project* in 2010 were breaking new ground, along with the research partner, as the earlier research had not ventured into a secondary setting before this project. An important strategy employed by the team was to introduce the mathematics faculty to accessible academic papers that did not overawe the teachers but stimulated discussion and sometimes vigorous debate. This was an important strategy to ensure buy-in of all stakeholders; imposed change rarely evolves to be sustainable (Hargraves, 2006; Timperley, 2009). Professional dialogue amongst the mathematic faculty was also informed by research which challenged assumptions about the use of assessment data. Using Timperley's (2009) observations, the teachers looked at the MAI data through a different lens and asked the question, "was the data more about the students' knowledge and understandings or was the data stimulating questions to reflect upon teacher effectiveness in aiding students' progress on the learning continuum?" Timperley and Parr (2009) argue that

> …making such changes is complex. Not only are changes in professional knowledge and skills of the use of assessment data required, but teachers also need deeper pedagogical content knowledge so that they are able to respond constructively to what the data is telling them about changes needed to their practice. (Timperley and Parr, 2009, p. 24)

In leading a faculty of very able and experienced Mathematics teachers, the team decided to use the *Inquiry and Knowledge Building Cycle* as a segue to explore the Australian Association of Mathematics Teachers (AAMT, 2006) *Standards for*

Excellence in Teaching Mathematics in Australian Schools. The call for a deeper Professional Knowledge in Domain 1 evoked conversations around how students learn mathematics and how indeed the mathematics teachers could enhance mathematics learning.

A significant moment in the learning journey occurred in the latter half of the first year of involvement in the *Numeracy Now Project* when one of the members of the mathematics faculty summed up a discussion about the use of the MAI data when he said, "We cannot possibly proceed with our programming for next year's Year 7 cohort unless we know what they know and can do."

Domain 1.3 (Knowledge of Students' learning of mathematics) in the AAMT standards helped the teachers and the team rationalise the need for new ways of knowing and new ways of teaching that in turn called for change.

> Excellent teachers of mathematics have rich knowledge of how students learn mathematics. They have an understanding of current theories relevant to the learning of mathematics. They have knowledge of the mathematical development of students including learning sequences... (AAMT, 2006, p. 2, 1.3 Knowledge of student learning of mathematics)

Further work by the team saw an investigation of Kagan's (1985) co-operative learning model. Moving from a competitive individualistic approach in achieving learning goals to a model where students worked together to accomplish shared goals required professional coaching. Workshops were co-planned by the team and professional learning was delivered by the Teaching Educator. The teachers were encouraged to employ the strategies in their classrooms and in the combined double lesson. These lessons incorporated a warm up activity, rich tasks and learning reflection. Students worked in teams to solve complex real world problems. One such double lesson saw students literally running to stations located throughout the College in an 'A-Math-zing Race' style of learning. The enthusiasm shown by the students exemplified the attitudinal shift that was taking place for both students and teachers.

Another area of inquiry that the team pursued was student and teacher efficacy in mathematics. A survey was developed and administered to gauge a wide range of responses including attitude about and relevance of mathematics. The following data (Table 2) is a snapshot of some of the survey data of the first student cohort involved in the *Numeracy Now Project*. The data demonstrates that after a year at the school, the students were much more likely to appreciate the relationship of mathematics learning to everyday life and to its usefulness when they leave school.

Table 2. Percentage responses from students about their attitudes to mathematics.

Survey statements	Yr 7 2010 (n=75)	Yr 8 2011 (n=75)
In my maths classes we relate what we are learning to everyday life.	68%	91%
I enjoy giving things a go in maths even if I don't know if they will work.	76%	84%
The maths I am learning will be useful to me when I leave the school.	89%	98%

Another significant learning for the team was influenced by the work of Robinson (2007) who asserts that when one sets a goal, it must be 'resourced strategically' in order to maintain the goal as a priority and to best ensure its success. The team resourced the *Numeracy Now Project* in a number of ways which included: funding of a Lead Teacher (Numeracy); prioritising of lesson times and rooms; designated EMU specialist rooms; meeting time for professional learning and co-planning; training of Specialist EMU Teachers; Leadership training for Lead Teacher, and acquisition of resources available for every mathematics teaching space.

Sustaining change, including planning for succession and engaging in a continual cycle of improvement, has been an ongoing feature of the work of the team. The CEDP's strategic plan, drawing on the work of Robinson (2007), cited in its *Theory of Action,* requires the development of an annual implementation plan. This plan, together with the Success Criteria, developed by the CEDP Numeracy Team, has provided the tools for the College to engage in frequent reflection and evaluation.

Emerging changes in mathematical environments

Hattie's (2009) Synthesis of over 800 meta-analyses relating to achievement, has also informed the work of the classroom teachers at the College. Teachers, knowing that they 'make a difference', have gained confidence in using the growth points for planning for and observing student achievements, become more willing to engage in co-teaching and frequently used 'critical friends' to provide feedback about their teaching. Dr Ann Gervasoni acted as a critical friend and spent some time in the College in 2012 engaging in instructional walks (Sharratt & Fullan, 2012) observing teacher practices and student engagement. The teachers all commented that they found her feedback extremely valuable and practical. As well as changing pedagogical practices from teacher centred to student centred learning using open-ended investigations, teachers have become more proficient in differentiating the learning for their students. They have been aided in this work by regularly using the differentiation planning grid, provided by Dr Ann Gervasoni, that included the following components:

- Brief description of the activity
- What is the mathematics?
- What is the growth-point focus?
- What do you want students to notice?
- Teaching adaptations—easier/more challenging
- Teacher questions to probe for understanding.

The mathematics teachers have also been developing their 'on the spot questioning techniques', aiming to assist student articulation of their thought processes, for example, "How do you know?", "Prove it!", "Explain how you know?". This powerful questioning gives both the teachers and students greater awareness of the students' mathematical knowledge and understanding. It creates feedback for the teacher which informs them how to progress the student from their Zone of Proximal Development (ZPD) (Vygotsky, 1978).

The deep questioning has also assisted the teachers to plan and deliver lessons that engage the maximum number of students in the maximum mathematical experiences for the maximum time.

One effective practice that emerged from this understanding was the collaboration between teachers to co-plan and co-facilitate the double lesson for the Year 7 cohort that occurred once a fortnight. The practice was first modelled by the Lead Teacher and the CEPD's assigned Teaching Educator. Through strategic resourcing and mentoring, the team ensured practical support and regular feedback for the development of this innovation. The traditional classroom environment is now more productive and supportive of student learning through the use of word walls, posters and easy student access to materials that aid their thinking and learning.

Creating opportunities for students to peer teach and to explore rich open-ended tasks in small group settings represented another major shift in pedagogical practice. Teaching strategies that were particularly useful to assist active student involvement in the learning enterprise included: Inside-outside circle, Jigsaw, Graffiti, Think-Pair-Share and Three Step Interviews.

An additional instance of team work that has emerged in the last two years has been a closer partnership between the class teacher and EMU Specialist to share information about specific student's learning needs and to plan and co-teach the activities needed to accelerate their mathematical progress.

Working with parents and the wider community

Parents continue to be acknowledged as one of the key factors in their child's learning. Through the work of the *Numeracy Now Project*, the team has raised the profile of the importance of parents supporting the development of numeracy skills. Since 2010 the College has used a variety of opportunities to encourage and support parents to actively assist their child's further numeracy development wherever possible in their daily experiences. This has occurred through:

- advice and information via the college newsletter and the student diary;
- workshops for parents of EMU students; and
- the display of concrete materials at parent information evenings, open days and
- student–parent–teacher conferences.

Changes in structures for teaching and learning

Dr Ann Gervasoni encouraged the team to plan for activities and professional learning that would act on Recommendations 1 and 12 from the National Numeracy Review Report (2008). Specifically these two recommendations made it clear that all teachers, no matter what year level or subject specialty, should acquire mathematical pedagogical content knowledge. To this end, the team continued with some preliminary work that had begun a year earlier in 2009 to enrich all staff members with a fuller understanding of their role as teachers of numeracy. Professional learning workshops have been held since 2010 with all staff focussing on different aspects of the *Numeracy Now Project* work including: the MAI instrument, the Growth Point Framework, and the MAI data and its implications for student learning in all Key Learning Areas (KLAs).

The team planned, from the outset, to develop a 'numeracy across the curriculum' teaching and learning disposition. Professional learning was undertaken to create awareness that every teacher is a teacher of numeracy. 'Numeracy moments' were identified and mapped in all KLA programs by the teachers. This mapping activity highlighted a number of numeracy skills common to all KLAs. At a series of workshops members of the mathematics faculty shared with their colleagues the pedagogical content knowledge needed to effectively teach numeracy skills commonly used across the KLAs. The mathematics teachers have remained connected with their assigned KLA expert adviser on mathematics in the curriculum.

Conclusion

The team at Delany College believe that the work undertaken to better meet the needs of all mathematical learners has implications for many secondary school leaders and mathematics teachers.

Through the work in implementing the *Numeracy Now Project* it has become evident that the following practices are worthy of consideration by those undertaking similar projects.

1. Devise or adopt a framework of inquiry and knowledge building.
2. Use research to inform the framework of inquiry.
3. Build a 'team' to lead the project which has expertise and spheres of influence.
4. Form powerful coalitions with academic partners and Professional Learning Communities at system level.
5. Strategically 'hook' the hearts and minds of all stakeholders.
6. Lead the community of teacher learners with precision to engender confidence in undertaking the challenge that change brings.
7. Resource strategically.
8. Plan for succession to sustain changes in culture.

One indicator of the success of the *Numeracy Now Project* at the College has been the change to teacher practice. One specific practice has been the programming for effective mathematics learning and teaching. The teachers are more cognizant of using student data, particularly the MAI data each year, to inform adjustments to the teaching plan and cycle of learning. As each Year 7 cohort commences, the process begins anew by:

- knowing the individual student's ZPD by using a diagnostic tool to assess the student's mathematical understandings and to program an appropriate course of teaching;
- knowing, through a structured numeracy lesson, that student reflection and response informs teaching adjustments to ensuing learning activities;
- challenging the learners with problems which create 'hard thinking' within a student's ZPD and provide mathematical thinking strategies and prompts to allow for multiple hits in understanding new concepts.

This new way of working has brought about some profound changes to student attitudes and learning behaviours. As two teachers recently commented,

> There are less students opting out of mathematics class work. The frequent use of concrete materials and investigations using teamwork has promoted student self-esteem, encouraged risk taking and enjoyment of the learning.

> It has been a truly exciting journey and my rewards are received every day on the smiling faces of the students, keen and eager to come and learn—that magical moment when the light goes on.

Students, when asked their opinion about the way in which they are learning mathematics, made the following comments.

> I like our lessons very much because we are learning things and figuring out what to do. I now have more ways of doing maths. In class now, I am more confident to try a question.
>
> My maths lessons have helped me get more strategies to solve problems. I know a lot more and I'm now not afraid to answer questions.
>
> I feel confident knowing that I'm not dumb any more. I am smart.
>
> I like my maths lessons because I've learnt to do things I couldn't do before.
>
> I didn't like maths in primary school because I couldn't do it. I love maths now. I hate missing a class.

These comments demonstrate the positive impact of the *Numeracy Now Project* for students and teachers. The College project leadership team, together with the mathematics faculty, believe that the essence of their work is 'launching confident numerate learners'. We are well on the way!

Note

A copy of the extended Launching Confident Numerate Learners paper is accessible on the Delany College website www.delanygranville.catholic.edu.au

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DATA AND MEASUREMENT IN YEAR 4 OF THE AUSTRALIAN CURRICULUM: MATHEMATICS

JANE WATSON
University of Tasmania
jane.watson@utas.edu.au

LYN ENGLISH
Queensland University of Technology
l.english@qut.edu.au

The activities introduced here were used in association with a research project in four Year 4 classrooms and are suggested as a motivating way to address several criteria for Measurement and Data in the *Australian Curriculum: Mathematics*. The activities involve measuring the arm span of one student in a class many times and then of all students once.

Introduction

The *Australian Curriculum: Mathematics* (Australian Curriculum, Assessment, and Reporting Authority [ACARA], 2013a) has had a mixed reception from mathematics educators (e.g., Atweh, Goos, Jorgensen & Siemon, 2012) and the debate will lead to continuing improvement and relevance over time. There are places in the curriculum, however, whether planned or not, where valuable links can be made that offer great opportunity for classroom activities. One of these appears in Year 4 in the strands of Measurement and Data. The relevant descriptors are presented in Table 1.

Table 1. Year 4 content descriptors, Australian Curriculum: Mathematics (ACARA, 2013a).

Measurement and Geometry
Using units of measurement
Use scaled instruments to measure and compare lengths, masses, capacities and temperatures (ACMMG084)
Statistics and Probability
Data representation and interpretation
Select and trial methods for data collection, including survey questions and recording sheets (ACMSP095)
Construct suitable data displays, with and without the use of digital technologies, from given or collected data. Include tables, column graphs and picture graphs where one picture can represent many data values (ACMSP096)
Evaluate the effectiveness of different displays in illustrating data features including variability (ACMSP097)

The final word in Table 1 is the key connecting concept across these descriptors. Variation is the concept underlying all of statistics (e.g., Moore, 1990; Watson, 2006) and research has suggested that students develop an appreciation for variation before an appreciation of expectation, where expectation is related to "expected values" such as probabilities or averages (Watson, 2005). These two concepts form the foundation of informal inference, which although not explicitly stated in the curriculum, must be the aim of the "interpretation" part of "Data representation and interpretation." Makar and Rubin (2009) define informal inference in terms of using data as evidence, generalising beyond those data, and acknowledging uncertainty in the generalisation. It is precisely the variation in the data that leads to the uncertainty in the expectation expressed in the generalisation.

Returning to the descriptors in Table 1, in measuring length students may have an expectation that their own measurement is precise and accurate, but when several students measured the same object, they saw that there is variation in the measurements. Variation will also have occurred if different students use different tools to measure the length of an object, linking to the second descriptor in Table 1 about methods of data collection. Variation is inherent in both the different methods of data collection and in the data collected. As intimated in the fourth descriptor, the suitable data displays from the third descriptor will have different qualities in displaying the variation in the data collected.

The activities described here, based on measuring arm span length (Watson & Wright, 2008), were the basis of a research project that will involve students in Years 4 to 6 over three years. Students were given workbooks (the questions from which are condensed in the Appendix) and worked in groups of two or three. The teachers were provided with extensive teaching notes, including instructions for carrying out and recording the measurements on a whiteboard in a list and with yellow stickies in a stacked "dot" plot. Students produced hand-drawn plots and later used the *TinkerPlots* software (Konold & Miller, 2011) for creating other plots of the data. The researchers were present observing and occasionally answering questions. The students were sometimes in their classroom and sometimes in a computer laboratory; the researchers fit in and were not considered an intrusion.

The lessons and student outcomes

The first lesson began with a discussion on how accurately the class could measure length, reviewing the units and tools they could use. The question was asked. "If we all measure the same object will we get the same answer for its length? Why or why not?" The scene was then set for measuring the arm span of one student, posing the question of everyone getting the same value, and being able to make a "best guess" (i.e., expected value) of the person's arm span. Students were given a choice of measuring tools, including rulers, tape measures, string, and metre sticks. The instructions in the workbook assisted students in recording the data on the whiteboard and in their workbooks; students answered questions Q3, Q4, and Q5 in the workbook (see Appendix) while the measurements were being made and recorded. One of the stacked dot plots of class data is presented in Figure 1.

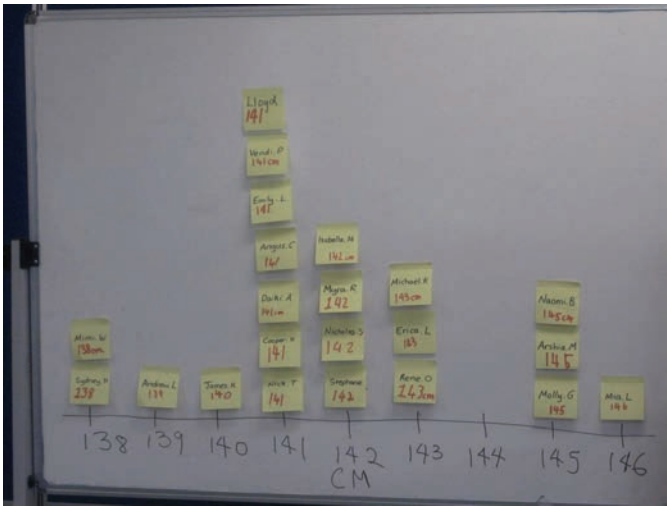

Figure 1. Stacked dot plot of one class's data.

In answering Q3, "Were all of the measurements the same? Why or why not?", students were well aware of the reasons for variation. Most answers focussed on the measuring tools or whether the measuring was carried out accurately.

> No, because the class was using different measuring materials.
>
> No, because some people didn't start at zero.
>
> No, because [student] could have moved her arms.
>
> No, because the units of measurement were different.

Other responses, however, were more vague.

> No, because everybody can get a different answer.

In answering Q4 "Were you surprised at some of the values? Which ones? Why?" there was variation across the classes because of the different data collected but well over half of the students identified an outlier, whereas others noticed the range or lack of variation.

> Yes, [student] did 99 cm and everybody else did over 110 cm.
>
> Yes, 146 and 138 because they were the biggest and smallest number.
>
> I wasn't surprised because the numbers are around the same.

Q5 was more comprehensive, asking for a summary of the accuracy of the measurements, a "best guess" of the arm span, and the student's confidence in the guess. Comments on accuracy mirrored some of the reasons given for Q3. Examples presented here hence focus on the reasons for the expectation associated with the "best guess" of the person's arm span, which varied considerably.

> I think it is 141 cm because there are more 141 cm in the measurements.
>
> I think that our guess was pretty good because we guessed 142 cm and it was in the middle of all of the guesses.
>
> I think 141 cm ... is the right size for [student].

When asked to create a representation of the data in a graph, picture or plot, the results were generally of two types. Figure 2 shows three representations that kept track of the values recorded for the measurements. On the left is an alphabetical list of the measurers and their measurements. In the centre is a horizontal line plot of unordered measurements with scale from 138 cm and on the right is a vertical value plot with the scale beginning at 10 cm.

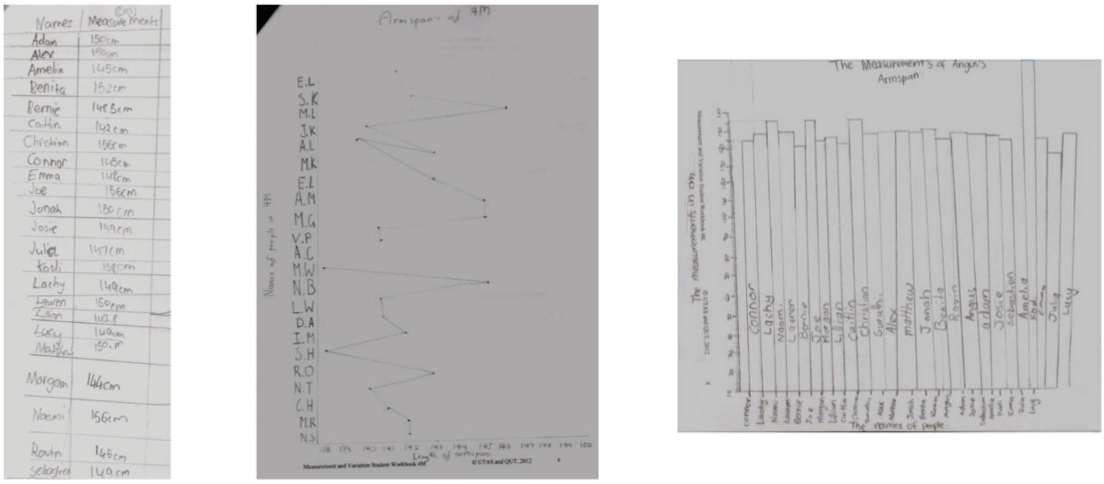

Figure 2. Three examples of representations of the values measured.

Figure 3 contains three representations of the frequency type. On the left is an unordered tally list; in the centre, an unordered frequency bar chart; and on the right, a frequency pictograph. The representations in Figures 2 and 3 illustrate the tremendous variation in tables and plots that are created by students when they are given a blank page rather than a labelled grid to fill in.

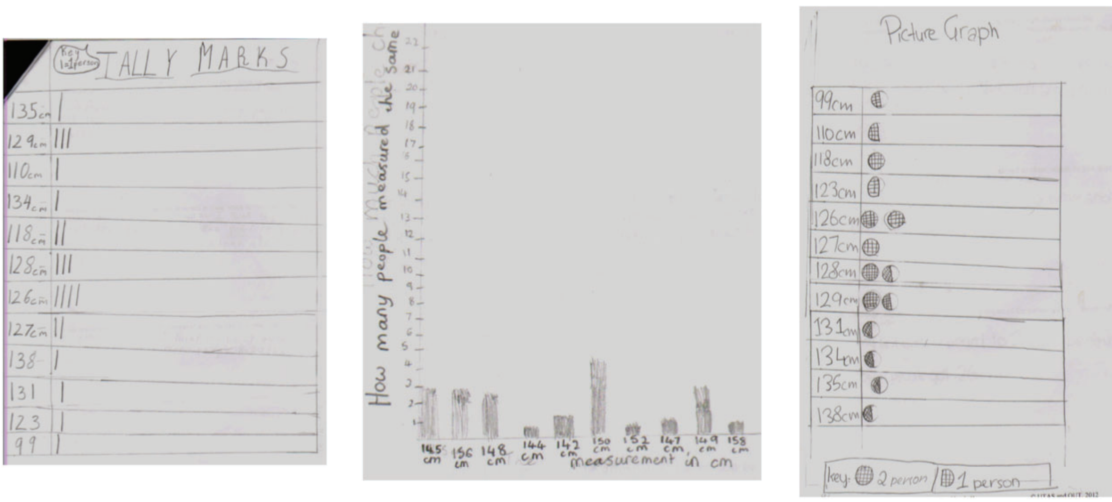

Figure 3. Three examples of representations of the frequencies of values measured.

For Q6, students were asked to write a summary of their representations, thinking about variation. Only a few students missed the point and talked about colour or people. The display of appreciation of variation itself revealed variation, from little recognition to quite meaningful suggestions.

I used a bar chart to show [student's] arm span.

My graph is counting in centimetres. It has the names of everyone on the horizontal axis and the measurement on the vertical axis.

It tells you from a number to a number e.g., 126 to 129 and how many people chose between the two numbers.

Bar graph and 141 is most likely to be the correct arm span.

I chose this graph because it shows the different variations of cm for one person's arm span. It also shows how many people got the same measurement.

I did a range of measurements to see which one is most popular.

Students then entered the data into *TinkerPlots* to consider other possible representations of the measurements. It was their first experience with data in *TinkerPlots* except for an introductory session in the computer laboratory a few weeks earlier. Students were keen to identify the people with the values they had measured and in this first exposure to the software unconventional representations were sometimes created. Two of the *TinkerPlots* graphs are shown in Figure 4, again representing measured values or frequencies of measured values.

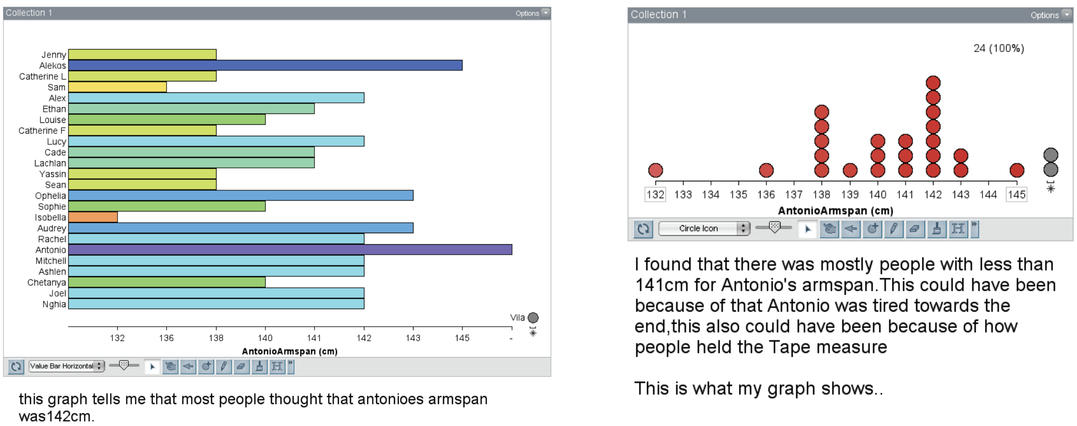

Figure 4. Two examples of TinkerPlots *representations of measurements for one student.*

Moving to the second lesson where students measured the arm spans of all members of their class once, the questions involved the accuracy of measurement and the class data were entered directly into *TinkerPlots* rather than the students creating their own representations first on paper. Following class discussion it was expected by the teachers and researchers that the issue of having many measurements to make a "best guess" of arm span, as in the first lesson, versus having only one measurement in the second lesson, would motivate students to question the accuracy of the single values for each student. In fact the circumstances of the lessons, where to speed the process, in most classes the researchers helped measure the arm spans of students along a tape measure attached to a wall, meant other salient features captured the attention of students. When answering Q4 on accuracy, most suggestions were of a practical rather than theoretical nature.

More accurate because we were putting the measuring tape on the wall.

They will be [more accurate] because the adults did it.

Yes. I think they will be because now we have had practice at measuring we might be more accurate than last time.

Only one response appeared to recognise the issue of repeated measurements.

I think [student's] arm span measurements were more accurate than ours because we were all measuring the same person.

For the final part of the activity students were asked to display the data from measuring the arm spans of the entire class. Because the data were entered into the same file that contained the measurements on the single student, students could be asked to describe differences between the plots for the two data sets. Two of the final plots and explanations are shown in Figure 5.

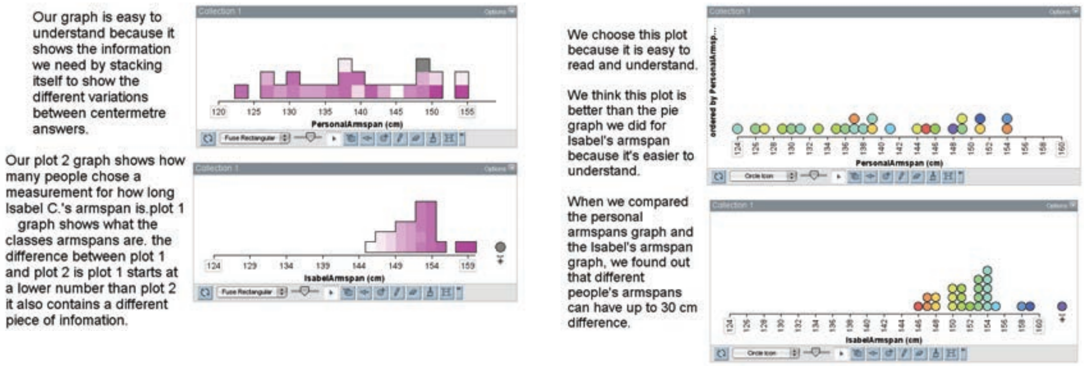

Figure 5. Two examples of summaries from the measurement activities.

For students who had completed their analyses, they were encouraged to consider other questions that might arise from the data collected, such as the possibility of gender difference in the arm span lengths. Figure 6 shows two examples. On the left is a plot showing the gender difference in one class and on the right is a plot showing the variation in measurements of a single student for the different tools used in another class. The foundation concept of variation was stressed throughout discussions with students and although not all written responses reflected it (43% of students were ESL students), the teachers expressed complete satisfaction with the outcomes.

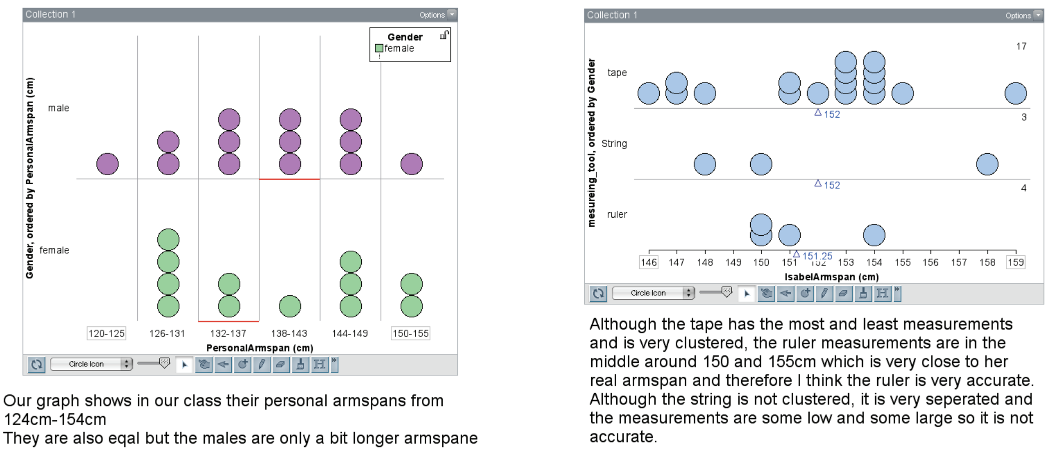

Figure 6. Two examples of extensions to the measurement activity.

Discussion

The activities described here have demonstrated the feasibility of linking Measurement and Data at Year 4. Without a context, statistics are meaningless (Rao, 1975) and measurement provides a realistic and motivating context within the *Australian Curriculum: Mathematics*. The activities also fulfil the detail of constructing "data displays, with and without the use of digital technologies" within Data Representation and Interpretation and reinforce the general capability of ICT across the curriculum (ACARA, 2013b). In particular, Investigating with ICT, Creating with ICT, and Communicating with ICT (p. 53) are illustrated with these activities. Further, there is potential for developing all four elements of the general capability of Critical and Creative Thinking: Inquiring, Generating ideas, possibilities and actions, Reflecting on thinking and processes, and Analysing, synthesising, and evaluating reasoning and procedures (p. 72).

Turning to the proficiency strands of the mathematics curriculum, these activities clearly contributed to understanding in building "a robust knowledge of adaptable and transferrable mathematical concepts" (ACARA, 2013a, p. 5), particularly in relation to linear measurement and variation as a foundation for statistical investigations. Variation must be one of the most transferable concepts in all of mathematics! Fluency in terms of "choosing appropriate procedures [and] carrying out procedures flexibly, accurately, efficiently and appropriately" (p. 5) was observed to develop during the lessons in the new techniques encountered by the students. Students also were developing "the ability to make choices, interpret, formulate, model and investigate problem situations" (p. 5) as they worked through the activities presented to them. Finally most students displayed progress in reasoning as they used logical thought patterns in the measurement context to infer, justify and generalise, explaining their outcomes (p. 5).

For those who would meet the content, proficiency and capability standards of the new curriculum, these activities provide an excellent starting point, as well as a foundation for informal inference in later years. Although not all students performed at the level displayed in the figures, all students were engaged, discussed difference and variation with members of their groups and the teacher, and could answer casual questions posed by the researchers.

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Appendix: Student Workbook (Condensed): Measuring a person's arm span

Name: [Other information]

How easy is it to get an accurate measurement of something? Today, we will be discussing methods of measuring the arm span of one person in your class and everyone will have a chance to do the measuring.

Q1. Explain here the method you and your partner have decided to use:

Q2. Each person in your group is to record below their name and the measurement they took.

Name: ______________________________ Arm span (cm): ____________
Name: ______________________________ Arm span (cm): ____________
Name: ______________________________ Arm span (cm): ____________

Once you have your measurement, also record the information on the whiteboard when instructed by your teacher. [Table provided with names of students to record measurements.]

Q3. Were all of the values the same?
Why or why not?
Which was the largest?
Which was the smallest?

Q4. Were you surprised at some of the values? Which ones? Why?

Q5. Write a summary of how accurate you think the measurements in the table are. What is your "best guess" of the arm span of the person the class measured? How confident are you of this value?

Use the next page to create a graph or plot or picture to represent the values in the table. [Blank page provided.]

Q6. Write a summary statement about what your representation shows about the measurements your class made of the arm span of the person you measured. Think about the variation that is seen in your plot or picture.

Q7. After looking at the other representations presented in class, which do you think is the best way to tell the story of the person's arm span? Why?

Q8. What is your new "best guess" for the arm span of the person the class measured? Explain why you chose this value.

Q9. [Instructions given for entering class data into *TinkerPlots*, which students complete and create plots.] Has your "best guess" changed for the person's actual arm span? If so, explain why?

Q10. After looking at the other *TinkerPlots* files created by your classmates, write a summary below of which representation you think shows the variation in the measurements the best.

Measuring our arm span

Last lesson we measured the arm span of a single person. This lesson we are going to measure and plot the arm span of all members of the class.

Q1. Describe the method your class have decided to use:

Q2. All people in your group are to record below their names and their arm span measurements. [space provided]

Once you have your measurement, also record the information on the whiteboard when instructed by your teacher.

Q3. Do you think all the values will be the same?
Why/why not?

Q4. How accurate do you expect your results to be compared to our last lesson?

[Table provided with names of students to record measurements.]

Q5. In your groups, enter the data into the *TinkerPlots* file.
You will need to:

- Create Data Cards and enter their data
- Create a plot that best describes the data set and tells the story
- Create a text box and write:
  - i. A summary statement about what your plot shows and
  - ii. At least two sentences that describe the differences between this plot and the earlier one.

Extension question: Do boys have longer arm spans or do girls? How might this be explored? Can you make a plot showing the difference between boys' and girls' arm spans?

PROFESSIONAL PAPERS

NEGOTIATING DEVELOPMENTAL ASSUMPTIONS IN EARLY YEARS MATHEMATICS CURRICULA: LESSONS FROM THE LANGUAGE OF SPACE

CRIS EDMONDS-WATHEN

Charles Darwin University

cris.edmonds-wathen@cdu.edu.au

Students who begin school with different everyday language to that used in the curriculum and in the school do not have the advantage of their early mathematics lessons building on the language and concepts which they use at home. Some of the discord between the sequencing of location in Early Years mathematics curricula and the understandings of Indigenous language speaking students can be explained using a typology of spatial frames of reference from cognitive linguistics. It shows how developmental progressions in early mathematics can contain culturally and linguistically specific assumptions which may be implicit in curricula and teaching practices.

Introduction

The sequence of locational language and concepts in Australian mathematics curricula generally begins with 'everyday language' and progresses to more technical or abstract concepts. The mathematics section of the *Northern Territory Curriculum Framework* (NTCF) is explicitly informed by the SOLO taxonomy and the Van Hiele levels (Northern Territory Department of Education, 2009). A close analysis of the sequencing of the location element of the NTCF also showed a relationship between its sequence and the stages of spatial development described by Piaget and Inhelder (1948/1956), hereafter referred to as the Piagetian spatial stages, although there is no reference to Piaget in the NTCF.

The sequence of spatial language in the NTCF also parallels the usual order of acquisition of spatial language in Indo-European languages, a language family which includes most European languages such English and also the Indo-Iranian languages such as Hindi and Persian (Johnston & Slobin, 1979; Levinson, 2003). While this sequence appears 'natural' for speakers of these Indo-European languages, spatial language and thinking is not acquired in the same order or even in the same categories in all languages (Levinson, 2003). A semantic typology of spatial frames of reference developed by cognitive linguists (Pederson et al., 1998) is proving useful in understanding this cross-linguistic variation because it provides a way to talk about these differences without using a universalising hierarchy.

The *Australian Curriculum Mathematics 4.1* (Australian Curriculum Assessment and Reporting Authority [ACARA], 2013) is currently replacing state and territory based curricula across the nation. As part of its more streamlined approach, it describes in less detail than the NTCF the specific language that might be used at each year level. From a cross-linguistic perspective this could be advantageous in that varied linguistic sequences could be used. However, it is likely that most teachers will continue to use a sequence that seems ‘natural’ to them because is concords with the normal sequence of acquisition in their own languages. Mathematics textbooks, workbooks and other support material are also likely to continue to reflect this sequence, which is an appropriate sequence for many children. It is important for teachers of students from non-Indo-European language backgrounds to understand both that the everyday language of space of these students may differ from that assumed in the mathematics curriculum and that the order of acquisition may vary. For children whose linguistic backgrounds are from some different language families, especially children who are first language speakers of Indigenous Australian languages, this sequence may not be appropriate. As well as describing how these sequences may vary, this paper makes suggestions for how teachers might teach spatial language in a more linguistically appropriate manner for these students.

Piagetian spatial stages

There are three main Piagetian spatial stages*: topological*, *projective* and *Euclidean* (Piaget & Inhelder, 1948/1956). These stages are believed to develop through the child’s perceptual experiences. The *topological* stage is egocentric and includes sensitivity to proximity, separation, order, enclosure and continuity; it is “purely internal to the particular figure whose intrinsic properties it expresses” (p. 153). In the *projective* stage, points of view begin to be taken into account and relationships “presume the inter-co-ordination of objects separated in space” (p. 154). This includes the concept of the straight line and the visual effects of perspective. Finally, in the *Euclidean* stage, space is conceived abstractly as a ‘container’ for the objects within it, and is organised by parallel and orthogonal straight lines. These stages are shown in Table 1.

Table 1. Piagetian stages of spatial development.

Piagetian stage	**Ages**	**Qualities**	**Descriptor**
topological	2–7 years	proximity, separation, order, enclosure, continuity	egocentric
projective	7–12 years	straight lines, perspective	points of view
Euclidean	12+ years	parallel and orthogonal axes	abstract

Piaget and Inhelder (1948/1956) based their theory on studies of European children. There is a resemblance between the Piagetian stages and the order of acquisition of spatial language in European languages (Brown & Levinson, 2000). Topological concepts such as ‘in’, ‘at’ and ‘on’ are acquired early, followed by projective concepts such as ‘in front of’, ‘in back of’, ‘to the left of’ and ‘to the right of’ and finally Euclidean concepts such as the geocentric terms ‘north’, ‘south’, ‘east’ and ‘west’. Note that there are both topological and projective senses of ‘behind’ and ‘in front’, with the projective sense acquired later than the topological (Johnston & Slobin, 1979). Brown and Levinson (2000) comment:

> Because of this correspondence between acquisition order and the predicted Piagetian order, it is generally held that *the order of language acquisition is driven by conceptual development*. In other words, the presumption is that language does not facilitate or influence the course of conceptual development by depends on it. (p.173, emphasis in original)

The applicability of these Piagetian stages to the mathematical development of children of non-European cultures has been extensively questioned. Studies using Piagetian theory have found differences between Indigenous Australian children and the European children of Piaget's study (Dasen, 1973; Seagrim & Lendon, 1980). Speakers of Warlpiri, spoken in the Central Australian desert begin to understand cardinal directions such 'north', 'south', 'east' and 'west' within the first few years of life (Laughren, 1978). In Tenejapan Tzeltal, a Mayan language spoken in Mexico, projective terms are not used at all, and geocentric (Euclidean) terms are acquired at same age or possibly before the topological (Brown & Levinson, 2000). Because space as conceived by the Navajo is developed and structured differently to that described by Piaget, "the educational procedures that would be appropriate and satisfactory of the teaching if spatial notions in Navajo cultural settings cannot take the Western Hierarchical spatial structure for granted" (Pinxten, van Dooren & Harvey, 1983, p. 161).

The languages which children hear and learn as they mature affect the course of their conceptual development with respect to space. These examples suggest that to some extent *the order of language acquisition drives conceptual development* of space and that this is culturally and linguistically specific.

Early years curriculum stages

The same sequence of the Piagetian stages and the order of acquisition of spatial language in European languages can be observed in mathematics curricula. In the NTCF (NTDET, 2009), topological relations can be seen at Key Growth Point 2 (KGP2) (preschool - Foundation Year) such as 'in', 'on top' 'beneath' and 'behind' and 'in front'. Key Growth Point 3 (KGP3) (Foundation - Year 1) introduces projective relations requiring descriptions of positions of self and objects in relation to other objects and in Band 1 (approximately Year 2) 'left' and 'right' are used along with grids and coordinates. By Band 2 (approximately Year 4), Euclidean relations are used such as the more abstract cardinal directions. A mapping of the Piagetian stages against the NTCF levels is shown in Table 2.

The *Australian Curriculum Mathematics* (ACARA, 2013) is less specific about terminology. However, it still progresses from topological concepts at Foundations level such as 'near' and 'next to' (ACMMG010), through to Euclidean concepts such as the use of grids in Year 5 (ACMMG113). It is harder to identify the projective stage, but is included in Year 1's 'clockwise' and 'anticlockwise' (ACMMG023) and Year 2's 'relative positions' (ACMMG044).

These curriculum sequences concord with how English speakers are taught, acquire and use the language of location. However, this sequence is not how speakers of many Australian Indigenous languages acquire and use spatial language. For Warlpiri children, with cardinal directions part of their everyday language before school entry age, 'left' and 'right' may never be everyday language (Laughren, 1978).

Table 2. Piagetian spatial stages, NTCF levels and indicative vocabulary.

Piagetian stage	Descriptor	NTCF level	English vocabulary
topological	scene internal	KGP2 – KGP3	‘in’, ‘on top’, ‘beneath’, ‘behind’, ‘in front’
projective	person’s point of view	Band 1	‘left’, ‘right’
Euclidean	space as container, abstract	Band 2	‘north’, ‘south’, ‘east’, ‘west’

Variation in the development of spatial concepts and language is significant in cross- or inter-cultural situations, in terms of whether the mathematics curriculum follows the same or a different sequence of spatial concept acquisition as the child. Investigating the sequence of acquisition of spatial concepts in a particular language could help determine whether the sequence of the mathematics curriculum is suitable for speakers of that language.

Frames of reference

The terminology of spatial frame of reference provides another way to describe variation in spatial languages and in acquisition of spatial language. Spatial frames of reference describe where things are located with respect to each other. They essentially involve coordinate systems which provide angular information about location (Levinson, 2003). Levinson (2003) describes three frames of reference: *intrinsic*, *relative* and *absolute*. Definitions of the frames of reference make use of the distinction between *figure* and *ground*, where the figure is the topic of the locational description and the ground is the reference object with respect to which the figure is located (Talmy, 1983).

Intrinsic

In the *intrinsic* frame of reference, descriptions are scene internal and the figure is located with respect to a feature of the ground, for example, “the pen is beside the cup”. Key terms in English are ‘front’, ‘back’ and ‘sides’. The intrinsic frame of reference is generally the first acquired and seems to be present in all languages (Johnson & Slobin, 1979). It evolves out of and can form a continuum with the topological part of spatial language (Levinson, 2003). Mopan, a Mayan language spoken in Belize and Guatemala, appears to only have this frame of reference (Danziger, 1996).

Relative

In the *relative* frame of reference, the location of the figure is described by relating the point of view and body of the speaker to the ground, for example, “the pen is to the left of the cup”. Key English terms are ‘in front of’, ‘behind’, ‘to the left of’ and ‘to the right of’, where these are from the speaker’s perspective. Terminology used in this frame of reference is often derived from the intrinsic frame of reference. While the relative frame of reference is used extensively in European languages, there are other languages which do not use it at all (Levinson, 2003).

Absolute

In the *absolute* frame of reference, the location of the figure is described in relation to a fixed direction or landmark, for example, “the pen is to the north of the cup”. The cardinal directions such as ‘north, ‘south’ ‘east’ and ‘west’ are the key terms in English. Other types of absolute direction systems have evolved in response to the environment, such as using direction of river drainage (Schultze-Berndt, 2006), or the use of an inland/seaward axis in coastal and island languages such as Iwaidja (Edmonds-Wathen, 2011).

Acquiring frames of reference

The order of acquisition of the frames of reference in Indo-European languages corresponds to the order of acquisition of the Piagetian stages for the European child: firstly the intrinsic, followed by the relative, with the absolute acquired last (Brown & Levinson, 2000; Levinson, 2003). Table 3 shows how the frames of reference can be mapped against the Piagetian spatial stages and the NTCF.

Table 3. Frames of reference, Piagetian spatial stages and NTCF level.

Frame of reference	Piagetian stage	Qualities	NTCF level	English vocabulary
Intrinsic	topological	scene internal	KGP 2 – KGP 3	‘in’, ‘on top’, ‘beneath’, ‘behind’, ‘in front’
Relative	projective	person’s point of view	KGP 3 – Band 1	‘left’, ‘right’
Absolute	Euclidean	space as container, abstract	Band 2	‘north’, ‘south’, ‘east’, ‘west’

In these terms, Warlpiri children acquire the absolute frame of reference at an early age, being receptive to it before the age of two, but Warlpiri does not use the relative frame of reference (Laughren, 1978). This appears to be widespread (Schultze-Berndt, 2006) although not universal (Edmonds-Wathen, 2011) among indigenous Australian languages. The frames of reference are not inherently sequential like the Piagetian spatial stages. Spatial frame of reference thus provides us with a non-hierarchical way to compare how different languages describe spatial location.

Implications and conclusion

As the *Australian Curriculum: Mathematics* is introduced across Australia, it is timely to examine possible assumptions underlying our practice and take the opportunity to make our teaching more suitable for our students. Spatial thinking and spatial language are acquired developmentally and the order of acquisition varies cross-culturally and cross-linguistically. The order of acquisition of spatial language in Indo-European languages appears to have influenced both the Piagetian spatial stages and the sequencing of school mathematics curricula. However, students with different everyday language to that used in the curriculum and for instruction do not have the advantage of their early mathematics lessons building on the language and concepts which they use at home. The benefit of using the frames of reference typology over the Piagetian stages to describe some of this variation is that the frames of reference are not hierarchical and the approach does not assume all languages and people use all three frames of reference.

Evidence from languages in which the absolute frame of reference is the predominant frame of reference used shows that children can acquire the absolute at a much earlier age than it appears in Australian mathematics curricula (Brown & Levinson, 2000; Laughren, 1978). All mathematics teachers, particularly teachers of students who speak these absolute focussed languages, could introduce and use English cardinal terms in the early years in direction and location associated activities. This is recognised in one of the *Australian Curriculum Mathematics* Year 2 work samples in which the task is to describe a bike track on a grid (ACARA, 2013). The student uses both relative terminology—“Go left”—and absolute—“Keep going south, then turn round, go north”.

The absolute frame of reference can be used to refer to the location of familiar object in the environment such as school buildings. The understanding that east and west can be determined from sunrise and sunset is an environmentally salient way to start to learn these directions. Some cities and towns also have street grids aligned to the cardinal directions which can be used.

Another suggestion is to use the absolute frame of reference to facilitate the learning of left and right, as a kinaesthetic mnemonic. Once absolute directions have been learnt, one faces towards a direction such as east (facing towards the direction where the sun rises in the morning). Left is then the side that is towards the north and right is the side facing towards the south. The directions can be reinforced using locally salient cues. To begin with, one assumes the actual facing direction to recall left and right, and then eventually uses a mental representation of facing in the direction.

These are preliminary suggestions but they are informed by a cognitive linguistic framework of demonstrated applicability in diverse contexts. It is important to take the opportunity offered by a new curriculum to examine preconceptions we may hold of ‘natural’ sequences of mathematical learning and where relevant adapt our teaching to make it more suitable for particular cohorts of students. An appreciation of diverse learning sequences can also open up new approaches to the learning capabilities of all our students.

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PRIME EXPLORATIONS WITH MATHEMATICA

DAVID LEIGH-LANCASTER

Victorian Curriculum and Assessment Authority

leigh-lancaster.david.d@edumail.vic.gov.au

This paper considers some of the ways in which a computer algebra system (CAS) such as *Mathematica* could be used to further study of prime numbers in the Year 6–12 mathematics curriculum.

Introduction

The study of prime numbers is typically part of the school mathematics curriculum in the upper primary and early secondary years. For example, from the *Australian Curriculum: Mathematics* Foundation – Year 10:

> **Year 5**
> Identify and describe factors and multiples of whole numbers and use them to solve problems (ACMNA098)
> **Year 6**
> Identify and describe properties of prime, composite, square and triangular numbers (ACMNA122)
> **Year 7**
> Investigate index notation and represent whole numbers as products of powers of prime numbers (ACMNA149)
> (ACARA http://www.australiancurriculum.edu.au/Mathematics/Curriculum/F-10).

Related content usually covers factors of natural numbers, classification of types of these numbers, their representations and the fundamental theorem of arithmetic (http://en.wikipedia.org/wiki/Fundamental_theorem_of_arithmetic). In later secondary years students may also study prime numbers as part of enrichment and extension of the curriculum in the area of number theory, or as a component of school-based assessment, for example exploration of Gaussian integers as part of work on complex numbers in advanced mathematics.

Applications involving prime numbers, such as the RSA cryptosystem (http://en.wikipedia.org/wiki/RSA_(algorithm); http://www.mathaware.org/mam/06/Kaliski.pdf) have also had occasional popularity, principally as mathematical investigations in the senior years as they also involve modulo arithmetic and other aspects of number theory.

While many students will have seen application of the *Sieve or Eratosthenes* (http://en.wikipedia.org/wiki/Sieve_of_Eratosthenes) (Figure 1) to the numbers from

1 to 100 they usually do not get to work with large prime numbers (ask any student what is the largest prime number that they *know* for sure) or indeed, large lists of prime numbers. This is certainly likely to be the case when the relevant computations for prime number identification are carried out by hand or using a scientific calculator. The *Sieve of Eratosthenes* is efficient for finding 'small' prime numbers. For a given natural number, n, the sieve tests potential divisors up to $\sqrt{n}$.

	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1. Sieve of Eratosthenes from 1 to 100.

As well as powerful numerical computation capability CAS such as *Mathematica* have specific number theoretic functionality that can be used to support a more detailed study of prime numbers. This paper explores some of these possibilities. There are a range of related *Demonstrations* (dynamic pre-developed *Mathematica* files) available from Wolfram Research that use this functionality to illustrate aspects of mathematics such as divisibility networks, the *Sieve or Eratosthenes*, prime factorisation or the distribution of primes (see: http://demonstrations.wolfram.com).

Some background

A natural number is an element of the set $N = \{1, 2, 3 \ldots\}$. In the following discussion the word 'number' is taken to refer to a natural number. The set of prime numbers sometimes designated P, is a proper subset of N. A number is said to be a *prime* if it has exactly two distinct factors, 1 and itself, while a number with more than two distinct factors is said to be *composite*. A special type of composite number is a factorial where 'n factorial' is $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$, for example $5! = 120$.

The number 7 is prime as its only factors are 1 and 7 (exactly two distinct factors); while 18 is composite as it has six distinct factors {1, 2, 3, 6, 9, 18}. The number 1 is a special case—it is neither prime nor composite as it has only one factor—itself (this is the only number that is neither prime nor composite). Prime numbers are important in mathematics because they can be used to build up (compose) and uniquely represent (up to order) other numbers using only multiplication, for example $2013 = 3 \times 11 \times 61$.

It is not immediately clear whether an arbitrary number such as 1 234 567 is prime or not. In February 2013 a new largest known prime number $2^{57\,885\,161} - 1$ was announced. This and the last few previous 'largest' prime numbers are a particular type of prime number called a *Mersenne* prime (http://en.wikipedia.org/wiki/Mersenne_prime).

Some interesting related mathematical questions are: How can one identify whether a number is prime or not? Is there a rule for generating all prime numbers? How are prime numbers used to build up composite numbers? How many prime numbers are there anyway? Are prime numbers easy to find? How are they distributed in N?

Using Mathematica functionality

Mathematica has several built in number functions related to prime numbers and factors.

Is *n* prime?

`PrimeQ[n]` tests whether n is prime or not, for example:

```
{PrimeQ[57], PrimeQ[59], PrimeQ[1234567]}

{False, True, False}
```

It is possible to use this to select all the primes from within a given set:

```
Select[Table[n, {n, 1, 100}], PrimeQ]

{2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,
 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97}
```

And also obtain the composites:

```
Complement[Table[n, {n, 1, 100}], Select[Table[n, {n, 1, 100}], PrimeQ]]

{1, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 30, 32,
 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 49, 50, 51, 52, 54, 55,
 56, 57, 58, 60, 62, 63, 64, 65, 66, 68, 69, 70, 72, 74, 75, 76, 77, 78,
 80, 81, 82, 84, 85, 86, 87, 88, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100}
```

What is the n^{th} prime?

`Prime[n]` gives the n^{th} prime number, for example:

```
{Prime[10], Prime[2013]}

{29, 17491}
```

This can be used to generate a list of the first n primes, for example:

```
Table[{n, Prime[n]}, {n, 1, 100}]

{{1, 2}, {2, 3}, {3, 5}, {4, 7}, {5, 11}, {6, 13}, {7, 17}, {8, 19}, {9, 23},
 {10, 29}, {11, 31}, {12, 37}, {13, 41}, {14, 43}, {15, 47}, {16, 53},
 {17, 59}, {18, 61}, {19, 67}, {20, 71}, {21, 73}, {22, 79}, {23, 83},
 {24, 89}, {25, 97}, {26, 101}, {27, 103}, {28, 107}, {29, 109}, {30, 113},
 {31, 127}, {32, 131}, {33, 137}, {34, 139}, {35, 149}, {36, 151}, {37, 157},
 {38, 163}, {39, 167}, {40, 173}, {41, 179}, {42, 181}, {43, 191}, {44, 193},
 {45, 197}, {46, 199}, {47, 211}, {48, 223}, {49, 227}, {50, 229}, {51, 233},
 {52, 239}, {53, 241}, {54, 251}, {55, 257}, {56, 263}, {57, 269}, {58, 271},
 {59, 277}, {60, 281}, {61, 283}, {62, 293}, {63, 307}, {64, 311}, {65, 313},
 {66, 317}, {67, 331}, {68, 337}, {69, 347}, {70, 349}, {71, 353}, {72, 359},
 {73, 367}, {74, 373}, {75, 379}, {76, 383}, {77, 389}, {78, 397}, {79, 401},
 {80, 409}, {81, 419}, {82, 421}, {83, 431}, {84, 433}, {85, 439}, {86, 443},
 {87, 449}, {88, 457}, {89, 461}, {90, 463}, {91, 467}, {92, 479}, {93, 487},
 {94, 491}, {95, 499}, {96, 503}, {97, 509}, {98, 521}, {99, 523}, {100, 541}}
```

And also to investigate the distribution of primes, as illustrated in Figure 2:

```
ListPlot[Table[{n, Prime[n]}, {n, 1, 50}] , AxesLabel → {n, "Prime[n]"}]
```

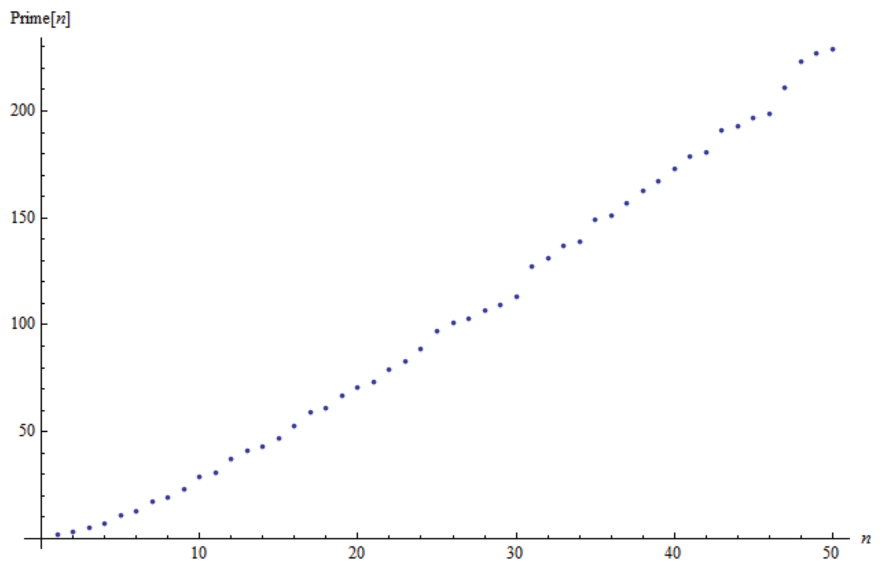

Figure 2. The graph of the n^{th} prime in terms of n for n = 1 to 50.

How many primes are there in a given interval?

PrimePi[n] gives the number of primes less than or equal to n, for example:

```
PrimePi[100]
25
```

PrimePi[n] is approximated by $n/\log_e(n)$, for example, if $n = 100$ then $100/\log_e(100) \approx 22$, and can also be used to investigate the distribution of primes, for example http://demonstrations.wolfram.com/DistributionOfPrimes. **PrimePi[n]** can also be used to find numbers of primes in an interval, for example:

```
Factorise[10! + 1]
11^1 × 329891^1
```

```
PrimePi[1000] - PrimePi[100]
143
```

As the size of prime numbers increases so does the gap between consecutive primes. For $n > 1$ it is known that there is at least one prime between n and $2n$. For $n > 3$, $\{n! + 2, n! + 3 \ldots n! + n\}$ forms a sequence of $n - 1$ consecutive composite numbers.

Prime factorisation

FactorInteger[n] computes the prime factors of n and their power in its prime factorisation, for example:

```
{FactorInteger[9864], FactorInteger[1234567]}
{{{2, 3}, {3, 2}, {137, 1}}, {{127, 1}, {9721, 1}}}
```

Thus the prime factorisation of 9864, that is, its representation as a product of powers of primes, is $2^3 \times 3^2 \times 137^1$. The following is a short program in *Mathematica* which defines a new function **Factorise[n]** that does the re-writing:

```
Factorise[n_] := If[Length[FactorInteger[n]] > 1,
  Cross @@ (Superscript @@@ FactorInteger[n]),
  First[Superscript @@@ FactorInteger[n]]]
```

```
Factorise[9864]
```

$2^3 \times 3^2 \times 137^1$

The corresponding table for n from 1 to 100 shows that at most four prime factors are used for a given number and that the powers of these are generally small. The *Demonstration* http://demonstrations.wolfram.com/FactoringAnInteger/ provides the prime factorisation of a number while the *Demonstration* http://demonstrations.wolfram.com/FactorTrees/ gives its factor tree, as illustrated in Figure 3.

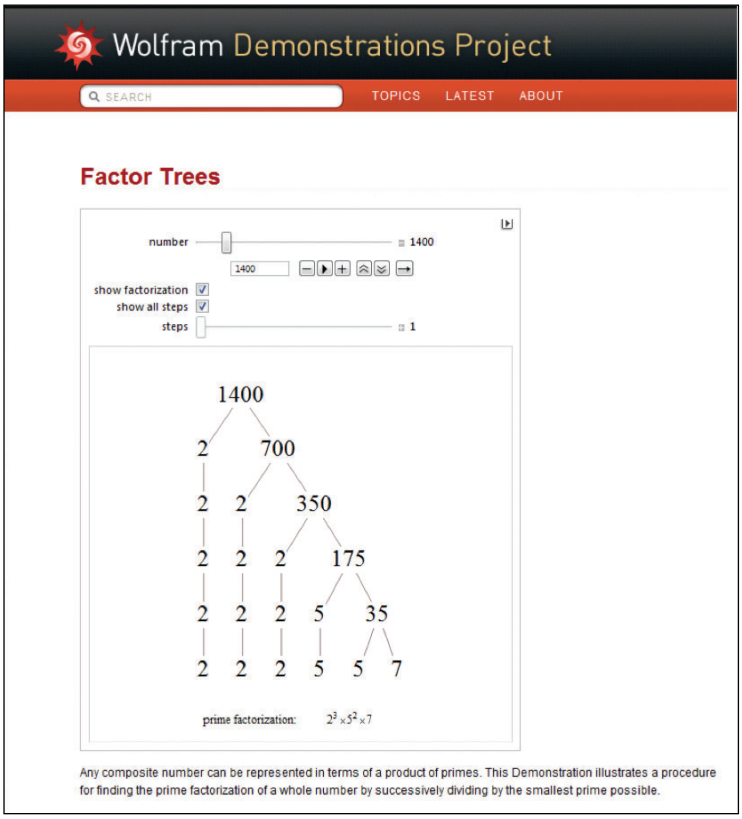

Figure 3. Wolfram Research Demonstration of a factor tree.

Finding primes

There are several proofs that P is an infinite set (that is, there is no largest prime), some of them such as Euclid's proof provide a method for finding a 'new' prime (see: http://primes.utm.edu/notes/proofs/infinite). A particular type of prime number is a factorial prime (see: http://mathworld.wolfram.com/FactorialPrime.html).

Factorials can be used to illustrate how one can find some 'new' primes. Suppose one has already identified all the primes less than or equal to n. Consider $n! = n \times (n-1) \times (n-2) \times (n-3) \times \ldots \times 3 \times 2 \times 1$. This is clearly composite, but what about $n! + 1$? None of the numbers 2, 3 ... n can be a factor of $n! + 1$, since there will be a remainder of 1 on division. Nor can they be a factor of any other number that is a factor of $n! + 1$. It may be the case that $n! + 1$ is prime, for example if $n = 3$ then $3! + 1 = 7$ which is prime.

On the other hand $n! + 1$ may not be prime, for example, if $n = 4$ then $4! + 1 = 25$. Here 5 is a factor of 25 and is a 'new' prime. If a factor of $n! + 1$ which has been found is not prime then the same reasoning can be applied to this number and so on, however this process must stop since the factors found in this sequence are getting smaller and all such factors must be greater than n. In the worst possible scenario it will turn out that $n + 1$ is a 'new' prime as is the case for $n = 4$. However when $n = 10$, and hence $10! + 1 = 3628801$ the 'new' prime found is 329 891.

```
{10 ! + 1, PrimeQ[10 ! + 1], Factorise[10 ! + 1]}
```

$\{3\,628\,801,\ \mathrm{False},\ 11^1 \times 329\,891^1\}$

The largest currently known factorial prime is 150209! + 1 identified in late 2011.

Not quite a formula

There is no known simple polynomial function $p: N \to R$ where $p(n)$ = the n^{th} prime number. However there are a several low-order polynomial functions that generate prime numbers for domain that is an initial sub-sequence of N (see: http://mathworld.wolfram.com/Prime-GeneratingPolynomial.html). Perhaps one of the best known of these is from Euler: $p(n) = n^2 - n + 41$ which generates 40 distinct primes for {0, 1, 2 ... 40}. There is an even simpler function from Legendre: $p(n) = 2n^2 + 29$ which generates 29 distinct primes for {0, 1, 2 ... 28}.

```
Table[2 n^2 + 29, {n, 0, 30}]
```

```
{29, 31, 37, 47, 61, 79, 101, 127, 157, 191, 229,
 271, 317, 367, 421, 479, 541, 607, 677, 751, 829, 911,
 997, 1087, 1181, 1279, 1381, 1487, 1597, 1711, 1829}
```

```
PrimeQ[Table[2 n^2 + 29, {n, 0, 30}]]

{True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, True, True,
 True, True, True, True, True, True, False, False}
```

```
{Factorise[1711], Factorise[1829] }

{29^1 × 59^1, 31^1 × 59^1}
```

Students could investigate the efficacy of various linear, quadratic and other polynomial functions for generating distinct primes on initial sub-sequences of N. There are quite a few such quadratic functions, what about linear functions? The function $f(n) = n^6 + 1091$ does *not* generate primes for $n = 1$ to 3095. What are some other 'prime-avoiding' polynomial functions?

Gaussian integers

The notions of 'integer' and 'prime' can be extended to complex numbers. A Gaussian integer is any complex number of the form $a + bi$ where a and b are integers. Clearly the natural numbers are Gaussian integers where $a \in N$ and $b = 0$. The notion of a prime number in C, or a Gaussian prime, is analogous to, but a bit more complicated than in N. However one can readily observe that some numbers that are prime in N are not prime in C. For example 2 and 5 are prime in N but have factorisations $(1 + i)(1 - i)$ and $(2 + i)(2 - i)$ respectively in C. It can be shown that any prime natural number of the form $4n + 3$ is also a Gaussian prime. Two Gaussian primes with non-zero imaginary parts are $2 + 3i$ and $3 + 5i$, as also are their conjugates. The Gaussian primes are symmetrically distributed about both real and imaginary axes in C as illustrated in Figure 4:

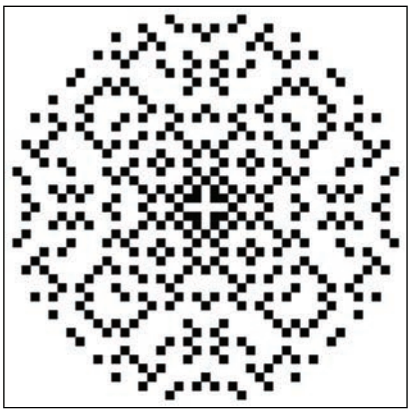

Figure 4. Pixel image of Gaussian primes in a radial subset of the complex plane.

Mathematica includes an option for number theoretic functionality to apply for Gaussian integers, for example:

```
Table[{n, FactorInteger[n, GaussianIntegers → True]},
 {n, 1, 20}]

{{1, {{1, 1}}}, {2, {{-i, 1}, {1 + i, 2}}}, {3, {{3, 1}}},
 {4, {{-1, 1}, {1 + i, 4}}}, {5, {{-i, 1}, {1 + 2 i, 1}, {2 + i, 1}}},
 {6, {{-i, 1}, {1 + i, 2}, {3, 1}}}, {7, {{7, 1}}},
 {8, {{i, 1}, {1 + i, 6}}}, {9, {{3, 2}}},
 {10, {{-1, 1}, {1 + i, 2}, {1 + 2 i, 1}, {2 + i, 1}}},
 {11, {{11, 1}}}, {12, {{-1, 1}, {1 + i, 4}, {3, 1}}},
 {13, {{-i, 1}, {2 + 3 i, 1}, {3 + 2 i, 1}}},
 {14, {{-i, 1}, {1 + i, 2}, {7, 1}}},
 {15, {{-i, 1}, {1 + 2 i, 1}, {2 + i, 1}, {3, 1}}},
 {16, {{1 + i, 8}}}, {17, {{-i, 1}, {1 + 4 i, 1}, {4 + i, 1}}},
 {18, {{-i, 1}, {1 + i, 2}, {3, 2}}}, {19, {{19, 1}}},
 {20, {{i, 1}, {1 + i, 4}, {1 + 2 i, 1}, {2 + i, 1}}}}
```

and

```
{PrimeQ[2 + 3 i, GaussianIntegers → True],
 PrimeQ[3 + 4 i, GaussianIntegers → True]}

{True, False}
```

These applications and related computations provide teachers and students with the opportunity to readily generate a range of examples and counter-examples that can be used as a basis for formulating and testing conjectures. They can be naturally complemented by related proofs as applicable, in particular where these are of a constructive nature. A range of results and proofs involving prime numbers can be found at the Australian Mathematical Sciences institute (AMSI) website: http://www.amsi.org.au/teacher_modules/Primes_and_Prime_Factorisation.html.

This also provides the opportunity to outline some well-known conjectures (that is mathematical statements for which there is neither a proof nor a known counter-example) such as:

- every even natural number greater than two can be expressed as a sum of two
- primes (Goldbach's Conjecture);
- there are infinitely many primes p such that p + 2 is also prime (the twin prime conjecture);
- there is an odd perfect number (a perfect number is a natural number that is the sum of its positive divisors excluding itself)
- the *abc* conjecture (see: http://news.sciencemag.org/sciencenow/2012/09/abc-conjecture.html)
- the Gaussian moat problem (see: http://en.wikipedia.org/wiki/Gaussian_moat).

MEASURING CHANGE IN STUDENTS' ATTITUDES TOWARD MATHEMATICS OVER TIME

AYSHA ABDUL MAJEED
The University of Adelaide
aysha.abdulmajeed@adelaide.edu.au

Students' attitude towards mathematics has been known to influence students' participation, engagement and achievement in mathematics. *Attitudes Toward Mathematics Inventory* (ATMI) was employed to measure change in middle school students' attitudes toward mathematics over time. The instrument was administered to 544 Year 7 and Year 8 students in 13 schools in South Australia to collect three data points over the academic year 2012. Confirmatory factor analysis (to examine factorial invariance) and Rasch analysis (to examine item-level invariance) were used. Concurrent equating was employed which rendered the scales comparable between occasions. Repeated measures ANOVA was used to measure change in mean scaled scores (logits) of students on the four-sub scale of ATMI. The results of the study show that students' attitudes toward mathematics generally decline over time during middle school years.

Introduction

Australia takes part in a variety of international assessment of mathematics achievement. The most influential has been the *Programme for International Student Achievement* (PISA), which assesses 15-year-old students. The other popular international survey is *Trends in International Mathematics and Science Study* (TIMSS), which assess students in Year 4 and Year 8. These international studies of mathematics achievement over the years have contributed to the observations made about falling standards and increasing number of Australian students scoring at or below the low benchmark.

At national level, all Australian states and territories have testing programmes to monitor student achievement. *The National Assessment Program—Literacy and Numeracy* (NAPLAN) tests have been conducted since 2008 across Australia in Years 3, 5, 7 and 9. The use of a common scale across year levels allows student progress in numeracy to be monitored, and also the levels of achievement to be charted over time (South Australia's Action Plan for Literacy and Numeracy, 2010).

The *South Australian Curriculum Standards and Accountability Framework* defines the 'Middle Years Band' as years 6–9. The achievement levels of middle school students in South Australia in international and national student assessments are continually showing downward trend.

State	TIMSS 2011 Mean	TIMSS 2011 SE	TIMSS 2007 Mean	TIMSS 2007 SE	2011 - 2007 difference	TIMSS 2003 Mean	TIMSS 2003 SE	2011 - 2003 difference	TIMSS 1995 Mean	TIMSS 1995 SE	2011 - 1995 difference
ACT	532	9.9	518	22.4	–	507	9.6	–	528	11.4	–
NSW	518	11.1	500	10.0	–	530	12.0	–	512	8.6	–
VIC	504	8.0	503	8.5	–	495	6.4	–	500	6.4	–
QLD	497	8.0	491	4.9	–	490	6.1	–	506	8.5	–
SA	489	5.8	490	6.7	–	501	11.3	–	513	5.6	↓
WA	493	10.6	485	8.3	–	487	7.6	–	527	6.7	↓
TAS	475	6.9	485	6.8	–	477	12.3	–	496	11.5	–
NT	462	14.4	483	13.9	–	449	14.2	–	470	19.9	–

Figure 1. Year 8 mathematics achievement in trends in International Mathematics and Science Study (TIMSS) (Thomson, Hillman & Wernet, Schmid, Buckley, & Munene, 2012, p.25).

State	PISA 2003 Mean score	PISA 2003 S.E.	PISA 2006 Mean score	PISA 2006 S.E.	PISA 2009 Mean score	PISA 2009 S.E.
SA	535	4.9	520	4.3	509	5.3
TAS	507	9.4	502	3.8	487	5.1
ACT	548	3.5	539	5.6	528	6.4
WA	548	4.1	531	6.5	529	7.2
NSW	526	4.3	523	5.0	512	5.2
NT	496	4.9	481	6.2	487	4.9
QLD	520	6.9	519	4.4	518	7.5
VIC	511	5.1	513	4.0	512	4.9

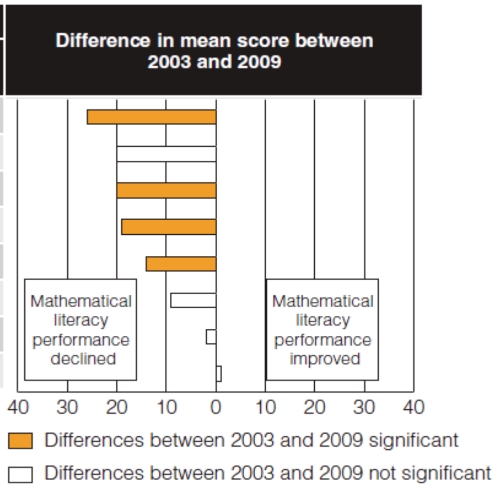

Figure 2. Mean mathematical literacy scores of 15 year olds for Programme for International student Assessment (PISA) Thomson, De Bortoli, Nicholas, Hillman, & Buckley (2011 p. 198).

		NSW	Vic	Qld	WA	SA	Tas	ACT	NT	Aust
Mean scale score/(S.D.)	2008	551.3 (78.3)	552.3 (69.4)	539.0 (70.4)	533.7 (68.7)	536.2 (67.7)	533.8 (67.5)	556.2 (71.0)	488.1 (84.0)	545.0 (73.2)
	2011	548.6 (79.5)	550.9 (70.0)	538.7 (68.3)	544.6 (72.0)	534.9 (67.9)	532.2 (70.5)	555.5 (71.8)	481.3 (90.1)	544.6 (73.7)
	2012	543.2 (80.5)	544.2 (70.3)	531.7 (67.4)	534.7 (71.1)	528.8 (66.8)	526.0 (68.1)	546.2 (72.1)	472.9 (92.2)	537.9 (73.8)
Significance of Difference: 2008 vs. 2012		▼	▼	▼	■	▼	■	■	■	▼
Significance of Difference: 2011 vs. 2012		■	▼	▼	▼	■	■	■	■	▼

Figure 3. Achievement of Year 7 Students in Numeracy, by state and Territory, 2008, 2011 and 2012 (NAPLAN National Report, 2012, p. 36).

The performance of South Australian middle school students in mathematics in the national (NAPLAN) and international assessments (TIMSS & PISA) is consistently declining. These results are considered disappointing and students considered at risk, but there is an absence of research studies investigating students' views on mathematics teaching and learning.

In Australia, many studies have investigated students' attitudes toward mathematics (e.g., Leder & Forgasz, 2006; Pierce, Stacey & Barkatsas, 2007; Beswick, Watson, Brown, Callingham & Wright, 2011), however, such studies are scarce in South Australia. This paper reports on students attitudes toward mathematics over time to represent 'student voice' investigating change in the pattern of their attitudes as they move through a range of challenging mathematics curriculum in the middle school.

Attitudes toward mathematics

Attitudes have powerful impacts on effective engagement, participation and achievement in mathematics (Reynolds and Walberg, 1992; Zan, Brown, Evans & Hannula, 2006). They are equally important as content knowledge for students to make informed decisions in terms of their willingness to use this knowledge (Wilkins & Ma, 2003) and their influence range from individual mathematical learner and the classroom teacher to the success or failure of massive curricular reforms (Goldin, Rösken & Törner, 2009). Middle school years are particularly important when examining students' attitudes toward mathematics. Many students in these years experience difficulties that often breed disengagement and negative attitudes toward school and reduced self-confidence and motivation particularly in mathematics (Sullivan, Tobias & McDonough, 2006). During this crucial period they tend to make many long term decisions about themselves and develop an understanding of their abilities, leading them to make their future choices (Manning, 1997; Garcia & Pintrich, 1995). Many students seem to become less optimistic over time about their chances to succeed in mathematics (Watt, 2004). Therefore, even though, in general students consider mathematics to be a subject of immense importance, many students tend to develop unfavourable attitudes toward this domain. There is a fall in student attitude and achievement in the middle school years of schooling (Midgley & Edelin, 1998), and roughly half of the Year 8 student population taking mathematics courses is lost (Schoenfeld, 1992). A meta-analysis of studies revealed that negative disposition toward mathematics reaches its peak in Years 9 and 10 with Years 7 and 8 identified as significant in its development (Hembree, 1990). Beswick, Watson and Brown (2006) observed that the students' attitudes toward mathematics declined as they progressed through school and Yates (2009)suggests that the middle years students should receive particular attention.

The instrument: Attitudes Toward Mathematics Inventory (ATMI)

In the study, a shorter version of the Attitudes Toward Mathematics Inventory (ATMI) (Tapia & Marsh, 2004), which measured four subscales was used;

- The Self Confidence subscale comprised of 12 items and measured students' confidence and self-concept of (their) performance in mathematics
- The Value subscale comprised of 9 items and measured students' beliefs on the usefulness, relevance and worth of mathematics to their lives.
- The Enjoyment subscale comprised of 7 items and measured the degree to which students enjoy working (on) mathematics
- The Motivation subscale comprised of 4 items and measured students' interest in mathematics and (their) desire to pursue further studies in mathematics

Scoring was done with a five-point Likert Scale. The response options ranged from 1 (strongly disagree) to 5 (strongly agree). All responses to negatively worded statements were reversed prior to the data analysis.

Participants

A total of 544 students participated in all three cycles of this study in the academic year 2012, each cycle was spaced out by 3 months. The respondents were Year 7 & 8 students in 4 primary and 9 secondary schools of South Australia. The sample included government schools, catholic and independent schools. In the sample there were 64% female students and 36% male students.

Method

The Confirmatory Factor Analysis (CFA) was conducted using SPSS 19 and AMOS 16.0 which offers a viable method for evaluating construct validity of the instrument. The results of CFA of ATMI showed a good model fit for attitudes toward mathematics measurement model using several fit indices which include CFI, TLI, GFI, AGFI, and RMSEA. The findings revealed that fit indices criteria were satisfied. To assess internal consistency, Cronbach's alpha coefficients for each subscale were estimated and a very high Cronbach's alpha values were obtained for the overall scale and all the individual subscales. Concurrent equating is undertaken in order to give equal value to all responses on the three occasions and to prepare the data for further analysis. The Rasch model was employed for scaling and equating procedures. Tables 1, 3, 5 and 7 show mean scores for the sub-scales for ATMI that are obtained using the CONQUEST programme, logit is the unit of measurement. SPSS 19 was used to run Repeated measures ANOVA which compared the mean scaled scores (logits) at multiple time periods for this single group of respondents.

Results

Sub-scale Self confidence

Table 1. Descriptive statistics.

	Mean	Std. Deviation	N
SLFCON1	1.0537	1.53394	541
SLFCON2	1.0569	1.55197	541
SLFCON3	.9911	1.57329	541

Table 2. Pairwise comparisons.

Measure: MEASURE_1

(I) SLFCON	(J) SLFCON	Mean Difference (I-J)	Std. Error	Sig.[a]	95% Confidence Interval for Difference[a]	
					Lower Bound	Upper Bound
1	2	-.003	.044	1.000	-.110	.103
	3	.063	.048	.582	-.053	.178
2	1	.003	.044	1.000	-.103	.110
	3	.066	.038	.242	-.024	.156
3	1	-.063	.048	.582	-.178	.053
	2	-.066	.038	.242	-.156	.024

a. Adjustment for multiple comparisons: Bonferroni.

Sub-scale Value

Table 3. Descriptive statistics.

	Mean	Std. Deviation	N
VAL1	2.7479	2.26005	540
VAL2	2.7905	2.30423	540
VAL3	2.7533	2.43176	540

Table 4. Pairwise comparisons.

Measure: MEASURE_1

(I) Val	(J) Val	Mean Difference (I-J)	Std. Error	Sig.[a]	95% Confidence Interval for Difference[a]	
					Lower Bound	Upper Bound
1	2	-.043	.079	1.000	-.233	.148
	3	-.005	.087	1.000	-.215	.204
2	1	.043	.079	1.000	-.148	.233
	3	.037	.079	1.000	-.152	.226
3	1	.005	.087	1.000	-.204	.215
	2	-.037	.079	1.000	-.226	.152

a. Adjustment for multiple comparisons: Bonferroni.

Sub-scale Enjoyment

Table 5. Descriptive statistics.

	Mean	Std. Deviation	N
ENJ1	.5648228	1.79996221	538
ENJ2	.5137171	1.67239423	538
ENJ3	.3447304	1.73866437	538

Table 6. Pairwise comparisons.

Measure: MEASURE_1

(I) ENJ	(J) ENJ	Mean Difference (I-J)	Std. Error	Sig.[b]	95% Confidence Interval for Difference[b]	
					Lower Bound	Upper Bound
1	2	.051	.052	.968	-.073	.175
	3	.220*	.057	.000	.084	.356
2	1	-.051	.052	.968	-.175	.073
	3	.169*	.044	.000	.064	.274
3	1	-.220*	.057	.000	-.356	-.084
	2	-.169*	.044	.000	-.274	-.064

*. The mean difference is significant at the .05 level.

b. Adjustment for multiple comparisons: Bonferroni.

Sub-scale Motivation

Table 7. Descriptive statistics.

	Mean	Std. Deviation	N
MOT1	.7637175	1.75650691	533
MOT2	.6455214	1.68777225	533
MOT3	.5976291	1.79052892	533

Table 8. Pairwise comparisons.

Measure: MEASURE_1

(I) MOT	(J) MOT	Mean Difference (I-J)	Std. Error	Sig.[b]	95% Confidence Interval for Difference[b]	
					Lower Bound	Upper Bound
1	2	.118	.058	.129	-.022	.258
	3	.166*	.065	.031	.011	.321
2	1	-.118	.058	.129	-.258	.022
	3	.048	.053	1.000	-.080	.175
3	1	-.166*	.065	.031	-.321	-.011
	2	-.048	.053	1.000	-.175	.080

*. The mean difference is significant at the .05 level.

b. Adjustment for multiple comparisons: Bonferroni.

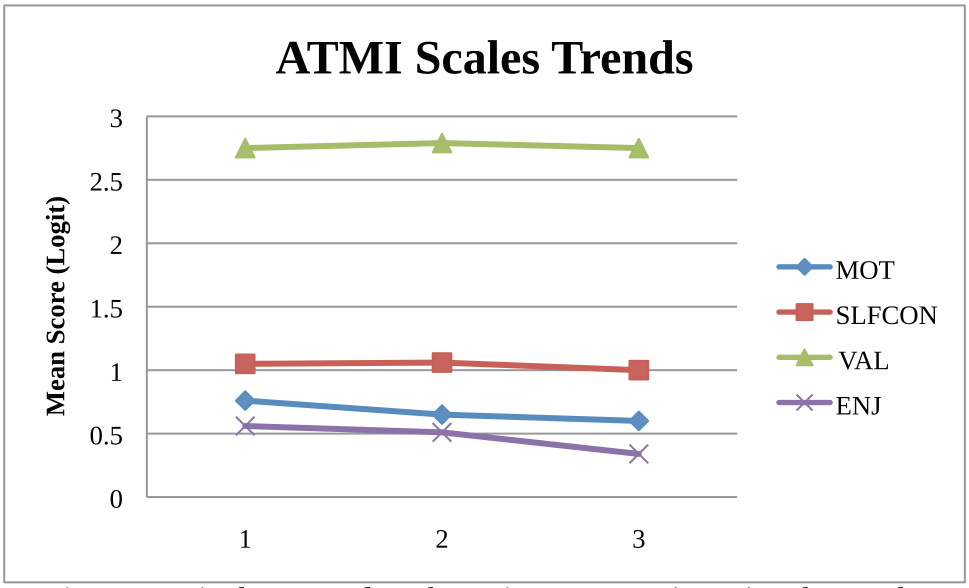

Figure 4. Attitudes Toward Mathematics Inventory (ATMI) scales trends.

Conclusion

The analysis using Repeated measures ANOVA showed a decline in the mean scaled score of the sub-scales of ATMI, except for sub-scale of value which remained stable from data point 1 to data point 3 (see Table 4). The mean scaled score of the sub-scale of Self- Confidence (see Table 2) declined over the academic year though did not show statistical significance. The sub-scales of Enjoyment (see table 6) and Motivation (see table 8) showed decline in the mean scaled scores which reach statistical significance at 95% confidence interval for difference of mean scaled scores. The ATMI scales trend (see fig.4) show that students of Year 7 and 8 consider mathematics to be a useful subject which has relevance and worth in their lives. The confidence and self-concept of their performance in mathematics show a decline but does not establish statistical significance. The degree to which students enjoy mathematics shows a fall of statistical significance. Students' interest in mathematics and desire to pursue further studies in mathematics post compulsory level also declines which is statistically significant. Therefore, it can be concluded on a sound experimental basis that students of Year 7 and 8 express lack of enjoyment and motivation in mathematics which is likely to influence their decisions about pursuing mathematics at post compulsory level and hence likely to further contribute to the already declining enrolments in advanced mathematics courses in South Australia.

Ma and Kishore (1997) acknowledge the complexity of the issue and state that the middle school students are at a point where 'alienation from mathematics' is at a serious level, however, they add that it is also 'optimally alterable towards engagement.' The analysis confirms that a negative disposition sets in and gains strength when students experience challenging mathematical experience in the middle years of schooling. Therefore, interventional strategies need to be put in place coupled with inspirational teaching to prevent or reverse decline of student attitudes toward mathematics.

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CONSTRUCTING A TETRAHEDRON IN MICROWORLD: POTENTIAL FUTURE RESEARCH AND PRACTICE

ANDY YEH
Queensland University of Technology
a.yeh@qut.edu.au

MATTHEW PEILE
Queensland University of Technology
matthew.peile@student.qut.edu.au

Traditionally, the teaching and learning about 3-D shapes has utilised concrete materials such as solid wood, plastic connectors and paper nets for modelling. While they are good materials, limitations such as time, accuracy, manipulability and creativity , etc. could apply. This paper presents how a 3-D microworld named VRMath 2.0 (VRMath2) can be utilised to construct a tetrahedron from within a cube, which would be difficult to do with real materials. The use of virtual materials, in addition to real materials, would have implications on how the teaching and learning about 3-D geometry could be enhanced, and how the human spatial cognition and abilities can be further researched.

Introduction

Human spatial ability is a factor of intelligence and is an important predictor of future career paths, especially in scientific research, engineering and the arts (Ivie & Embretson, 2010). Pittalis and Christou (2010) also pointed out that spatial abilities are closely related to academic achievement, particularly in mathematics and geometry. In other words, learning about geometry would contribute to the development of spatial abilities.

The teaching and learning about three-dimensional (3-D) shapes (a topic in Measurement and Geometry) in primary school classrooms usually starts with real objects in the natural environment, then formalises into more regular shapes such as sphere (ball), cube (box), cylinder (can), cone, prism and pyramid. Typical activities include students constructing drawings, interpreting diagrams, and constructing physical models using nets or concrete materials. Concrete materials play an important role for school children to recognise, identify, visualise, and examine the properties of shapes. However, the use of concrete materials could be time consuming for preparation and construction. Flexible materials (e.g., sticks and foam balls, rubber bands, straws and pipe-cleaners) can have accuracy issues when measuring and they are not easy to put together. Solid materials (e.g., timber, plasticine, paper and cardboard) may prevent students from visualising invisible edges. The use of commercial shape connectors with some standard units could also be inflexible for construction (e.g., designed only for certain 3-D shapes, could only connect in certain angles, no diagonal length). These conditions of concrete materials could limit the creativity and further development of geometrical understanding of 3-D shapes and spatial abilities such as spatial visualisation, spatial orientation, and spatial relations (for details about the three spatial abilities, see Lohman, 1988).

The use of information and communication technology (ICT) tools can address most of the issues above. ICT tools such as LOGO microworld and Dynamic Geometry Software (DGS) are intuitive to use, accurate when constructing geometric objects, easy to manipulate once the geometric objects are created, and expressive[13] to create geometric objects. The power of LOGO is its ability to express natural geometric movement and the programming language, which links multiple representations of mathematical entities (Hoyles, Noss & Adamson, 2002). However, most of the LOGO environments are lacking interactive graphics, particularly on 3-D capabilities. DGS, on the contrary, has great interactive 3-D graphics that allow direct manipulation of virtual geometric objects, but it often has no natural expressions of movement nor a focus on programmability. It is often perceived that DGS operates on more formal geometric thinking and reasoning as it is mostly based on Euclidean geometry. Constructing 3-D shapes using ICT tools usually force and require learners to use and develop their understanding of geometrical relationships to produce dynamic “figures”, rather than static “drawings” (Jones, 2000). This is one key aspect for developing deep geometric understanding.

Based on the information presented above, it was thought that an ICT tool empowered by natural expressions of geometric movement, programming ability and a 3-D interactive graphic, could not only enable children to develop deeper understanding about 3-D geometry, but also develop a wide range of spatial abilities through new ways of thinking and doing in the new computational environment (Resnick, 1996). Therefore, the purpose of this paper is to demonstrate how to create a tetrahedron in such an ICT tool environment, named VRMath 2.0 (VRMath2), as an example, and discuss its implications for potential future research and practice.

The learning environment: VRMath2

VRMath2 is an open online learning environment. It has three main components: an interactive virtual reality (VR) 3-D interface, a 3-D programming interface, and a Web 2.0 style sharing and collaborating interface. This paper focuses on the first two interfaces (see Figure 1).

The VR interface is virtually an unlimited 3-D space, in which learners can navigate to see the virtual world from any perspective, and interact with the objects created in the virtual space. As a VR space, the objects in it include visible geometric objects with rich colours and textures, and invisible environmental objects such as light, sound and camera viewpoints. This interface helps develop *spatial visualisation* when manipulating geometric objects, particularly in small scale, and *spatial orientation and relations* when navigating, particularly in large scale. One thing to note about this 3-D VR graphic is that its unit of measurement is not based on pixels on a screen as other 2-D or 3-D computer graphics. As its name “virtual reality” implies, this VR 3-D space

13 The ICT tools provide facilities such as movement commands, mouse dragging actions, and colours and textures for users to express their ideas.

utilises units that can be interpreted as metres to reflect the real world settings and to work with the Cartesian coordinate system.

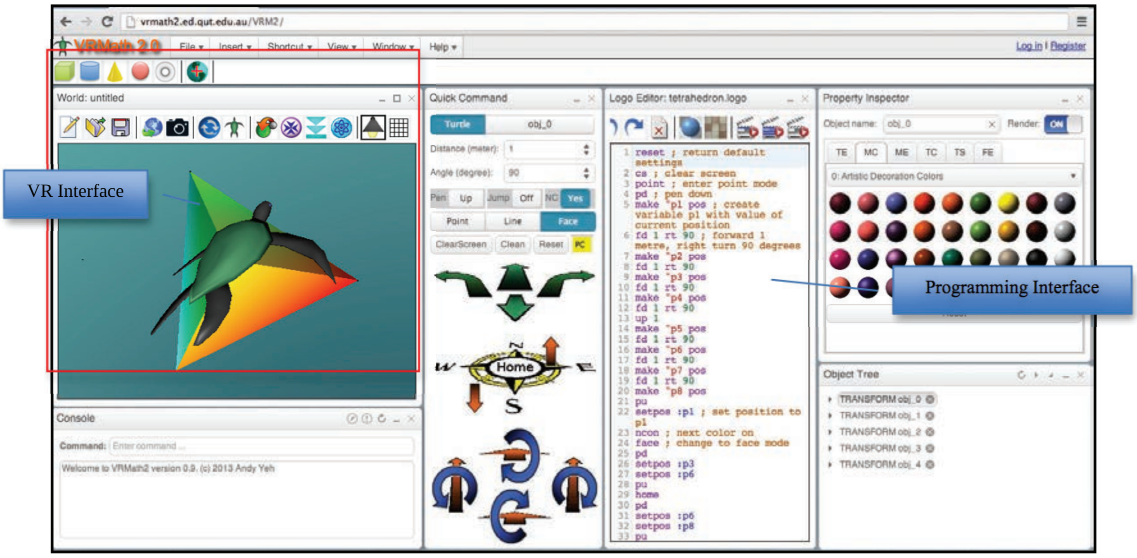

Figure 1. VRMath2.

The programming interface includes fully functional LOGO programming language and a set of graphic user interfaces (GUI) to assist with the creation of virtual worlds. Because of 3-D, this LOGO has an extended set of 3-D movement commands (Figure 2) and 3-D geometric primitives commands.

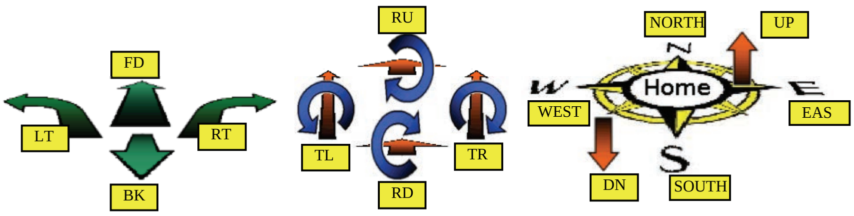

Figure 2. GUI of 3-D movement commands.

Using the concept of frame of reference (FOR), these 3-D movements can be classified into two categories: egocentric and fixed movements. The egocentric movements include FORWARD[14] (FD), BACK (BK), and the six turns of RIGHT (RT), LEFT (LT), ROLLUP (RU), ROLLDOWN (RD), TILTRIGHT (TR) and TILTLEFT (TL). These egocentric movements are based on the orientation of the turtle in the 3-D space. The fixed movements are the four compass points EAST, WEST, NORTH and SOUTH, plus the UP and DOWN. These movements express a relationship to a fixed direction and are not related to the turtle's orientation. HOME is a special fixed movement as it brings the turtle back to home—position (0, 0, 0), facing north with back up. There is also a set of coordinate commands (universal FOR) that changes the turtle's position (or location) by specifying the x, y and z coordinate. For example, SETPOS [x, y, z] will

14 The VRMath2 Logo commands are case insensitive. For ease of reading, we use capital letters for Logo commands in the paragraphs. If programmed in Lego Editor, we simply use small letters for Logo commands.

move the turtle to the specified coordinate [x, y, z]. With these natural expressions of 3-D movement, the turtle is able to turn in any direction and move to any location to create objects in the 3-D virtual space.

As a programming language, LOGO can create variables to store information and customise commands. For example, MAKE “p1 POS will store the turtle’s current position to the variable named p1. The command POS is a built-in command which returns the turtle’s coordinate. The MAKE command takes two inputs, and stores the value of the second input into the first input. Once this is done, SETPOS :p1 will then take the turtle to the coordinate (location or position) stored in p1.

It is also very easy to customise a command in LOGO. For example, if SETPOS is hard for young children to remember, we may create an easier command called GOTO instead. We can type in the Logo Editor with the following program and execute:

```
TO GOTO :place
SETPOS :place
END
```

The key words TO and END specify the beginning and end of a procedure or command. The GOTO is the procedure’s name. The :place specifies that the procedure GOTO will take in an input. This new GOTO procedure will then pass on the input :place to SETPOS command. Therefore, after the procedure GOTO is defined (executed), GOTO :p1 will be doing the same as SETPOS :p1.

Construction of tetrahedron in VRMath2

The tetrahedron is one of the five Platonic solids, which are among the regular 3-D shapes constructed in primary school classrooms. The construction of a tetrahedron can be a simple process when using paper nets or commercial shape connectors (see Figure 3).

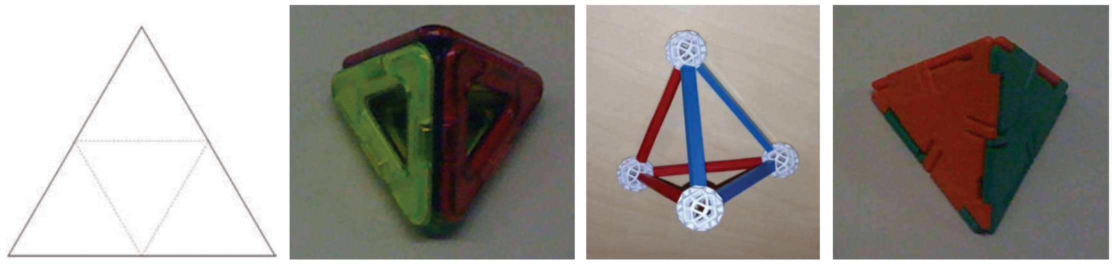

Figure 3. Tetrahedron net and models.

The use of concrete materials allows children to easily identify the properties of tetrahedron as four faces of equilateral triangles, four vertices and six edges. And as in most 3-D shapes, the angles (between two faces and between an edge and a face) within are not easy to find or measure. Therefore, it is in fact more complex and difficult to construct in ICT tools.

The construction of a tetrahedron from within a cube is an extension of learning about the relationships between the two. It is a good activity to develop children’s geometric understanding, and spatial visualisation and orientation abilities. The use of concrete materials can be beneficial, however, they start to reveal some limitations. As can be seen in Figure 4, the shape connectors do not have diagonal connectors. Therefore, blu-tack was used to secure the edges of tetrahedron on some vertices.

Although it is possible to use concrete materials to achieve this task, the cube frame could distract from the visualisation of the tetrahedron in the cube. The tetrahedron inside the cube is also only a frame. It could also be difficult for children to visualise the four triangle faces. Using concrete materials to create the four faces could be very difficult. ICT tools such as VRMath2 can address the limitations that arise from using concrete materials.

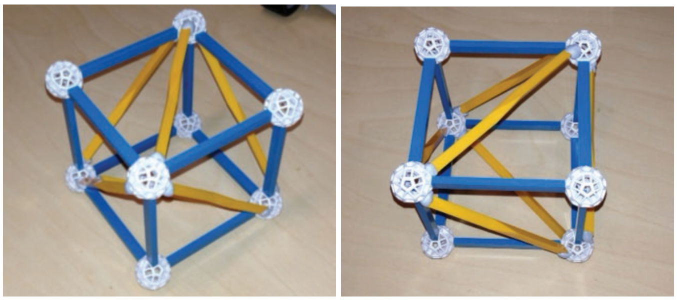

Figure 4. Tetrahedron in cube.

To construct a tetrahedron in VRMath2, one can use trigonometry to find out the angles to turn, then forward (move) the turtle to create edges or faces. For example, the angle between an edge and a face is

$$\arccos\left(\frac{1}{\sqrt{3}}\right) = \arctan\left(\sqrt{2}\right) \cong 54.7356°$$

and the angle between two faces (dihedral) is

$$\arccos\left(\frac{1}{3}\right) = \arctan\left(2\sqrt{2}\right) \cong 70.5288° .$$

In LOGO, one can simply type in RU acrtan(sqrt(2)) to pitch up the angle between an edge and a face.

For young children, who do not know about trigonometry, it is suggested that teachers start with discussions about a cube on a drawing or using concrete materials (Figure 5).

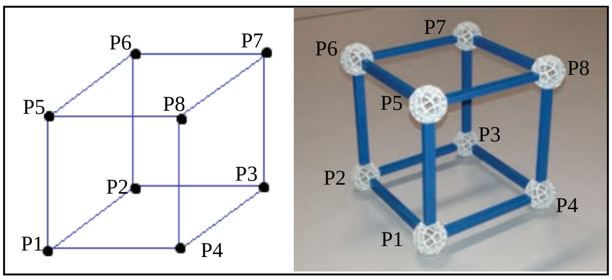

Figure 5. Cube with labelled vertices.

In these discussions, the eight vertices of the cube become the eight positions (or locations), which can be labelled as p1 to p8. The 3-D Cartesian coordinate system is not necessary here, although it may be used by upper primary or secondary students who have prior knowledge about 3-D Cartesian coordinates.

Then in VRMath2, these eight positions can be stored in eight variables named as p1 to p8. To do so, we need to command the turtle to move into those eight positions. This can be achieved by using the movement icons (Figure 2) in Quick Command and

Console (Figure 1). Or if students are familiar with movement commands, they can use their mental imagery (with or without the help of Figure 5) and program in the Logo Editor as below:

```
cs ; clear screen
point ; enter point mode
pd ; pen down
make "p1 pos ; create variable p1 with value of current position
fd 1 rt 90 ; forward 1 metre, right turn 90 degrees
make "p2 pos
fd 1 rt 90
make "p3 pos
fd 1 rt 90
make "p4 pos
fd 1 rt 90
up 1 ; fixed FOR command to move turtle up 1 metre without changing its direction.
make "p5 pos
fd 1 rt 90
make "p6 pos
fd 1 rt 90
make "p7 pos
fd 1 rt 90
make "p8 pos
pu ; pen up
```

In the above program, the semicolon sign (;) denotes that what follows are comments only. The POINT command in second line specifies the pen mode (turtle track) to be point only. Other pen mode commands include LINE and FACE. The pen mode commands need to be given before PENDOWN (PD), and during PD, the pen mode cannot be changed. As can be seen in the program, the turtle will be moving through the eight positions, and will leave eight points track in the 3-D space when executed (see Figure 6).

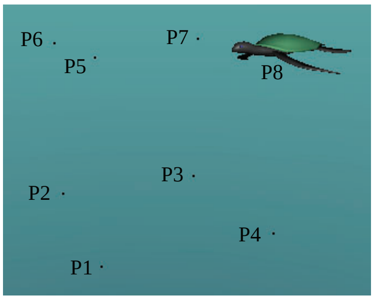

Figure 6. Cube vertices in VRMath2.

Next, the discussion will focus on the properties of tetrahedron. Since the faces of tetrahedron are equilateral triangles, we need to find four sets of three positions out of the eight and connect them to form four triangles. Another idea could be to find the four vertices in the eight positions, then connect the four vertices with lines, or utilise the four vertices to construct the four triangle faces. There could be a couple of possibilities and it is a good idea to try from the first position p1. With the help of concrete materials (e.g., Figure 5) or the navigation in the VR space, students should be able to find that it cannot be in adjacent positions and the four edges would be the diagonal of the square faces on the cube. The following program demonstrates how to start from p1 to create a tetrahedron in the cube.

```
setpos :p1 ; set position to p1
ncon ; next colour on, this will help create colour rich triangles
face ; change to face mode
pd ; pen down for first triangle
setpos :p3
setpos :p6 ; as the turtle has moved through p1, p3 and p6, a triangle face will be created
pu ; pen up for the end of first triangle
setpos :p1
pd ; this pd creates the second triangle with vertices in position p1, p6 and p8
setpos :p6
setpos :p8
pu ; end of the second triangle
pd ; this pd creates the third triangle with vertices in position p8, p6 and p3
setpos :p6
setpos :p3
pu ; end of the third triangle
pd ; this pd creates the fourth triangle with vertices in position p3, p1 and p8
setpos :p1
setpos :p8
pu ; end of the fourth triangle and the tetrahedron
```

When the above program is executed, a colourful tetrahedron will be created in the 3-D virtual space. We can hide the turtle by clicking an icon or giving a command HIDETURTLE (HT) for better views of the tetrahedron. The cube vertices and each of the four triangle faces can also be hidden to examine the construction process. Figure 7 below shows the process of construction.

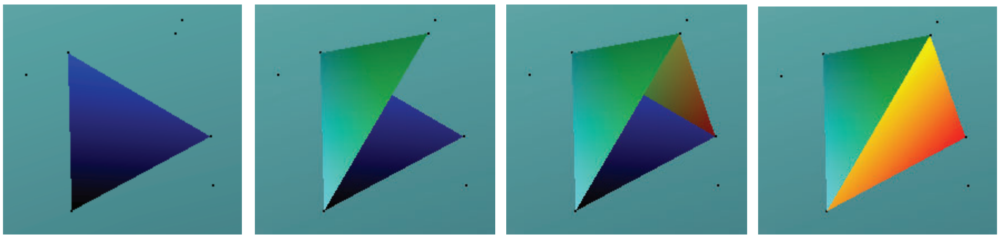

Figure 7. Construction process of tetrahedron in VRMath2.

Implications for future research and practice

In this paper, we have demonstrated the construction of a tetrahedron from within a cube in a powerful ICT tool or microworld named VRMath2. VRMath2 has a VR 3-D interface, a programming interface, and a social (web 2.0 style) interface. The VR 3-D interface allows users to navigate in virtual worlds and examine and interact with virtual objects. The programming interface allows users to use natural expressions of 3-D movement and programming language to create virtual objects in virtual worlds. The social interface enables users to share their creations of virtual worlds and collaborate with other users. We think that VRMath2 represents a new paradigm of computing, and this new paradigm of thinking and doing mathematics would have some profound implications for future research and practice.

For research, VRMath2 is a pertinent vehicle for investigating and developing human spatial cognition and abilities. Because of its VR 3-D interface, users will naturally tend to think 3-D and work 3-D. Unlike the 3-D interfaces in other ICT tools that focus on one or a few classical geometric shapes, this VR 3-D space enable users to think and reason on the larger scale of 3-D worlds as well as on the smaller scale of geometric objects. The larger scale of virtual worlds provides opportunity particularly for the operation and development of spatial orientation ability. The smaller scale of geometric objects will particularly facilitate the operation and development of spatial visualisation ability. The construction of virtual objects and worlds in VRMath2 using LOGO programming language will also assist the operation and development of spatial relations ability. VRMath2 is rich in facilities that encourage users to express their thinking and ideas. When users express their thinking and ideas via programming and/or blogging their creations, their artefacts will help research into how we think, reason and develop our spatial abilities.

For practice, VRMath2 is also a vehicle to foster creativity. The unlimited virtual space and programming means endless possibilities of design and creation. When creating a virtual world, there are always many ways of doing and solving problems. For example, to connect or move from p1 to p3 in the tetrahedron task, instead of SETPOS :p3, a more natural way of TOWARDS :p3 FORWORD DISTANCEBETWEEN :p1 :p3 can also achieve the same result. Furthermore, creating a virtual world in VRMath2 is not just for making classroom geometric objects. With rich textures and environmental effects, the creation and construction of virtual worlds in VRMath2 can have virtually any object (such as a building, a table, a chair or a tree) that appear in the real world. Learning in VRMath2 is truly by designing and doing. Teachers can design learning and assessment activities using VRMath2, with possibilities to integrate disciplines such as mathematics, arts, science, engineering and technology.

To conclude, this paper has presented a new microworld named VRMath2 and demonstrated the construction of a tetrahedron in this microworld. We hope that the construction of a tetrahedron in VRMath2 serves as a good example to encourage more future research and practice. And from more future research and practice, VRMath2 can evolve towards a better learning environment for all.

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EMPOWERING TEACHERS IN NEW WAYS

ALEXANDER (ALEC) YOUNG
Ingenious Technological Enterprises
akyoung5@bigpond.net.au

Empowering teachers to collaborate in new ways to see how they can:

- improve their 'assessment for learning' through the innovative use of their school photocopier as a high speed scanner obtaining exceptionally powerful insights into their assessment of written responses and multiple choice assessment;
- measure 'the effect size' of their teaching through the use of pre-test/post-test scenarios;
- reduce their workload and increase their students' rate of learning compared with the national average;
- identify gaps in student learning at any point in time not observable under conventional assessment;
- quantify question quality through the automatic application of classical test theory.

Introduction

I heard a keynote speaker at a recent conference ask the question, "What profession ignores its own research more than education?" With this in mind, how is it that we as a profession can sometimes get it wrong when it comes to improving literacy and numeracy outcomes? "The Australian National Audit Office found that an ambitious $540 million scheme introduced by the Rudd Government, in an attempt to lift literacy and numeracy skills, made no discernible difference to the results of the schools taking part." (Courier Mail, 1 August 2012, p. 23)

By contrast, Professor Helen Timperley more than doubled the learning gain of students in 300 New Zealand primary schools and the biggest winners were the schools' lowest-performing students—the bottom 20 per cent. Their achievement gains grew three to four times faster than the expected rate nationally (Milburn, 2009).

Laptop computers are seen by some as a solution to educational productivity. However, a ten million dollar research project in 2004 reported by Hu (2007) showed that there was no significant improvement in educational outcomes by students with laptops compared with those without laptops. This calls into question how the education dollar is best used.

On the sporting field, the Olympic Games are often seen as our nation's sporting benchmark, whereas, the Trends in International Mathematics and Science Study (TIMSS) can be seen as one of education's.

The average success of Australian students in Mathematics in TIMSS for the period 1995 to 2011 can be seen in Figure 1 (Thomson, Hillman & Wernert, 2012, p. 17). The fact that the trend is not going up is of major concern. When we compare trends of teaching success for Korea and Chinese Taipei (Figure 2; Thomson, Hillman & Wernert, 2012, p. 17) with Australia's, the result is alarming.

It can be seen therefore that Australia needs to take a really hard look at 'lifting its game'.

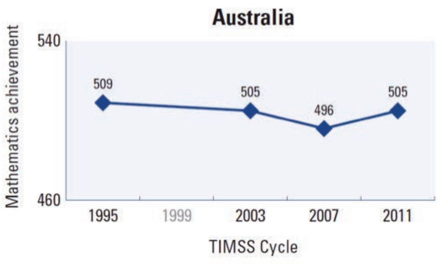


Figure 1.

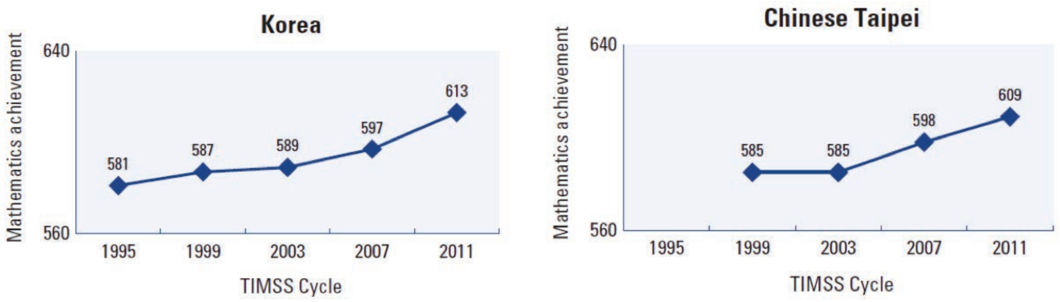


Figure 2.

At this point we might recall a quote from Winston Churchill when things were grim during WWII, "It is no use saying, 'We are doing our best'; we have to succeed in doing what is necessary."

AutoMarque

AutoMarque enables educators to obtain exceptionally powerful insights into their assessment of written responses, multiple choice assessments and practical work. Teachers will learn about the quality of their interventions as well as reducing some aspects of their workload and be able to improve students' outcomes to at least double the rate of the national average gain.

With this in mind a teacher is now able, with AutoMarque, to mark thirty students' paper based responses to multiple choice questions, in less than two minutes, using a photocopier connected to a computer.

Assessment of written work

One of the major challenges teachers are confronted with almost every day is how to improve students' capacities to write assignments. The author contends that this task can be more effectively achieved when teachers list in detail what is expected of students. This can be illustrated by showing students examples of quality work (Petty, 2006, p. 90; Hattie, 2009, p. 172).

However, showing students quality work is, in itself necessary but not sufficient. If a rubric sheet with scoring apportioned is used by teachers then the data produced can be used in two ways;

1. As explicit feedback to the student.
2. As forensic feedback to the teacher to obtain:
    - better understanding of her/his teaching effectiveness
    - identification of common learning gaps across a teaching group
    - the learning needs of each student.

When the learning needs of a large portion of a teaching group appear to be in common a digital rubric is used to explain the quality of each component in a piece of work. This is also used as an assessment instrument to aid in the forensic analysis of students learning needs. An example of a digital rubric can be found at Annex A.

You will note that the digital rubric consists of a number of questions all of which can be answered in binary terms. That is either ‘yes’ or ‘no’, or if you prefer ‘competent’ or ‘not competent’.

Once the digital rubric is developed by the teacher, it is initially used as a list of things to taught and then be used as a marking and feedback tool.

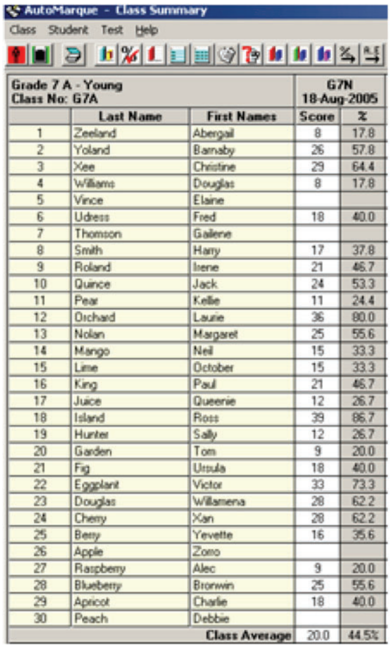
However, before giving the assessed digital rubric sheets to their students (as detailed feedback) teachers scan them a photocopier using AutoMarque to obtain powerful insights not otherwise observable.

This form of assessment enables the ready application of 21st century skills (Critical thinking, analytical, reasoning and problem solving skills) and the analysis of student progress, aside from the pedagogical challenges that they present.

Teacher self-assessment

“It is critical that teachers learn about the success or otherwise of their interventions” (Hattie, 2009, p. 24). According to Hattie (2009, p. 241), students already know about forty per cent of the material the teacher is planning to teach. That being the case, it is essential that teachers can quickly identify the student knowledge base and adapt their pedagogy accordingly.

How does AutoMarque inform the teacher of his her teaching effectiveness?



AutoMarque - Class Summary

Class Student Test Help

Grade 7 A - Young Class No: G7A			G7N 18-Aug-2005	
	Last Name	First Names	Score	%
1	Zeeland	Abergail	8	17.8
2	Yoland	Barnaby	26	57.8
3	Xee	Christine	29	64.4
4	Williams	Douglas	8	17.8
5	Vince	Elaine		
6	Udress	Fred	18	40.0
7	Thomson	Gailene		
8	Smith	Harry	17	37.8
9	Roland	Irene	21	46.7
10	Quince	Jack	24	53.3
11	Pear	Kellie	11	24.4
12	Orchard	Laurie	36	80.0
13	Nolan	Margaret	25	55.6
14	Mango	Neil	15	33.3
15	Lime	October	15	33.3
16	King	Paul	21	46.7
17	Juice	Queenie	12	26.7
18	Island	Ross	39	86.7
19	Hunter	Sally	12	26.7
20	Garden	Tom	9	20.0
21	Fig	Ursula	18	40.0
22	Eggplant	Victor	33	73.3
23	Douglas	Willamena	28	62.2
24	Cherry	Xan	28	62.2
25	Berry	Yevette	16	35.6
26	Apple	Zorro		
27	Raspberry	Alec	9	20.0
28	Blueberry	Bronwin	25	55.6
29	Apricot	Charlie	18	40.0
30	Peach	Debbie		
		Class Average	20.0	44.5%

Figure 3. Initial student results.

After scanning either, multiple choice sheets, practical digital rubric analysis sheets or written digital rubric analysis sheets you open AutoMarque on your computer and the results appear as per Figure 3, in two columns raw score and percentage.

Until now, a class list of results like this is all that teachers have had, limiting teacher understanding of student learning needs.

AutoMarque's power is in the icons at the top of the results sheet.

To obtain results by strand of learning click on , one of sixteen icons available. Figure 3 is a summarized version of the class' success by strand of learning. In this case, the teacher needs to reconceptualise his/her teaching of the three strands of learning in which the class performed poorly.

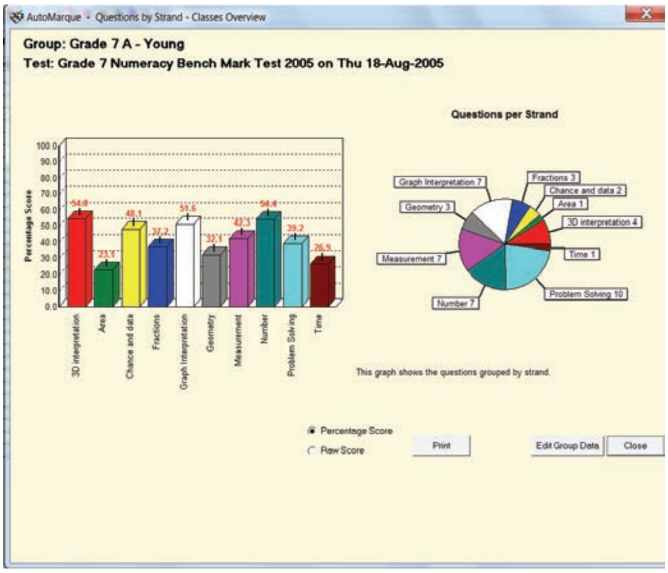
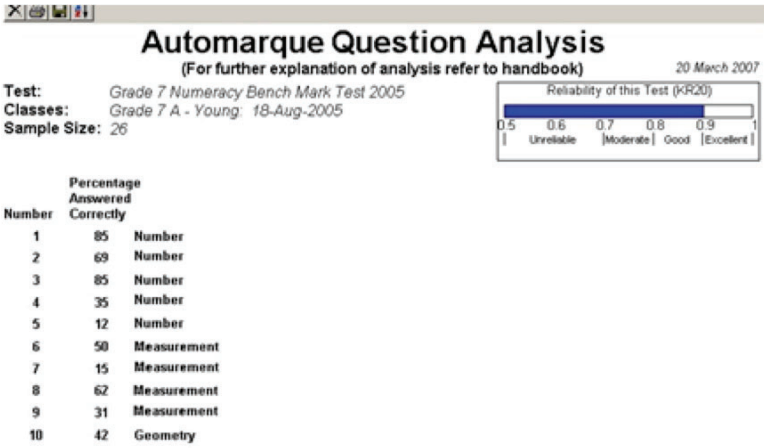


Figure 4. Class success by strand of learning

To obtain results per question, click on . Figure 5 shows the students' results per question and the strand of learning to which each question belongs and an indication of the overall reliability of the test.



Automarque Question Analysis

(For further explanation of analysis refer to handbook) 20 March 2007

Test: Grade 7 Numeracy Bench Mark Test 2005

Classes: Grade 7 A - Young: 18-Aug-2005

Sample Size: 26

Reliability of this Test (KR20)

0.5 0.6 0.7 0.8 0.9 1

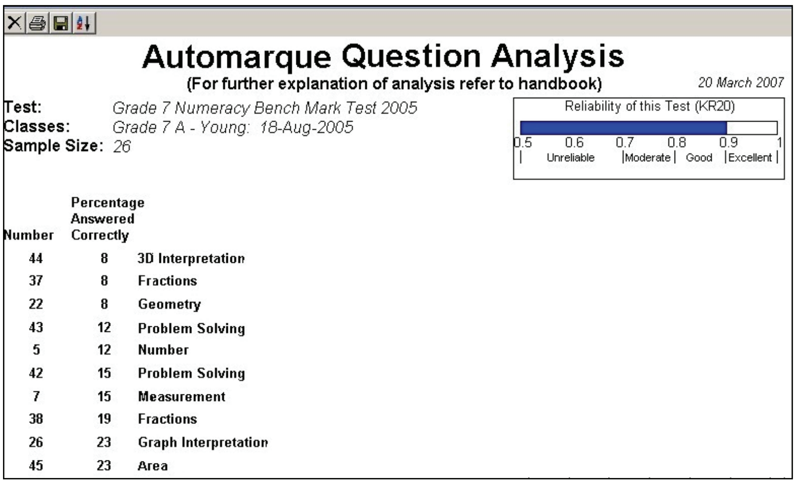
Unreliable | Moderate | Good | Excellent

Number	Percentage Answered Correctly	
1	85	Number
2	69	Number
3	85	Number
4	35	Number
5	12	Number
6	50	Measurement
7	15	Measurement
8	62	Measurement
9	31	Measurement
10	42	Geometry

Figure 5. Class results per question

Reliability of the test is calculated by AutoMarque using the Kuder Richardson formula 20 (KR20) test of reliability. KR20 is the most accurate of the practical Kuder Richardson formulas for estimates of reliability. It measures consistency of responses to all the items within the test. It is the mean of all possible split half coefficients Athanasou & Lamprianou, 2002).

If one or more of the questions require more than one response to obtain a correct answer then the reliability analysis used is Chronback's Alpha coefficient of reliability.



Automarque Question Analysis

(For further explanation of analysis refer to handbook) 20 March 2007

Test: Grade 7 Numeracy Bench Mark Test 2005

Classes: Grade 7 A - Young: 18-Aug-2005

Sample Size: 26

Reliability of this Test (KR20)

0.5 0.6 0.7 0.8 0.9 1

Unreliable | Moderate | Good | Excellent

Number	Percentage Answered Correctly	
44	8	3D Interpretation
37	8	Fractions
22	8	Geometry
43	12	Problem Solving
5	12	Number
42	15	Problem Solving
7	15	Measurement
38	19	Fractions
26	23	Graph Interpretation
45	23	Area

Figure 6. Results re-sorted to show the least well understood

Figure 6, identifies the questions that were least well performed by the class. In effect, Figures 4, 5 and 6 provide a teacher with insights into her/his teaching effectiveness enabling immediate redirection of the teaching effort, where necessary, to meet the class' needs. Similarly, if it is evident that the students know what is being taught then time is saved by immediately moving on to new work.

The detailed results of a digital rubric seen in this way transforms teaching to a new powerful level.

When compared with the traditional practice of revising the whole test, teachers are now able to define the areas of weakness in learning in a class as a whole and concentrate their teaching resources to address the class' specific learning needs.
The author understands this aspect of AutoMarque's output is one of the many 'world first' benefits it provides.

When pre-test/post-test analysis is conducted by AutoMarque it insights for teachers on the quality of their practice, (Figure 7) showing the difference per strand of learning and the effect size of the teaching group's learning gain (Hattie, 2009, p. 8).

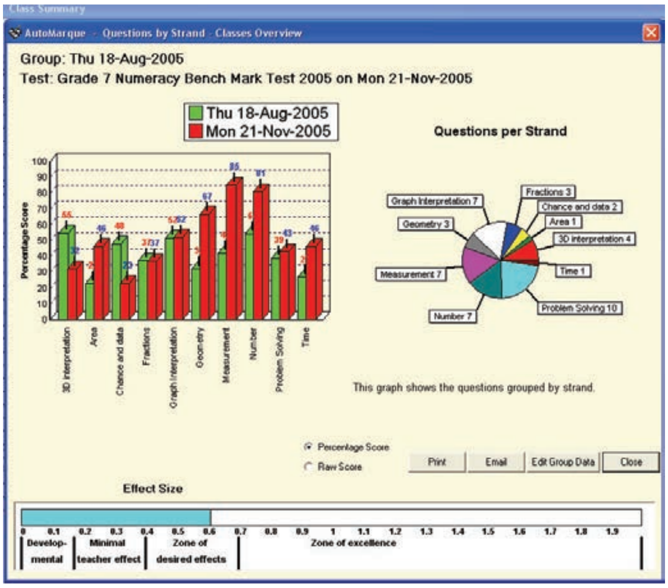


Figure 7. Teaching effectiveness

This print out is expected to be the subject of much discussion about how best to teach particular concepts; thus improving student outcomes in new ways.
When teachers aim to demonstrate excellence in teaching by regularly creating an effect size of better than 0.7, students results are likely to rocket ahead.

Student work

Prompt feedback to each student when assessed by multiple choice assessment is readily achieved by clicking on the print icon within AutoMarque and they appear as shown in Figure 8. This removes the need for hand marking multiple choice student work and yet provides detailed feedback. This is in line with Hattie's findings that feedback is one of the most powerful things you can do for your students in their process of learning.

Assessment for learning as detailed above, has great potential to assist teachers improve the quality of their teaching but will not necessarily deliver insights into individual student needs.

Individual student results, by strand of learning, compared with their peer group are obtained on the click of a mouse on . This provides a powerful insight into an individual student's needs (Figure 9). In this instance, the student's teacher would not have known that the student could not do any of the 'chance and data' questions even though he obtained 62% in the test overall. It can be seen therefore that this feedback provides powerful insight into a gap in learning that would otherwise have been invisible. How many other students learning needs like this are going unnoticed?

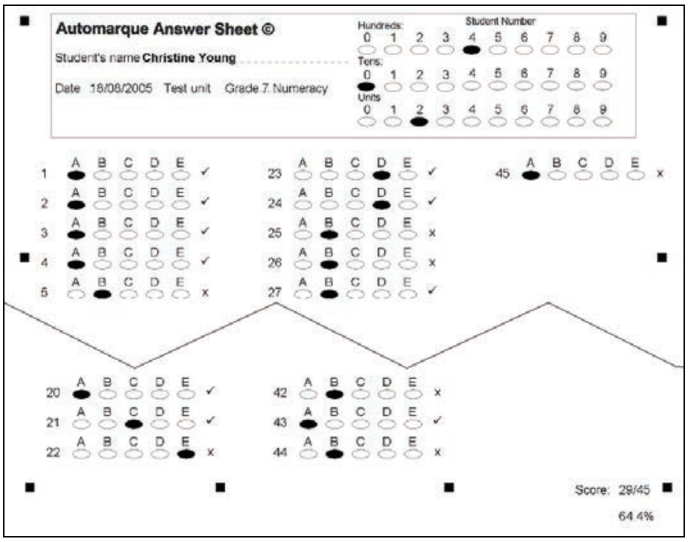


Figure 8. An example of automated feedback to students.

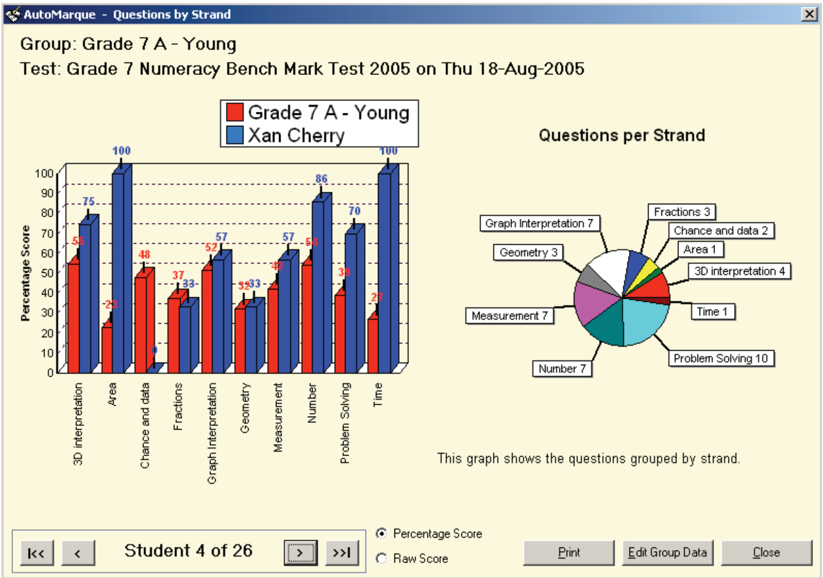


Figure 9. A Student's results per strand of learning compared with the class average.

AutoMarque's student strand analysis result sheet is of great assistance when counselling the teacher's supervisor that intervention is required. As AutoMarque stores test results, a history of student achievement is readily retrievable and student progress easily and clearly demonstrated.

This 'one off' glimpse is further improved when the same test is conducted a second time as shown in Figure 10. For teachers who wish to create a feeling of progress amongst their students this facility is particularly powerful. The inclusion of the index of educational growth (effect size) provides a further quantification of the learning that has taken place.

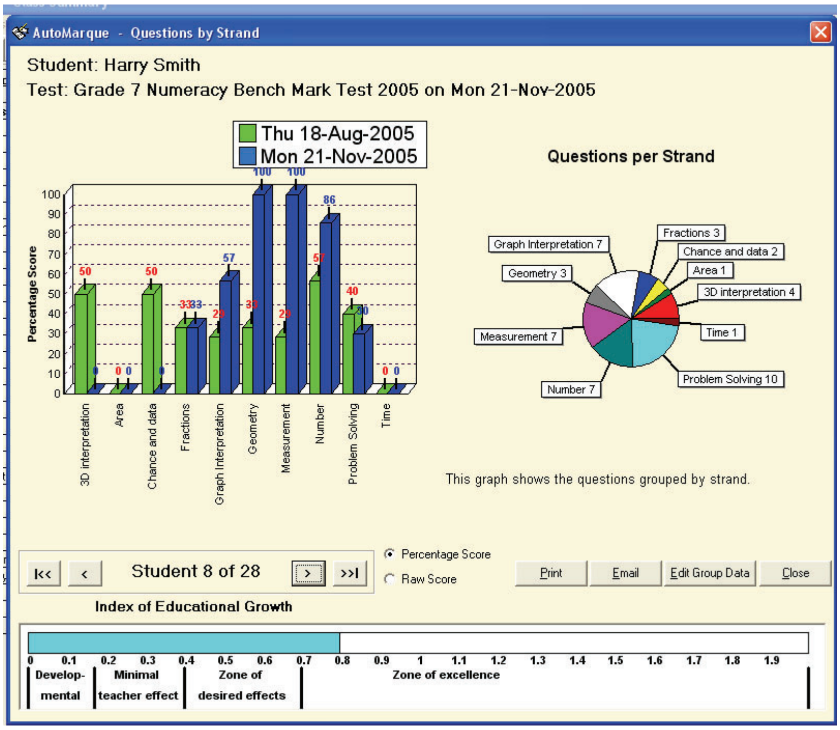


Figure 10. A student's results over time per strand of learning.

If you would like to see how a student has succeeded in one strand of learning over a series of assessments it is only a matter of clicking on to obtain an image like that in figure 11. So long as the strand has the same name it can analyse a student's success in a range of subjects. For example the strand "Number' could be tracked across Commerce, Science and Maths providing new powerful insights not previously available.

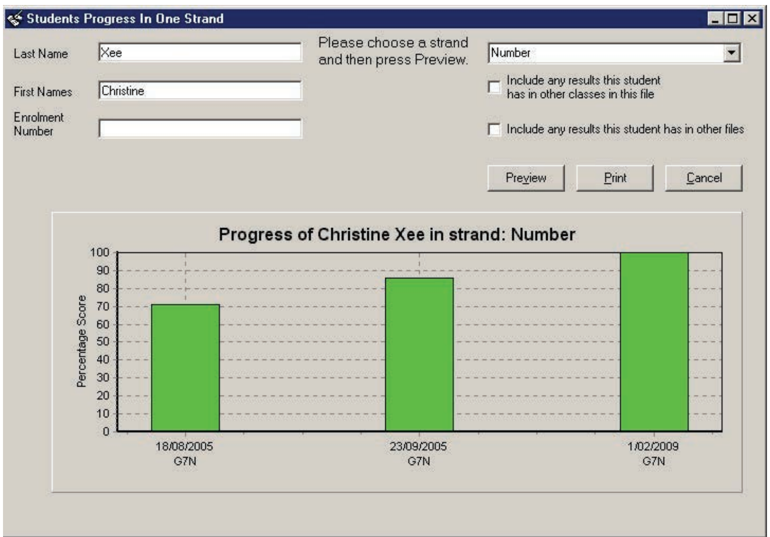


Figure 11. Tracking of a strand of learning (Number) across different assessments.

If a teacher wants to obtain an overview as to which students have similar gaps in learning, AutoMarque's 'needs analysis' shows this in Figure 12. It ranks students from the least successful to the most successful in a particular strand. AutoMarque also enables the ranking of a series of classes within an institution, based on a single strand of learning. Consequently, a more tactical approach can identify groups of students who have similar needs enabling targeted remediation.

AutoMarque Needs Analysis

Test: Grade 7 Numeracy Bench Mark Test 2005

Strand: Problem Solving

	Class	Date set	Percent correct
Douglas Williams	Grade 7 A - Young	18-Aug-05	0.0
Abergail Zeeland	Grade 7 A - Young	18-Aug-05	0.0
Tom Garden	Grade 7 A - Young	18-Aug-05	10.0
Sally Hunter	Grade 7 A - Young	18-Aug-05	10.0
October Lime	Grade 7 A - Young	18-Aug-05	10.0
Kellie Pear	Grade 7 A - Young	18-Aug-05	10.0
Yevette Berry	Grade 7 A - Young	18-Aug-05	20.0
Victor Eggplant	Grade 7 A - Young	18-Aug-05	70.0
Christine Xee	Grade 7 A - Young	18-Aug-05	70.0
Laurie Orchard	Grade 7 A - Young	18-Aug-05	90.0
Ross Island	Grade 7 A - Young	18-Aug-05	100.0

Figure 12. Learning needs analysis.

Identification of quality questions

There are considerable resources on the 'net', usually in pdf format, available for teachers to acquire. The quality of these questions can then be assessed by AutoMarque. AutoMarque requires a minimum of 100 students to have completed an identical test before the question quality analysis (item analysis) can take place.

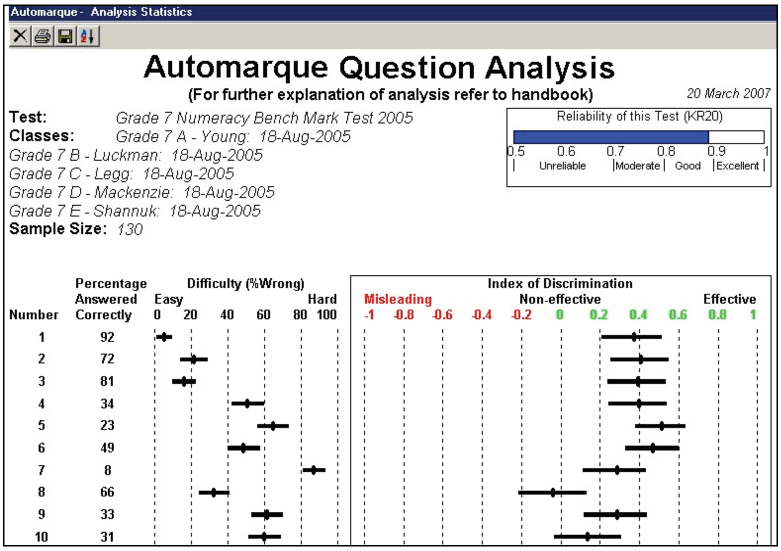


Figure 13. Item analysis.

In Figure 13, we see how five classes, 130 students, have completed an identical test and that an analysis of each question is displayed as well as an indication of the test's overall reliability.

AutoMarque expresses the difficulty of a question as a percentage of the students who answered incorrectly. For discrimination, the software uses a Point Biserial

Coefficient of Correlation between the correctness of the response to the given question and the students' result in the test as a whole (Athanasou & Lamprianou, 2002, p. 309). The confidence intervals displayed are indicated by the length of each line, per question, for difficulty and discrimination. The line's length is inversely proportional to the square root of the sample size. This helps raise the quality of a teacher's work and the consequent improvement in students' outcomes.

Surveys: Rating scale analysis

It is one thing to know a student could not correctly answer a particular question in an assessment, (as shown above); it is another thing to be able to understand why. AutoMarque provides two techniques to assist teachers to understand more about student misunderstanding or wrong thinking, they are:

- generation of a spreadsheet for detailed perusal, and
- segment analysis of student responses.

A spreadsheet is generated by clicking on  in AutoMarque, part of which is seen in figure14. The spreadsheet enables the teacher to identify the distracters chosen by students for each question. For example, in question five only 12 per cent of the students selected the correct answer, 'A', while 81 per cent chose 'B'. Looking at the nature of distracter 'B', the teacher can see which students did not understand this specific aspect of the subject and help them accordingly.

	A	B	C	D	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
1	Class: Grade 7 A - Young (G7A) Test: G7N (18/08/2005) Student: 26																	
2	**No**	**Enrolment Number**	**Student**		**01**	**02**	**03**	**04**	**05**	**06**	**07**	**08**	**09**	**10**	**11**	**12**	**13**	**14**
3	1	78,923,457	Zeeland	Abergail	B	A	B	B	B	B	B	B	B	B	A	D	E	A
4	2	78,923,458	Yoland	Barnaby	A	A	A	A	B	A	A	A	B	B	A	D	D	A
5	3	78,923,459	Xee	Christine	A	A	A	A	B	A	B	D	A	B	A	D	C	A
6	4	78,923,460	Williams	Douglas	A	B	A	B	B	B	B	A	B	B	B	X	E	B
7	6	78,923,487	Udress	Fred	A	B	A	B	B	B	B	A	B	A	A	B	B	B
22	22	78,923,490	Eggplant	Victor	A	A	A	B	B	A	B	C	A	A	A	D	C	A
23	23	78,923,476	Douglas	Willamena	A	A	A	A	A	B	B	A	A	A	A	D	C	B
24	24	78,923,477	Cherry	Xan	A	A	A	B	A	A	A	A	B	B	A	D	C	B
25	25	78,923,478	Berry	Yevette	A	B	A	B	B	A	B	C	A	B	B	B	D	
26	27	78,923,491	Raspberry	Alec	A		A	B	B	B	B	B	B	B	B	D	E	B
27	28	78,923,484	Blueberry	Bronwin	B	A	A	A	B	A	B	D	B	A	A	D	D	A
28	29	78,923,485	Apricot	Charlie	B	A	B	B	B	B	B	A	B	B	B	D	C	A
29																		
30																		
31																		
32		**% Answered**																
33		A			85%	69%	85%	35%	12%	50%	15%	62%	31%	42%	62%	0%	0%	54%
34		B			15%	19%	15%	54%	81%	50%	85%	12%	69%	54%	38%	15%	12%	31%
35		C			0%	0%	0%	0%	0%	0%	0%	8%	0%	0%	0%	4%	35%	0%
36		D			0%	0%	0%	0%	0%	0%	0%	19%	0%	0%	0%	73%	35%	0%
37		E			0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	19%	0%
38		No Answer			0%	12%	0%	12%	8%	0%	0%	0%	0%	4%	0%	0%	0%	15%
39		Multi			0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	8%	0%	0%
40																		
41																		
42		Correct Answer			A	A	A	A	A	A	A	A	A	A	A	D	C	A

Figure 14. Spread sheet of class responses.

This same process of data collection can be used for surveys of students to monitor student well being (minimise bullying), student sports carnival choices, optional subject choices and parent satisfaction surveys. School psychologists find AutoMarque's ability to save them time and provide them with insightful results a great advantage. AutoMarque's ease of data collection reduces teacher and teacher leaders' workloads and yet provides insights not previously obtainable without considerable time consumption. Hence the expression, “Let the photocopier do your work”. The use of segment analysis of student responses is even more powerful in analysing student thinking. This is obtained by clicking on  to obtain a graph describing the student’s thinking, Figure 15.

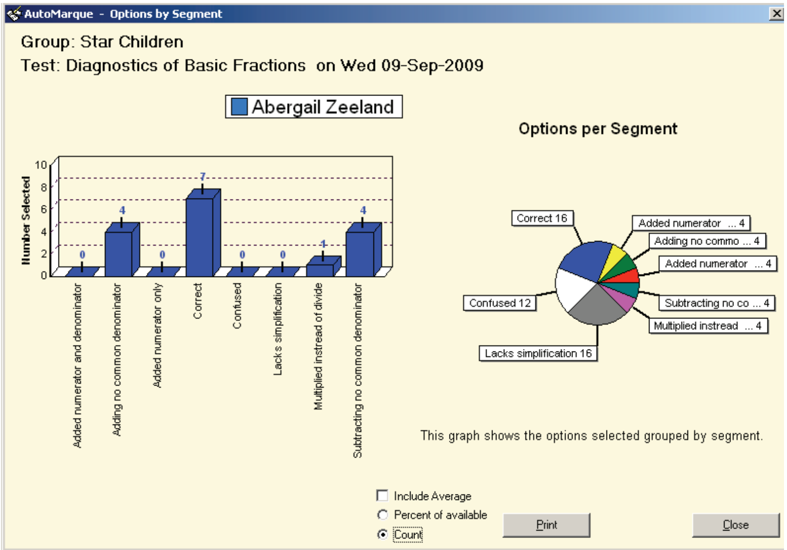


Figure 15. Segment analysis of student thinking.

In this instance the student had completed a diagnostic test on the manipulation of fractions. The observer will see that the student was adding and subtracting without using a common denominator.

Conclusion

Imagine, if you will, that you are alerted by AutoMarque, that the last unit you had taught was not understood by most of your students and that the pedagogical methodology you had applied at the time was the one you had been using for many years. In discussions with colleagues about how they taught such material you learn they use the same pedagogical methodology. Would not this be a handy wakeup call for your team to look afresh at how the unit could be better taught?

This has been a brief summary of *some* of the features of AutoMarque and its benefits for teachers and above all, their students. It is clear from the above that we have a resource that provides teachers with an efficient self-coaching/advice tool on their teaching effectiveness and student learning needs, enabling the development of effective individual learning plans well-grounded on a powerful data base.

When teachers master the development of digital rubrics in conjunction with AutoMarque they too will find their and their students’ lives changed for the better.

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