capital MATHS

Proceedings of the
26th Biennial Conference
of the Australian Association
of Mathematics Teachers Inc.

Edited by Valerie Barker,
Toby Spencer & Kate Manuel

aamt

Capital Maths
Proceedings of the 26th Biennial Conference of the Australian Association of Mathematics Teachers Inc.
Edited by Valerie Barker, Toby Spencer & Kate Manuel

ISBN 978-1-875900-80-0

Published by
The Australian Association of Mathematics Teachers Inc
GPO Box 1729
Adelaide SA 5001
08 8363 0288
office@aamt.edu.au
www.aamt.edu.au

aamt

CONTENTS

PREFACE

“Capital Maths” is a simple and striking theme for this 26th Biennial Conference of the Australian Association of Mathematics Teachers, and reflects that the conference is being held in Canberra, the nation’s capital. More than that, however, is that we are in a position to emphasise yet again the importance of mathematics to the building of our nation. Just as Canberra is the capital of Australia, indeed the ‘heart of the nation’ according to the registration plates of ACT’s vehicles, we must continue to assert that mathematics is at the heart of our county’s social, scientific and economic growth—truly a source of capital investment in our future. More than ever, education—particularly STEM education—is at the forefront of the minds of our politicians and the wider community. As passionate educators, we too have the responsibility of asserting that influence, of supporting it, and making the very best mathematics education available to all of our students in all sectors.

Two years ago, the proceedings lead editor identified major and public themes connected to education: national security, financial literacy, environmental change, STEM, the relatively poor performance of Australian students in major international rankings, the professionalism and quality of our teaching force, and the continuing marginalisation of Indigenous Australians. In particular, the editors noted that while schools have a fundamental role to play in dealing with social problems, many of these seemingly endless calls to schools to solve our social ills through the curriculum and the provision of extra programs are made without a genuine understanding of the complexities of education.

Today these words serve just as well; that is, these issues and more—school funding reform, the accountability of the teaching profession as examples—are a continuing source of concern and a challenge to us all as educators, not just in terms of our students but also in the context of our responsibilities to the broader community, none of whom is untouched by the work of our educators, from a local pre-school to primary school and secondary schools, or a university preparing our future teachers. I write today as a practising classroom teacher, a position that I continue to recognise as a great personal privilege. The opportunity to come together at this conference with such a large and diverse group of educators, to share knowledge and innovation, and to assess and confirm best practice from so many colleagues across all sectors, allows me and all of this diverse group to continue to sustain our understandings of mathematics education as being at the heart of education, and an investment in our society and its future. Our conference speakers and presenters, representing the broader community of educators, bring a breadth of interests, affiliations, expertise and experience, new ideas and alternative thinking to challenge us, to support change and growth, and to affirm the profoundly important work of all in education.

Part of that challenge to me, and I hope to all at the conference, is how we are able to take these ideas and challenges back to our own communities—to lead learning, innovation, influence, change and growth. My hope is that our conference experiences guide us in the leadership that we take to our own communities, allowing us to reflect on and reinvigorate, validate and reappraise, challenge and change our personal and collegial practices. We are a truly supportive community of each other as educators, whether it be as delegates at this conference or in our own workplaces and professional communities. It is appropriate for me to acknowledge with much gratitude the support and guidance I have been given in this editorial role, from many whom I have not met personally, as a particular example. It heartens and encourages me, as I am sure it will for all delegates, that in my everyday practice I am able to influence, lead and be guided by passionate, caring and committed educators; it is we small investors who can play a profound role in mathematics education as a critical component of securing a capital investment in our nation's future.

As this proceedings' lead editor, I would particularly like to thank Kate Manuel and Toby Spencer (AAMT) for their support and guidance throughout the editorial process.

Valerie Barker, Proceedings Lead Editor

Review process

Presentations at AAMT 2017 were selected in a variety of ways. Keynote and major presenters were invited to be part of the conference and to have papers published in these proceedings. A call was made for other presentations in the form of either a seminar or workshop. Seminars and workshops were selected as suitable for the conference based on each presenter's submission of a formal abstract and further explanation of the proposed presentation.

Authors of seminar and workshop proposals were also invited to submit written papers to be included in these proceedings. These written papers were reviewed without any author identification (blind) by at least two reviewers. Reviewers were chosen by the editors to reflect a range of professional settings. Papers that passed the review process have been collected in the 'Professional Papers' section of these proceedings.

The panel of people to whom papers were sent for review was extensive and the editors wish to thank them all:

Judy Anderson
Lorraine Day
Suzanne Garvey
Holly Gyton
Theresa Hanel
Greg Hine
Derek Hurrell
Chris Hurst
Ann Kilpatrick
Barry Kissane
Kate Manuel
Karen McDaid
Denise Neal
Robyn Pierce
Cath Pearn
Karen Perkins
Kate Quane
Howard Reeves
Monique Russell
Aimee Shackleton
Matt Skoss
Paul Turner
Jane Watson
Garry Webb
Bruce White

KEYNOTE PAPERS

‘FUNCTIONAL’ MATHEMATICS IN AN ELECTRONIC AGE: IMPLICATIONS FOR CLASSROOM PRACTICE

RHONDA FARAGHER

The University of Queensland

r.faragher@uq.edu.au

This paper explores the challenge of determining the mathematics curriculum for students with mathematics learning difficulties and disabilities in an age where the tools of mathematics are readily available. Aspects once considered essential for ‘functional mathematics’ such as written calculation, using cash and telling time can be undertaken easily with readily available devices such as calculators and smart phone applications. Deeper concepts such as generalising through algebra, making decisions about money, and engaging in mathematical problem solving and applications are now a possibility. This paper explores the challenge through the theoretical perspectives of Educational Quality of Life and Numeracy Development, presenting five principles for mathematics curriculum planning. The possibility exists of using year level appropriate mathematics curriculum to build lifelong numeracy—’functional’ mathematics for a new age.

Introduction

At some stage in curriculum planning for learners who struggle with mathematics, the conversation is likely to turn to a discussion of ‘functional mathematics’ with the view to what the student will need in the future. This ‘functional’ mathematics has tended to focus on basic arithmetic (often using written methods or rote learning of number facts), money (particularly tending currency and calculating change), and measurement (mostly reading time pieces). Perhaps these aspects were important for numerate adult citizenship in the past but what is ‘functional mathematics’ in our present age? How do we determine what and how mathematics should be taught in this electronic age where the tools of mathematics are so readily available?

Functional mathematics is changing in light of much recent research in the field of numeracy. Combined with improved understanding of approaches to inclusive mathematics education for students with mathematics learning difficulties, this means the time is right for a reconceptualisation of the mathematics that should be taught for all. The traditional functional mathematics topics need to be viewed in this new light. For example, calculating change is no longer functional in a world where electronic transactions are commonplace. Other aspects of financial literacy, such as making sound financial decisions, have increased importance.

When students have significant mathematics learning difficulties, the mathematics curriculum choice is particularly challenging. This paper explores the challenge and considers possible resolutions based on some theoretical perspectives: Educational Quality of Life, current understandings of numeracy development, and approaches to learning in general and mathematics in particular.

Implications for classroom practice are proposed with guiding principles for curriculum design to provide an indication of what might be possible to teach students with significant learning difficulties in mathematics, offering them a rich mathematics program and leading to lifelong numeracy development.

The story so far

Mathematics curriculum offerings for students with mathematics learning difficulties or disabilities have usually been focussed on 'functional' mathematics—the mathematics a person is going to need to 'function' in daily life. Butler and colleagues in undertaking a review of literature from the late 20th century note:

> Traditionally, basic skills instruction, including functional mathematics and life skills, has been the focus of mathematics curricula developed for persons with mental retardation. Although children without disabilities acquire basic skills with few problems, children with mental retardation often complete their schooling without mastering such skills ... Accuracy in counting, recognizing numerals, telling time, and understanding quantity are important if individuals with mental retardation are to achieve employment, independent living, competence in basic skills, and successful integration into school and community settings. (Butler, Miller, Lee & Pierce, 2001, p. 21)

Ten of the 16 articles included in Butler et al.'s review focused on improving students' computation skills, even though the articles were published after 1989—long after the routine availability of electronic calculators. Are those skills—counting, recognising numerals, telling time and understanding quantity—still necessary for a functional adulthood or indeed, are they sufficient? Butler and colleagues suggest not, "limiting mathematics instruction to rote computation practice will deprive students with disabilities from competence in important mathematics concepts and, thus, prevent them from succeeding in inclusionary settings and using mathematics effectively in real-world activities" (Butler et al., 2001, p. 20). In more recent times, these issues have been studied through research and practice in the area of numeracy development.

Numeracy development

At the heart of many conversations about curriculum design is a concern for what a student will need in the future. The use of mathematics in life contexts is numeracy and in Australia, a rich, well-conceptualised understanding of numeracy development has emerged over at least the last two decades. This has been encouraged by government policy and mathematics teachers and education researchers determined to understand the distinction between mathematics and numeracy, as well as to move beyond the misconception of numeracy as 'basic skills'.

Considered a General Capability in the Australian Curriculum, numeracy has been defined as follows:

> Numeracy encompasses the knowledge, skills, behaviours and dispositions that students need to use mathematics in a wide range of situations. It involves students recognising and understanding the role of mathematics in the world and having the dispositions and capacities to use mathematical knowledge and skills purposefully. (ACARA, n.d.-b)

In foundation work on the theorising of numeracy, the Australian Association of Mathematics Teachers (AAMT) stated that numeracy is:

> …the disposition to use, in context, a combination of: underpinning mathematical concepts and skills from across the discipline (numerical, spatial, graphical, statistical and algebraic); mathematical thinking and strategies; general thinking skills; [and] grounded appreciation of context (AAMT, 1997, p. 15).

The AAMT definition makes clear that numeracy involves mathematics from across the discipline and not just narrow areas of arithmetic. In addition, thinking skills and understanding the influence of the context in which the mathematics is being used are central aspects. Numeracy requirements vary according to context, culture and over time.

The importance of understanding the context is key to numeracy, as are cognitive dispositions such as willingness to do the mathematics required. Goos and colleagues have continued the theorising of numeracy and include five elements in their model (Goos, Geiger & Bennison, 2015) as shown in Table 1.

Table 1. Elements of numeracy (Goos et al., 2015, p. 12).

Mathematical knowledge	Mathematical concepts and skills; problem solving strategies; estimation capacities.
Contexts	Capacity to use mathematical knowledge in a range of contexts, both within schools and beyond school settings.
Dispositions	Confidence and willingness to use mathematical approaches to engage with life-related tasks; preparedness to make flexible and adaptive use of mathematical knowledge.
Tools	Use of material (models, measuring instruments), representational (symbol systems, graphs, maps, diagrams, drawings, tables) and digital (computers, software, calculators, internet) tools to mediate and shape thinking.
Critical orientation	Use of mathematical information to: make decisions and judgements; add support to arguments; challenge an argument or position.

Faragher and Brown (2005) demonstrated that numeracy development affects a person’s quality of life. Quality of life is a major framework in intellectual disability research (Schalock et al., 2002) from which has emerged a new branch, Educational Quality of Life (EQoL) theorised by Faragher and van Ommen (2017). In that model, there are five domains with related indicators. These are shown in Table 2, with a third column providing suggested connections to numeracy.

Table 2. Domains of educational quality of life with connections to numeracy.

Domain	Indicators	Numeracy implication examples
Learning	Aspects of well-being; student voice; student identity ...	Opportunities to learn mathematics, approaches to learning
Curriculum and teaching approaches	Teacher quality; response to individual learners; life-span perspective ...	Planned and delivered curriculum, Included mathematics
School organisation	Physical resources; staffing; timetabling ...	Policies for allocation to classes (heterogeneous vs ability grouping)
School community	Outside school experiences; school-community connections...	Opportunities for learning in context
Vision and culture	Shared moral purpose; attitudes to inclusion; approaches to inclusive practice...	Beliefs about mathematics and how it is learnt.

In mathematics, what is taught, how it is taught and what is learnt has a direct impact on EQoL and quality of life in general. The domains of Learning and Curriculum and teaching approaches, are particularly important to the impact of numeracy on EQoL. Vision and Culture is arguably as important, though, as beliefs teachers hold about learners are known to be particularly significant in mathematics education (Beswick, 2008).

In order to prepare students for a numerate adulthood, teachers need to ensure students are taught mathematics from across the discipline as well as the application of mathematics in other subject areas. In the next section, one key component frequently proposed in 'functional mathematics' courses is explored—that of financial literacy. This is intended to serve as an example of the changing requirements of areas that have traditionally been a foundation component of alternative courses offered instead of the standard mathematics program to students who are struggling or underachieving in mathematics.

Financial literacy: An example of changing requirements

Financial literacy provides an illustration of how functional mathematics is changing. The documented fall in cash being withdrawn from automatic teller machines in Australia (The Australian, 2016) correlates with a dramatic increase in cashless transactions, such as 'payWave'. The need to tender currency and calculate change has been replaced by the need to understand financial literacy concepts such as budgeting and transferring money between accounts.

Developments in digital technology are only one impetus for change in financial literacy requirements. The Organisation for Economic Cooperation and Development (OECD) reminds us of the impact of changing demographics and social trends, "The number of financial decisions that individuals have to make, and the significance of these decisions, is increasing as a consequence of changes in the market and the economy" (OECD, 2016, p. 80). Decision making is a critical factor. The Australian Securities and Investment Commission (ASIC) provided a review of literature (2011) indicating that adults were reasonably able to make decisions about simple aspects of finances such as credit, debt, interest rates etc. but were less competent with more complex aspects such as superannuation and saving for retirement. Many relied on the advice of others, such as financial planners, to assist their decision-making. Some members of our community, such as those with intellectual impairments, are likely to rely on the advice of others for most of their financial decisions. Assisting citizens to make good financial decisions based on advice of those they can trust could be regarded as an important aspect of building financial literacy and numeracy development.

Concepts for financial literacy

Building financial literacy for all students requires a radical review of what is needed. Devoting a great deal of curriculum time to fading skills and competencies such as tendering cash and calculating change cannot be justified when much more relevant and vital concepts need development. What, then, should be included? The National Consumer and Financial Literacy Framework (ASIC, 2011) identifies three interrelated dimensions: Knowledge and Understanding; Competence; and Responsibility and Enterprise. Similarly, the OECD includes Content, Processes and Contexts. These two frameworks remind us that financial literacy requires more than content knowledge.

Teaching financial literacy

Programs to teach the knowledge and understandings required for financial literacy are available and many are collected on websites such as ASIC's MoneySmart website (www.moneysmart.gov.au). Research has also been undertaken into techniques for teaching skills to students with intellectual impairment, such as the use of debit cards (Rowe & Test, 2012). Carly Sawatzki (2017) reports research on using rich tasks to explore financial dilemmas. This work indicates a move beyond teaching content and importantly, included students from diverse backgrounds and with low social and economic status. Much work is still needed into effective methods for teaching financial literacy in schools to all students (Blue, Grootenboer & Brimble, 2014).

Financial literacy as an aspect of functional mathematics

This section on financial literacy has been used as an example of the complexity of considering functional mathematics requirements in the 21st century. Traditional approaches have focussed on skills and concepts that are now insufficient. It is often thought that these skills are enablers or foundational and higher, more relevant concepts cannot be taught until these have been accomplished. Fortunately, evidence is mounting that mathematics is not hierarchical as once thought. In the next section, this will be explored moving beyond the context of financial literacy into mathematics in general.

Learning year level mathematics curriculum

Is mathematics really hierarchical?

A long-held view of mathematics development is that it is a hierarchical discipline. Developmental continua are common and well researched (see for example, the Early Numeracy Interview, First Steps Numeracy, Count Me In Too programs) (Bobis et al., 2005). Typical development in mathematics does seem to follow the path indicated by these continua. However, this may not be the only possible order of development. If we were to teach concepts in a different order, it may be possible that different learning outcomes might be achieved.

Bruner claimed, “that any subject can be taught effectively in some intellectually honest form to any child at any stage of development” (Bruner, 1960, p. 33). His ideas led to concepts such as the spiral curriculum where increasingly sophisticated concepts were developed over time. His theoretical position provides a basis for understanding how students can learn topics in mathematics from across the discipline, at a range of levels.

More recent work by Forgasz and Cheeseman (2015) further challenges the assumption of a mathematics hierarchy, as they explore approaches to inclusive practice in primary and secondary mathematics. They note, “Unlike many other subjects, mathematics is widely considered to be a cumulative study for which subsequent levels of learning are dependent upon pre-determined sequencing which is rarely questioned” (p.74). Learning theory in general and mathematics education theorising in particular have moved to a perspective of the possibility of learning mathematics from across the discipline without set order.

Teaching year level mathematics curriculum, though seemingly counter-intuitive, has its basis firmly in established educational theory.

Year-level curriculum

Education policy in Australia is underpinned by international conventions and commonwealth law. The Disability Standards for Education (Commonwealth Department of Employment Education and Training, 2005) provide explanation for educators to assist them to meet their legal obligations. The Australian Curriculum: Mathematics has been developed under this legal framework. Within the Disability Discrimination Act and explained in the Disability Standards, an important concept is education ‘on the same basis’. Explanation of how this concept has been accounted for in the Australian Curriculum is provided in the section on Students with Disability.

- ‘On the same basis’ means that a student with disability should have access to the same opportunities and choices in their education that are available to a student without disability.
- ‘On the same basis’ means that students with disability are entitled to rigorous, relevant and engaging learning opportunities drawn from the Australian Curriculum and set in age-equivalent learning contexts.
- ‘On the same basis’ does not mean that every student has the same experience but that they are entitled to equitable opportunities and choices to access age-equivalent content from all learning areas of the Australian Curriculum.
- ‘On the same basis’ means that while all students will access age-equivalent content, the way in which they access it and the focus of their learning may vary according to their individual learning needs, strengths, goals and interests. (ACARA, n.d.-a)

Therefore, the Australian Curriculum is designed as an age-equivalent or year-level curriculum. If students are in Year 7, they should be taught the Year 7 curriculum. As made clear, this does not mean that students are all taught in the same way. It is essential that adjustments are made to assist any student to learn. For many teachers, adjusting the mathematics curriculum is a challenging idea at first. Evidence of what can be achieved and how the teaching might be undertaken has emerged over many years (see, for example, Browder, Jimenez & Trela, 2012; Browder & Spooner, 2006). A large collection of examples now exists, including a fascinating example of using the distance formula to understand length measurement (Monari Martinez & Benedetti, 2011). A recent example of adjustment of mathematics is described in Box 1.

Box 1. Adjusting a year 9 mathematics program.

A year 9 student with an intellectual disability attends a regular secondary school in Brisbane. He is unable to reliably add single digit numbers and his mathematics teacher was providing worksheets with simple addition exercises such as 5+ 1, written vertically. The others in his class were commencing a unit on linear algebra. The teacher decided to try adjusting the unit and prepared a worksheet where the student practised substituting a range of values for x to find the y value for linear functions such as $y = x + 7$. The mother of the student reported with pride that he was able to complete these tasks with the support of a calculator and learnt to do so remarkably quickly.

There are a number of features of this lesson adjustment that are important to note.

- The student is learning concepts from his year level.
- His teacher is adjusting the work she is already planning for her class. She does not have to prepare completely separate content.
- The materials were able to be used by other students in the class.
- The underlying skill of adding single digit numbers is now being practised in an age-appropriate context.
- The tools of mathematics (use of a calculator) and concepts (substituting into formulae) were being explicitly taught.

What we see from examples such as that in Box 1 is that it is indeed possible for students whose attainment is behind their age peers to learn adjusted content from the mathematics curriculum of their year level.

Implications for practice

Requirements for a numerate adulthood are changing. In this section, the focus will be on mathematics in the compulsory schooling years and on students who have lagged behind their age peers in mathematics, for whatever reason, building on the Goos et al. model of numeracy (2015). Two aspects of that model, Mathematical Knowledge and Tools, are the focus. Mathematical knowledge forms the basis of the aspects of the model that focus on applying mathematics in context: Contexts and Critical Orientation. The more mathematics a person knows, the broader the possibilities for engaging with a variety of life's contexts (Faragher & Brown, 2005). As we have seen, it is indeed possible for even students with significant mathematics learning difficulties to accomplish mathematics from their year level and so form the basis of lifelong numeracy development as this knowledge is applied in life contexts. The following guiding principles, in Table 3, can be used to underpin curriculum design for mathematics.

Table 3. Five guiding principles for curriculum design.

Element	Principle
Mathematical knowledge:	1. currency 2. focus on concepts—the 'big ideas' 3. taught through explicit teaching, and rich tasks to allow problem solving.
Tools:	4. use of the most efficient tools should be explicitly taught and practiced 5. tools such as calculators can act as prosthetic devices—tools for overcoming learning difficulties—and should be routinely available as needed.

1. **Currency**

   The principle of currency refers to ensuring the mathematics included is what is important now, and not included because it has always been taught. An example would be aspects of arithmetic. The use of basic arithmetic in life contexts is rapidly changing and shrinking. In a world where the use of cash is disappearing and being replaced by PayWave, where calculators are within hands reach of most and where transport card systems and funds transfer are in regular use, a focus on arithmetic is no longer necessary or sufficient.

2. **Focus on concepts**

   It is critical that students are taught as much mathematics in the school years as possible in order that they can be taught application in life contexts where the mathematics is needed.

   A focus on concepts, the 'big ideas' of the field of study, guards against a concentration on outdated techniques. Instead, a focus on concepts is responsive to new methods. For example, if aspects of financial literacy, such as paying bills and budgeting, are taught, the techniques used may change but the underlying idea does not. Cheque books may give way to online bank transfer which may give way to a smart phone application but the concept and the underlying mathematics is the same. Our curriculum can be responsive to current methods if the concepts are the focus.

3. **Taught through explicit teaching, and rich tasks to allow problem solving**

   Numeracy attainment depends not only on mathematical skills and concepts but also on the ability to apply mathematics, solve problems, make decisions and exhibit dispositions such as the preparedness to adapt mathematical knowledge to contexts. As Hwang and Ricconimi (2016) point out, students with mathematics learning difficulties should "receive opportunities to engage in unstructured realistic problems with appropriate instructional supports or the gap between their mathematical knowledge and their real-life use will continue to widen" (p.179). Unfortunately, some writers in special education mistake constructivism for discovery learning and advocate for direct instruction alone. A focus on direct instruction without the opportunities to engage in mathematical thinking removes the opportunity to develop required elements of numeracy. A balance is needed in provision of explicit teaching of techniques as well as opportunities to learn through rich tasks. Sullivan and colleagues demonstrate that this is possible and effective (Sullivan, Mousley & Zevenbergen, 2006).

4. **Use of the most efficient tools should be explicitly taught and practiced**
   The tools for learning mathematics are varied. Providing learners with opportunities to engage with a range of tools is critical for application and for learning of the mathematics in the first place. Graphing software, spreadsheets, manipulatives and other mathematics tools are a feature of effective mathematics pedagogy for all learners.
5. **Tools as prosthetic devices**
   Many learners with mathematics learning difficulties struggle with arithmetic. This should not prevent them learning other areas of the discipline. Understanding what operations (such as subtraction) do is a functional necessity, whereas the ability to perform the operation is not, when the tools of mathematics such as electronic calculators are readily available. In situations where a student is unable to perform a calculation, a calculator becomes a prosthetic device. A prosthesis is a device that can perform a function when the body is unable. The ready availability of calculators makes them ideally suited to supporting numeracy throughout life, however, students need to be explicitly taught how to use them effectively.

The five principles described above, provide an indication of how curriculum decisions for students with significant mathematics learning difficulties can be guided by established models of numeracy development.

Conclusion

A numerate adulthood is attainable by all learners, with a realistic understanding of functional mathematics in an electronic age. An essential foundation is a rich mathematics curriculum including topics from across the discipline. Even more so are those mathematical dispositions of grappling with problems, persisting, exploring, being challenged, even seeing the beauty of mathematics. Those of us who have chosen to work in mathematics know the sheer joy of our discipline. Of course, we also know all too well that for many students, they do not experience mathematics the way we do. However, it is possible for all learners to come to appreciate mathematics in this way with good teaching and the right support. This is best achieved by teaching *all* learners the year level curriculum in inclusive classrooms, adjusted as required, with a focus on teaching the mathematics through explicit teaching and through providing rich tasks and contexts for students to learn to apply their mathematics.

References

AAMT. (1997). *Numeracy = everyone's business. The report of the numeracy education strategy development conference*. Canberra: Commonwealth of Australia.

ACARA. (n.d.-a). *Australian Curriculum: Student Diversity*. Retrieved from http://www.australiancurriculum.edu.au/studentdiversity/students-with-disability

ACARA. (n.d.-b). *Numeracy. Australian Curriculum*. Retrieved from http://www.australiancurriculum.edu.au/generalcapabilities/numeracy/introduction/introduction

ASIC. (2011). *National consumer and financial literacy framework*. Retrieved from http://www.curriculum.edu.au/verve/_resources/National_Consumer_Financial_Literacy_Framework_FINAL.pdf

Beswick, K. (2008). Influencing teachers' beliefs about teaching mathematics for numeracy to students with mathematics learning difficulties. *Mathematics Teacher Education and Development, 9*, 3–20.

Blue, L., Grootenboer, P. & Brimble, M. (2014). Financial literacy education in the curriculum: Making the grade or missing the mark? *International Review of Economics Education, 16*, 51–62. doi:http://dx.doi.org/10.1016/j.iree.2014.07.005

Bobis, J., Clarke, B., Clarke, D. M., Thomas, G., Wright, R., Young-Loveridge, J. & Gould, P. (2005). Supporting teachers in the development of young children's mathematical thinking: Three large scale cases. *Mathematics Education Research Journal, 16*(3), 27–57.

Browder, D. M., Jimenez, B.& Trela, K. (2012). Grade-Aligned Math Instruction for Secondary Students with Moderate Intellectual Disability. *Education and training in autism and developmental disabilities, 47*(3), 373–388.

Browder, D. M.& Spooner, F. (Eds). (2006). *Teaching Language Arts, Math & Science to Students with Significant Cognitive Disabilities*. Baltimore, MA: Brookes.

Bruner, J. S. (1960). The Process of Education. Cambridge, MA: Harvard University Press.

Butler, F. M., Miller, S. P., Lee, K. & Pierce, T. (2001). Teaching mathematics to students with mild-to-moderate mental retardation: A review of the literature. *Mental Retardation, 39*(1), 20–31.

Commonwealth Department of Employment Education and Training. (2005). *Disability standards for education 2005 plus guidance notes*. Canberra, ACT: Author.

Faragher, R. & Brown, R. I. (2005). Numeracy for adults with Down syndrome: It's a matter of quality of life. *Journal of Intellectual Disability Research, 49*(10), 761–765.

Faragher, R. & Van Ommen, M. (2017). Conceptualising Educational Quality of Life to Understand the School Experiences of Students With Intellectual Disability. *Journal of Policy and Practice in Intellectual Disabilities, 14*(1), 39–50. doi:10.1111/jppi.12213

Forgasz, H. & Cheeseman, J. (2015). Effective and inclusive mathematics teaching and learning. In J. M. Deppeler, T. Loreman, R. Smith & L. Florian (Eds), *Inclusive pedagogy across the curriculum* (Vol. 7, pp. 73–97): Emerald Group.

Goos, M., Geiger, V. & Bennison, A. (2015). Conceptualising and enacting numeracy across the curriculum. In K. Beswick, T. Muir & J. Wells (Eds), *Proceedings of 39th Psychology of Mathematics Education conference* (Vol. 3, pp. 9–16). Hobart, Australia: PME.

Hwang, J. & Riccomini, P. J. (2016). Enhancing mathematical problem solving for secondary students with or at risk of learning disabilities: A literature review. Learning Disabilities Research and Practice, 31(3), 169–181. doi:10.1111/ldrp.12105

Monari Martinez, E. & Benedetti, N. (2011). Learning mathematics in mainstream secondary schools: experiences of students with Down's syndrome. *European Journal of Special Needs Education, 26*(4), 531–540. doi:10.1080/08856257.2011.597179

OECD. (2016). *PISA 2015 Financial Literacy Framework PISA 2015 Assessment and Analytical Framework: Science, Reading, Mathematic and Financial Literacy* (pp. 79–100). Paris: OECD Publishing.

Rowe, D. A. & Test, D. W. (2012). Effects of simulation to teach students with disabilities basic finance skills. *Remedial and Special Education, 34*(4), 237–248. doi:10.1177/0741932512448218

Sawatzki, C. (2017). Lessons in financial literacy task design: authentic, imaginable, useful. *Mathematics Education Research Journal, 29*(1), 25–43. doi:10.1007/s13394-016-0184-0

Schalock, R. L., Brown, I., Brown, R. I., Cummins, R. A., Felce, D., Matikka, L., ... Parmenter, T. (2002). Conceptualization, measurement, and application of quality of life for persons with intellectual disabilities: Report of an international panel of experts. *Mental Retardation, 40*(6), 457–470.

Sullivan, P., Mousley, J. & Zevenbergen, R. (2006). Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International journal of science and mathematics education, 4*, 117–143.

The Australian. (17 March 2016). Cashless tap dance puts ATMs in shade. *The Australian*. Retrieved from http://www.theaustralian.com.au/business/financial-services/cashless-tap-dance-puts-atms-in-shade/news-story/42805944a7ebe94bb40ca34147341879

THE JOURNEY TO STATISTICAL LITERACY AT THE SCHOOL LEVEL: PAST, PRESENT AND FUTURE

CHRISTINE FRANKLIN
The American Statistical Association
christine@amstat.org

This paper is a summary of the closing keynote at the AAMT conference 2017. It describes my reflections on an almost 40-year journey teaching statistics—feeling guilt about how poorly I must have taught my students those early years, attempting to stay current with pedagogy and assessment, and predicting the future. As teachers, we want our students to experience the beauty of statistics and also the importance of being a healthy sceptic of statistics. This paper highlights taking a journey to explore the evolution of teaching statistics over the past 50 years internationally and lessons we have learned that will positively impact the statistical literacy of our students in the future.

Introduction

Thank you for this tremendous honour of being with you today. I am thrilled to be back in Australia after spending two fabulous weeks in 2015 at the University of Tasmania working with Jane Watson, Helen Chick, and other amazing Australian mathematics and statistics educators. At that time, I was experiencing one of the highlights of my career, being selected as a US Fulbright Scholar to learn more about school level statistics ‘down under’. I spent 6 months in New Zealand with the Fulbright program also supporting travel for me to experience two weeks in Australia. I learned so much from the mathematics educators and school level teachers in Australia and New Zealand. I have been able to take these experiences back to the US and hopefully better impact the integration of statistics into the US school level curriculum. After completing the Fulbright, I made the decision to retire from higher education and the classroom (and yes, I miss the students) so that I could begin a new journey, helping to structure and grow a new position the American Statistical Association felt was a high priority, advocating as a K–12 Statistical Ambassador. Since being in this position, I often fall back on the knowledge I gained during my time in Australia and New Zealand.

Retiring from a long career of teaching and research brings reflection. I have cherished each day of my career as a statistics educator. Let’s journey together exploring the evolution of how we teach statistics—the past, present, and the future.

Data everywhere

Statistics educators are now faced with acknowledging and adapting to the reality that our traditional concept of data has changed. Data are no longer simply "numbers in context" (Cobb and Moore, 1977), stored in static spreadsheets and collected to answer specific research—driven questions. Today, data are also dynamic, complex, highly structured collections of pictures and sounds—data sets are vast and readily available. The hot new curriculum is data science combining the skills of statistics, mathematics, and computer science. Statistics integrated into the school level curriculum is key for students developing the statistical reasoning skills to make sense of the massive data that surround them on a daily basis, much of which students generate themselves. According to the American Statistical Association's (ASA) Pre-K–12 GAISE Framework Report, "Every high-school and college graduate should be able to use sound statistical reasoning to intelligently cope with the requirements of citizenship, employment, and family and to be prepared for a healthy and productive life" (Franklin, et al., 2007, p.1). We have come a long way to meeting this goal over the past 50 years but there is still much work necessary for the future.

Early years of teaching

When I first taught statistics 40 years ago, the introductory course was just making its way to the undergraduate level at university, most certainly not at the school level. The study of statistics was for the graduate level. The first 'statistics' book I taught out of focused more on mathematical probability than on statistical reasoning. Because statistics was being taught in mathematics departments, the mathematical underpinning of statistics was overpowering the practical applications and context of statistical reasoning. I found this puzzling given that the roots of statistics as a field are not mathematics but in such fields as agriculture and the social sciences. The field of statistics developed to solve real problems (Franklin, 2013). Although I was both a political science and maths major, I discovered the study of statistics through political science not maths. It bothered me that introductory statistics courses emphasised computations but not concepts and interpretation of findings. No doubt, the lack of technology at that time contributed to the emphasis on calculations. However, in 1986, I landed in a magical place, the University of Florida, with Richard Scheaffer as my department chair. Scheaffer is a pioneer in statistics education and advocate for statistics at the school level. He is the individual who set me on the journey to devote my career to bettering the teaching of statistics at the school and undergraduate level and he continues to mentor me.

Change of culture

My first year at University of Florida, I was asked to teach out of a visionary introductory text authored by Freedman, Pisani, and Purvis (1978) now in its fourth edition. This book was about conceptual understanding and being able to reason statistically. The first chapter was about study design, unheard of at that time in an introductory course. This is the one reference book I always have on my shelf.

It was also during this time that statistical software was beginning to appear such as *Minitab*. Technology was becoming an option to use pedagogically. My second year at

UFL, I was asked to class test a forthcoming textbook, *Introduction to the Practice of Statistics* by Moore and McCabe (1989). This book was visionary in that it balanced procedural with conceptual understanding. The book is now in its 8th edition.

Before moving to the University of Georgia in 1989, Dick Scheaffer convinced me that I should focus part of my career on integrating statistics at the school level. Scheaffer was leading efforts through the American Statistical Association (ASA) and College Board to develop resources and statistics standards for the school level.

Early school level efforts

While president of ASA in 1968, Frederick Mosteller (Professor at Harvard) reached out to the National Council of Teachers of Mathematics (NCTM) to establish a joint committee with ASA. NCTM is the USA equivalent organization to AAMT. Mosteller was a leader in statistical research who had an appreciation for statistics education. One of Mosteller's Princeton colleagues, John Tukey, was at this time steering much of the emphasis in the statistics profession away from mathematical theory and toward data analysis often referred to as exploratory data analysis. Tukey gave us the boxplot, the stem and leaf plot, and the 1.5*IQR criterion for outliers (Tukey, 1977).

The Joint Committee of ASA and NCTM developed in the 1980's the ground breaking Quantitative Literacy Project (QLP) that consisted originally of four booklets, *Exploring Data, Exploring Probability, The Art and Technique of Simulation, and Exploring Surveys and Information from Samples* (Landwehr & Watkins, 1986). Workshops were held throughout the US providing teachers with professional development. Building on the work of the ASA-NCTM Joint Committee, NCTM published the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), the first major curriculum document internationally with Statistics as a significant component at all school levels. These recommendations were enhanced with the publication of *Principles and Standards for School Mathematics* (NCTM, 2000). These documents were followed by similar curriculum documents including statistics and probability in other countries (e.g., Australia, Japan, Korea, and New Zealand).

It was in 1996 that a visionary book was published that heavily influenced the pedagogy in my classroom, *Activity-Based Statistics* (Scheaffer et al., 1996). This book inspired me to prioritise hands on activities to promote investigative and conceptual learning. It was also at this time that statistical calculators were becoming available to more readily bring technology to the classroom. Activity based learning was ideal for carrying out simulation especially at the school level.

AP Statistics

One of the most successful efforts in the US of introducing statistics at the school level has been Advanced Placement (AP) Statistics (http://apcentral.collegeboard.com/apc/public/courses/teachers_corner/2151.html) with the first exam administered in 1997. Students take an AP course in high school that is considered equivalent to a university course. If the student makes a certain score or higher on the nationally administered exam, the student receives course credit at the university level. In 1997, approximately 7500 exams were scored. In 2017, 21 years later, approximately 217 000 exams were scored. AP Statistics has maintained a nice linear growth over the years. Oversight for AP exams is provided by College Board, a non-profit organisation that develops curricula and standardised exams used by the school level to encourage college readiness (https://www.collegeboard.org). Richard Scheaffer provided the leadership to bring about AP Statistics in the early 90's. He asked me to become involved in its beginning stages and implementation and I have benefitted greatly from my long involvement with AP Statistics. My work with AP gave me an understanding of (1) the urgency of teacher preparation in statistics at the school level (2) the importance of sound assessments in statistics and (3) the importance that all school level students develop statistical reasoning skills, not just post-secondary bound students. AP Statistics motivated my professional efforts at the school level for the past 20 years (Rossman and Franklin, 2013).

Pre-K–12 GAISE

The vision of ASA emerged again in 2007 with the publication of the *Guidelines for Assessment and Instruction for Statistics Education (GAISE) Report* (Franklin et al.). Building upon earlier documents, *GAISE* provides a statistical problem-solving framework with the concept of variability as its foundation:

- Formulate Questions – Anticipating Variability;
- Collect Data – Acknowledging Variability;
- Analyze Data – Accounting of Variability;
- Interpret Results – Allowing for Variability.

GAISE also stresses the importance of understanding the difference between mathematical and statistical thinking. As stated in *GAISE,* p.6, "Statistical thinking, in large part, must deal with the omnipresence of variability; statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in the data. It is this focus on *variability in data* that sets apart statistics from mathematics." Not only is variability essential in statistical reasoning but so is context.

GAISE advocates that students in the lower grades are ready to explore the design of studies and to begin using the statistical problem-solving process. How could this be, given that traditional introductory statistics course would omit these topics**?**

GAISE reflects the guidelines of other frameworks such as those proposed by Holmes (1980) and Wild and Pfannkuch (1999). *GAISE* is complemented by Makar and Rubin (2009), who introduced the phrase "informal inference" to describe the situation for school students to make generalizations from a sample to a wider population without using the formal theoretical tools available to practicing statisticians.

Since the publication of these different frameworks, research is occurring on statistical learning at the school level with much of the effort being 'down under' in Australia and New Zealand. Recently the focus is on data modelling and complete statistical investigations reflecting GAISE and engaging students in the practice of statistics with meaningful contexts (e.g., Ben-Zvi, Ardor, Makar & Bakker, 2012; Watson & English, 2015, 2016, 2017; Pfannkuch, 2011).

Technology

We have come a long way with technology from my first years of teaching with no technology. We have evolved from statistical calculators in the 90s to powerful statistical software packages, applets, and amazing data visualization tools such as

Gapminder. Simulation is as easy as accessing a public applet (on your smart phone no doubt) and clicking a mouse or pushing buttons. We are able to use randomization tests and bootstrapping to build sampling distributions for making inferential statements, this all happening from school level to post-secondary. We don't need computer labs—just internet access. Investigative learning is easy now—just push the correct buttons. Who needs hands on simulation? Here is where I believe we need to go back into the past and keep the hands on before moving to technology. Anecdotally, in teaching, especially with teachers, I observe those light bulb moments when they understand a fundamental statistical concept by going through a hands-on activity whereas before they were going straight to technology. For example, what is the reason we randomly assign treatments in an experiment? By first setting up the simulation of the experimental situation with a deck of cards, then randomizing the cards and assigning the cards to treatments, the teachers realise what we intend when we ask the question, "Is the difference we observe in the treatments due to random chance or is there a real difference?" Technology is amazing but we still need to first lay the groundwork for our students with visualizing the concepts hands on.

Statistical education of teachers

Statistics at the school level provides us with a huge opportunity: Statistical literacy for all! If not at the school level, how can we expect individuals to become data literate? We need our teachers to teach students to navigate this world of data (Bargagliotti and Franklin, 2015). In the US and internationally, school level teachers and teacher educators express their lack of confidence in being prepared to teach statistics. In 2015, ASA published the *Statistical Education of Teachers (SET)* report (Franklin et al., 2015). SET is a must-read document and provides recommendations for pre-service teacher education and provides examples of activities that satisfy its recommendations. Two chapters of particular interest are Chapter 3 discussing the mathematical practices through a statistical lens and Chapter 7 discussing the appropriate way to assess statistical reasoning.

The eight mathematical practices (MP) describe ways students of maths and stats should engage with the subject content—the MP's are processes and practices that complement content knowledge. The practices emphasise problem solving, reasoning, communications, connections, and representations (Franklin et al., 2015). I think of the practices as habits of mind.

Chapter 7 shares examples of good statistics assessment items for the school level from the Levels of Conceptual Understanding in Statistics (LOCUS) project (https://locus.statisticseducation.org). The work of LOCUS was based upon the outstanding model of assessment provided by AP Statistics. It was AP Statistics and LOCUS that made me realise the importance of assessment as a priority for teaching. Assessment needs to happen alongside the planning of delivery content with clearly defined learning outcomes. Assessment must not be an afterthought as is traditionally the case.

I have a wish

I have a wish that the nurturing community of maths and stats educators can gently push us as teachers out of our comfort zone—to take the leap and embrace new pedagogy and professional development to teach statistics and how to make sense of data. It is important that this community not only encourage taking risks but also provide the safety net. I know in America, ASA is an example of that community and I experienced observing that nurturing community during my time in Australia. Collaborations are happening internationally to support teachers of statistics. I often share that without my mentors and colleagues providing a safety net, I would not have easily moved out of my comfort. But I am grateful I did.

I have a wish that teachers of statistics will enthusiastically say, "Teaching statistics is fun and important!" Finally, I have a wish that teachers can embrace the positive lessons we have learned from the past, the research, the technology, the assessment models, and the pedagogical tools of the present, and in the future, encourage students to use resources available for generating and collecting meaningful data to analyse—to practice being data scientists and making sense of the world around them. Statistical reasoning is magical but real, beautiful, and brings mathematics into the practical world of useful applications.

Thank you for spending time with me to share my passion for teaching statistics.

My deepest appreciation to friend and colleague, Jane Watson for her valuable contributions to this paper.

References

Bargagliotti, A. & Franklin, C. (2015). The statistical education of teachers: Preparing teachers to teach statistics. *Chance, 28*(3), 19–27.

Ben-Zvi, D., Aridor, K., Makar, K. & Bakker, A. (2012). Students' emergent articulations of uncertainty while making informal statistical inferences. *ZDM Mathematics Education, 44*, 913–925.

Cobb, G. & Moore, D. (1997). Mathematics, statistics, and teaching. *The American Mathematical Monthly, 104*(9), 801–823.

Franklin, C. (2013). Common core state standard and the future of teacher preparation in statistics. *The Mathematics Educator, 22*(2), 3–10.

Franklin, C., Bargagliotti, A., Case, C., Kader, G., Scheaffer, R. & Spangler, D. (2015). *Statistics Education of Teachers*. Alexandria, VA: American Statistical Association. Retrieved from http://www.amstat.org/education/SET/SET.pdf

Franklin, C., Kader, G., Mewborn, D., Moreno, J., Peck, R., Perry, M. & Scheaffer, R. (2007). *Guidelines for assessment and instruction in statistics education (GAISE) report: A preK-12 curriculum framework*. Alexandria, VA: American Statistical Association. Retrieved from http://www.amstat.org/education/gaise/

Freedman, D., Pisani, R. & Purves, R. (1978). *Statistics*. New York: W.W. Norton & Co.

Holmes, P. (1980). *Teaching Statistics 11–16*. Slough, UK: Schools Council Publications and W. Foulsham & Co.

Landwehr, J.M. & Watkins, A.E. (1986). *Exploring data: Quantitative literacy series.* Palo Alto, CA: Dale Seymour.

Makar, K. & Rubin, A. (2009). A framework for thinking about informal statistical inference. *Statistics Education Research Journal, 8*(1), 82–105. Retrieved from http://iase-web.org/documents/SERJ/SERJ8(1)_Makar_Rubin.pdf

Moore, D.S. & McCabe, G.P. (1989). *Introduction to the practice of statistics*. New York: W.H. Freeman.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: Author.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Pkannkuch, M. (2011). The role of context in developing informal statistical inferential reasoning: A classroom study. *Mathematical Thinking and Learning, 13*(1&2), 27–46.

Rossman, A. & Franklin, C. (2013). Interview with Christine Franklin. *Journal of Statistics Education, 21*(3). Retrieved from http://ww2.amstat.org/publications/jse/v21n3/rossmanint.pdf

Scheaffer, R., Gnanadesikan, M., Watkins, A. & Witmer, J. (1996). *Activity-based statistics*. New York: Springer-Verlag.

Tukey, J. (1977). *Exploratory data analysis*. New Jersey: Pearson.

Watson, J. & English, L. (2015). Introducing the practice of statistics: Are we environmentally friendly? *Mathematics Education Research Journal*, *27*, 585–613. DOI 10.1007/s13394-015-0153-z

Watson, J. & English, L. (2016). Repeated random sampling in Year 5. *Journal of Statistics Education*, *24*(1), 27–37. DOI: 10.1080/10691898.2016.1158026

Watson, J. & English, L. (2017). Reaction time in Grade 5: Data collection within the Practice of Statistics. *Statistics Education Research Journal,16*(1), 262–293. Retrieved from http://iase-web.org/documents/SERJ/SERJ16(1)_Watson.pdf

Wild, C. & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review, 67*(3), 223–248.

HANNA NEUMANN MEMORIAL LECTURE:

INSPIRING TEACHERS; CHALLENGING CHILDREN

PETER M. NEUMANN

The Queen's College, Oxford University

Introduction: Inspiring teachers

My presidential address to The Mathematical Association (UK) in April 2016 was entitled 'Inspiring Teachers'. The concept was a talk that might trace my mathematical experiences from grammar school sixth form, through my development in retirement as a contributor to masterclasses for the UK Mathematics Trust and the Royal Institution, learning from the students and from the inspiring teachers at whose masterclasses I assist, to the small understanding of Key Stage 2 Mathematics that I am gradually acquiring through an hour a week with some Year 6 students and their remarkable teachers in a local primary school.

Three months later I was greatly honoured to be invited to deliver the Hanna Neumann Memorial Lecture to the AAMT biennial conference 2017. Late in October 2016 a request came for a title. I was stumped. But just then there arrived out of the blue—by airmail—a delightful letter from Deanne Whittleston of Sydney. It was dated 2 November 2016 and I have permission to quote from it:

> She [Hanna Neumann] was remarkable for many reasons but to me she was remarkable as she managed to get me through Pure Mathematics I and in all my years of study (13 at school, 5 at ANU, 2 at Canberra University and 5 studying law in Sydney) she was the best teacher I ever had. How she did it I do not know.
>
> In my Pure Maths I class at ANU in 1967, there were about 200 students (only 4 of whom were women, of whom I was one). Hanna taught that class as though she knew us all intimately—she was incredible.

The lovely coincidence of the arrival of this letter just when I was at a loss for a lecture title tempted me to re-use that of my earlier address—though I was clear in my mind that the actual address would have to be very different. The title is deliberately ambiguous. So is the second half that I have added later.

Another point about the title is that no abstract is needed; yet another, that the words can be permuted in several ways, such as 'Challenging Children Inspiring Teachers' and 'Teachers Inspiring Challenging Children'. And of course, as so often, when a title is requested long in advance, what was needed was something that would restrict the content on the day as little as possible.

As I have written in *The Mathematical Gazette* (November 2016), my lectures are designed to be ephemeral oral presentations. They are not designed to be written down and published as articles. Please bear that in mind gentle reader, and judge accordingly. If you find something of value here I shall be delighted; if not, I shall not be surprised.

Challenging children 1: A Year 6 discovery

Eleven-year old Nathan on 9 June this year:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

Choose four numbers, one from each row, one from each column. Whatever the choice you make, they add to 34.

I have no idea where this came from. Possibly from one of his family. Quite probably a discovery of his own. The context was that two weeks earlier, on the last Friday before the week-long half-term holiday, eleven high-ability Year 6 children, a teacher and I, had started from Durer’s famous 4 × 4 magic square and had investigated other, mostly smaller, magic squares. These squares captured the children’s imagination. And so it was that Nathan, refreshed by his holiday, came up with his discovery.

It was new to me. The response had to be “Lovely! Can you explain to me WHY it works? What about squares of other sizes?” By the end of the session (45 minutes) the children had investigated squares of sizes down to 1 × 1 (one of the girls did this) and up to 10 × 10, and three or four of them had understood why it works. Those were challenged to write their explanations down. But writing mathematical explanations is, in my somewhat limited experience, not something that eleven-year olds, even bright ones, put high on their agenda.

Challenging children 2: A Year 10 masterclass

Year 10 children are fourteen or fifteen years old. The masterclasses offered by the UK Mathematics Trust and by the Royal Institution of Great Britain are aimed at a group consisting of two of the ablest children in each of twenty to thirty schools. Since average cohorts will be between three and four classes per year (smaller in independent schools) the clientele of these masterclasses might be expected to come from the top 2% or 3% of the ability range. Of course, ability does not work quite that way; even so, the children will be among those who (in England and Wales, possibly also in Scotland and Northern Ireland) are deemed to be ‘Gifted & Talented’ (in a semi-formal political sense). These masterclasses are intended to show able children that there is mathematics outside of their syllabuses, that mathematics is not done and dusted, that there are areas where mathematicians in industry and in universities are still struggling to gain understanding. Here is an example of a Year 10 masterclass on combinatorics of words, that hides some deep group theory in which there are still many open research problems.

We start with an alphabet {a*, b, c…*}, as few or as many letters as we wish, and we focus on words. In this context words are any strings of letters such as;

a, aa, aaabaccba, bbababaaaab,

They are meaningless—all that is of interest is the combinatorics associated with linear arrangements of symbols. The *length* of a word w is defined simply to be the number of letters in w. The examples above have lengths 1, 2, 9 and 11 respectively. An important convention is that we allow length 0, no letters! But this word needs to be seen on the page, so we write 1 for the 'empty' word.

Before we move on to transformation rules, the main topic, let's digress briefly (but usefully).

Challenge

Think of a good way to list the words in the two-letter alphabet {*a, b*}.

It does not take children long to realise that a list starting

a, aa, aaa, aaaa, aaaaa,…

as a dictionary might, will never reach words that involve the letter *b*. And then some children have the idea of *length-lex* (also known as *shortlex*) listing:

1*, a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb,*
aaaa, aaab, aaba, aabb, abaa, abab, abba, abbb, baaa, baab…

Exercises

1. What comes next after *abababb*?
2. What is the 64^{th} word? The 100^{th} word? The 2017^{th} word?
3. Can you find a general rule?

Once we are all happy that we know what is meant by 'words', that a word (in this area of mathematics) is just a meaningless string of letters, of any length, including length 0, then we move on.

In this branch of mathematics, we have *transformation rules* to change words into other words. We want to find out what effect these changes have.

First example

Focus on words w in the one-letter alphabet {*a*}. Choose a number—let's say 4. Then we have two rules:

- *Expansion rule:* Choose any word u and *expand* w by inserting $uuuu$ between two adjacent letters of w, or at the front or at the back of w.
- *Contraction rule:* If we see $uuuu$ somewhere in w (a part of w consisting of 4 consecutive non-overlapping instances of some word u) we may contract w by deleting it and closing up.

So for example, with $u = aa$ expansion may change aaa to $aa\underbrace{aaaaaaaa}a$, and with $u = aaa$ contraction can change $aa\underbrace{aaaaaaaaaaaa}aaa$ to $aa\,aaa$.

Equality of words

We say that words v and w have the same value, and we write $v = w$ if word v can be changed to word w by expansions and contractions any number of times (using the same or different words u), in any order. Since every expansion may be undone by a contraction, and vice-versa, if word v can be changed to word w by expansions and contractions any number of times then also w can be changed to v.

Exercise

Organise the following words into groups so that those in a group have the same value:

1, *a, aaa, aaaaa, aaaa, aaaaaaaaaaa* (11),
aaaaaaaaaaaaa (13), *aaaaaaaaaaaaaaaa* (16)

Problem

For this transformation rule, how many *different* values are there?

Answer

For this transformation rule there are 4 different values: any word has the same value as one of 1, *a, aa, aaa*, and these are different.

Similar examples of transformation rules

Stay with the one-letter alphabet {a}, but what if we replace 4 by 5? Now we have words in the one-letter alphabet {a} and our transformation rules are insertion or deletion of *uuuuu*. Equality of values is defined as before: we write $v = w$ if word v can be changed to word w by expansions and contractions of this kind.

Problem

How many different values are there now?

It is not hard to see that the answer is answer 5.

Problem

What is the answer if we replace 4 by 3 instead of by 5? Or by 12? Or by n?

Summary of first examples of transformation rules

Focus on words in the one-letter alphabet {a}; permitted transformations are insertion or deletion of u^n (that is, n consecutive non-overlapping instances of a word u); write $v = w$ if word v can be changed to word w by expansions and contractions; and the basic problem is, how many different values are there? The answer is that words v and w have the same value if their lengths leave the same remainder when divided by n. That is, any word has the same value as just one of these words:

$$1, a, aa, aaa \ldots a^{n-1}$$

So, there are *n* different values.

Let's move on to larger alphabets

Focus on words w in the two-letter alphabet {a, b}. For our next transformation rule we again choose a number—let's say 2 . Then our rules are:

Expansion rule

Choose any word u and expand w by inserting uu between two adjacent letters of w, or at the front or at the back of w.

Contraction rule

If we see uu somewhere in w (a part of w consisting of 4 consecutive non-overlapping instances of some word u) we may contract w by deleting it and closing up.

As before, we write $v = w$ if word v can be changed to word w (or, equivalently, w changed to v) by expansions and contractions (any number of times, in any order, using the same or different words u).

Problem

With this transformation rule for words in the two-letter alphabet {a, b}, how many different values are there?

Important note

When this topic is used for a Year 10 masterclass, it is usually spread over two sessions, each of 70 or 80 minutes, on successive days. The break is planned to come here. The three-part exercise about the length-lex listing of words in the two-letter alphabet, together with this problem, are offered as "homework", and are discussed at the start of the second day.

Have you found, given a little time, that any word w has the same value as one of these:

$$1, a, b, ab, ba, aba, bab?$$

But *are these all different*?

Perhaps about 400 or 500 children have been exposed to this question (a question posed in different language to first-year undergraduates at my university) and the number who have worked out that the answer is 'no' is positive but small (around 10):

$$ba = \underline{aa}ba = aaba\underline{bb} = a\underline{abab}b = ab$$

Thus $ba = ab$, and it then follows quickly that $aba = b$ and $bab = a$. Therefore, for this example of a transformation rule for words in a two-letter alphabet there are values. Listed in length-lex order they are 1, a, b, ab.

If time permits we investigate what happens when we keep the same rule but work with words in larger alphabets. It does not take the majority of the children long to realise that they can use what they have just discovered (or been shown) to treat words in $\{a, b, c\}$. by focussing first on those that do not involve c, then those that do not involve b, then words that do not involve a, and the upshot is that, with the rule 'insert or delete uu', any word has the same value as one of:

$$1, a, b, c, ab\ ac, bc, abc$$

and these are all different, so there are 8 values. And if the alphabet has 26 letters then there will be 2^{26} different values; if the alphabet has m letters then there will be 2^m different values.

Next, we return to words w in the two-letter alphabet $\{a, b\}$ and we investigate the rule where 2 is replaced by 3. That is:

Expansion rule

Choose any word u and expand w by inserting uuu between two adjacent letters of w, or at the front or at the back of w.

Contraction rule

If we see uuu somewhere in w (a part of w consisting of 4 consecutive non-overlapping instances of some word u) we may contract w by deleting it and closing up.

As before, we write $v = w$ if word v can be changed to word w (or, equivalently, w changed to v) by expansions and contractions (any number of times, in any order, using the same or different words u). And as before, the fundamental problem is: how many different values are there now? The answer now (two-letter alphabet, insertion or deletion of three consecutive instances of words u) is that there are 27 different values of words. *But* this would be a hard problem for third-year or fourth-year university students!

The research context

Start with two positive whole numbers m, n. We work with words w in an alphabet that has m letters; n is known as the *exponent*.

Expansion rule

Choose any word u and expand w by inserting u^n between two adjacent letters of w, or at the front or at the back of w.

Contraction rule

If we see u^n somewhere in w (a part of w consisting of n consecutive non-overlapping instances of some word u) we may contract w by deleting it and closing up.

As before, we write $v = w$ if word v can be changed to word w (or, equivalently, w changed to v) by expansions and contractions (any number of times, in any order, using the same or different words u). Call the resulting list (organised in length-lex order) of different values $B(m, n)$.

The Burnside Problem

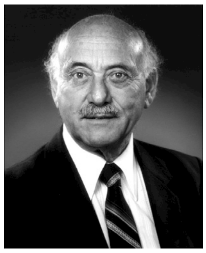
Is the list $B(m, n)$ finite? If so, how long is it?

William Burnside and his problem.
'On an unsettled question in the theory of discontinuous groups'
Quarterly Journal of Mathematics, 1902:
is $B(m, n)$ finite? Now, in 2017, much is known, much remains unknown.

At this point in a Year 10 masterclass I would spend fifteen or twenty minutes showing the children something of what we know, something of what is still unknown despite great efforts by many mathematicians. The fact that (in spite of the clue in the title of Burnside's paper) this is all a part of modern group theory remains hidden.

But what is the hidden group theory? In fact, $B(m, n)$ forms a group. It is the "freest" group generated by the m letters subject only to the condition that $u^n = 1$ for every word u in the m generators (that is, every element of the group). Technically, $u^n = 1$ is an identical relation in $B(m, n)$ in the sense of B. H. Neumann, *Identical relations in groups*, PhD thesis, Cambridge (1935), and article published in *Mathematische Annalen* (1937).

BHN in Cardiff, circa 1937.



BHN in Canberra, circa 1970.

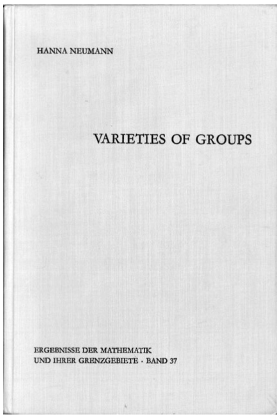
There is more hidden group theory, though. The groups that satisfy a given set of identical relations form what is called a variety of groups.

Example

The abelian variety consisting of the groups in which $ab = ba$ (or equivalently $a^{-1}b^{-1}ab = 1$) for all elements a, b.

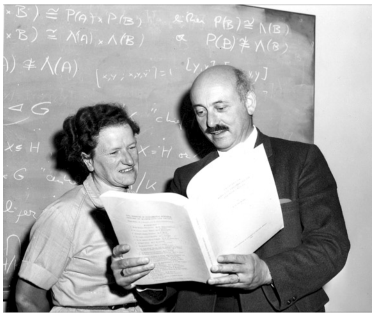
Example

The Burnside variety consisting of all groups in which $u^n = 1$ for all elements u.



The 1967 monograph.

Hanna in Canberra, 1959.



The inaugural Vice-President and President of AAMT.

Conclusion: The Burnside Problem for Exponent 5

Take words in a two-letter alphabet $\{a, b\}$. Words v, w are deemed to have equal value if one can be obtained from the other by insertion or deletion of words of the form *uuuuu* (written u^5). The resulting list of different words is known as $B(2, 5)$:

$$\text{Is } B(2, 5) \text{ finite?}$$

It is unknown. That way fame and fortune lie!

COLOUR BY NUMBERS

CHRIS WETHERELL

Radford College, ACT

cjtwetherell@gmail.com

Pascal's Triangle, which is named for 17th century French mathematician Blaise Pascal, is a simple triangular arrangement of numbers which finds important application in the fields of algebra, probability, financial arithmetic and calculus, to name just a few, and contains within it many well-known patterns such as the hidden encodings of the Fibonacci numbers and the hockey-stick theorem. It also has a fractal-like structure: colouring the odds and evens black and white results in the famous Sierpinski Triangle, if you zoom out far enough. In this paper we will investigate the surprisingly rich structure that it exists within an even simpler arrangement of numbers: the Times Tables.

Inspiration

I have the great pleasure of spending several weekends a year sitting around a table discussing (arguing about) draft problems for various competitions run by the Australian Mathematics Trust. There is always a sense of excitement about the problems that will pique my interest at those meetings (and a healthy dose of embarrassment about the ones I am unable to solve). One seemingly innocuous problem involved adding up a pattern of numbers in a grid—a simple enough task, if not a little tedious. However, the genius of the question was that anyone who spots 'the trick' should be able to perform the calculation in less than a minute, instead of 10, leaving them extra time to spend on other questions in the paper. Analysing that trick a little more closely revealed it to be nothing more than a repeated application of the so-called *distributive property*:

$$a \times (b + c) = a \times b + a \times c$$

Yes, it is the same rule for 'expanding brackets' or 'removing grouping symbols' that we torture our students with when they start learning some serious algebra in about Year 8. Already knowing something about the connection between triangles, squares and cubes, it dawned on me that there was a lot more going on in that 'pattern of numbers in a grid' and that the draft competition question could easily be made a lot more interesting (which is code for 'harder'). Having fulfilled my duties, I left the meeting with more ideas than I had managed to bring to it and proceeded to spend the next four or five months obsessing over those numbers in a grid.

Some interesting sequences

When you are old enough (say, in Year 11) you might learn that a sequence is just an infinite list of numbers. That's it, just a list. Of course, it is human nature to be more fascinated by the ones which have some kind of interesting pattern—and you can easily spend hours of your life looking for them on *The Online Encyclopaedia of Integer Sequences* (http://oeis.org). The following sequences are deemed to be 'interesting' enough for further discussion.

Counting numbers

These are also known as the natural numbers or positive integers. The sequence begins

1, 2, 3, 4...

and the formula for the n^{th} counting number is, not surprisingly, n. Perhaps this is stretching the definition of 'interesting' just a little, but we can think of the counting numbers as the result of adding one extra dot to an existing pattern of dots, as illustrated in Figure 1. This idea will be extended in the other interesting sequences to follow.

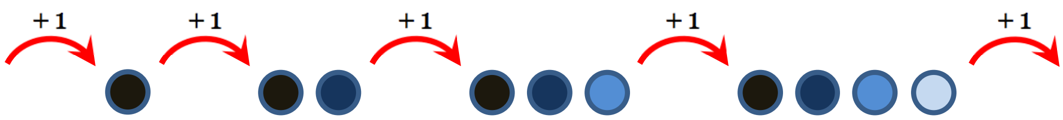


Figure 1. Counting numbers formed by adding ones.

Square numbers

The sequence begins

1, 4, 9, 16...

and the formula for the n^{th} square number is, also not surprisingly, n^2. Square numbers can be arranged into a square array of dots, grouped into n rows of n dots each. Figure 2 illustrates that square numbers can also be formed by adding the next odd number.

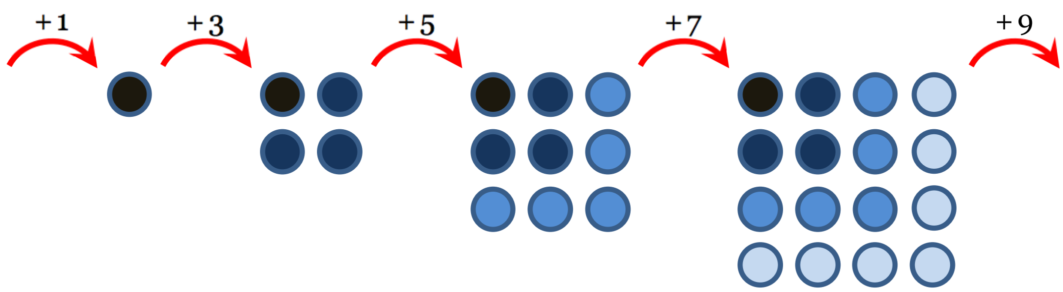


Figure 2. Square numbers formed by adding odd numbers.

Algebraically, this pattern is explained by the fact that the $(n-1)^{st}$ square number plus the n^{th} odd number equals the n^{th} square number, or in symbols

$$(n-1)^2+(2n-1)=n^2.$$

Cube numbers

The sequence begins

1, 8, 27, 64...

and the formula for the n^{th} cube number is n^3. These are the three-dimensional versions of the square numbers. That is, by stacking n squares each with n^2 dots we can create a cube of n^3 dots in three-dimensional space. The diagrams are omitted.

Triangular numbers

Returning to two-dimensional shapes, another pattern of dots can be achieved by starting with a single dot and adding the next counting number in each successive row. This forms a triangular array of dots as shown in Figure 3.

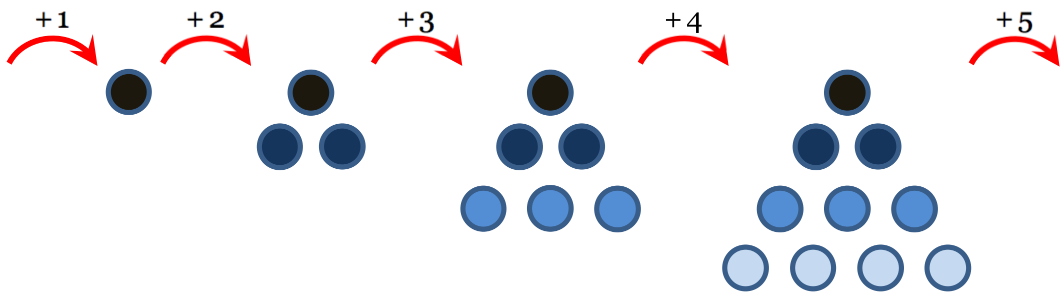


Figure 3. Triangular numbers formed by adding counting numbers.

The sequence begins

1, 3, 6, 10, 15...

and the well-known formula for the n^{th} triangular number is $T_n=\frac{1}{2}n(n+1)$. Using combinations notation, this can also be written $T_n={}^{n+1}C_2$. The defining property of the triangular arrays is described by

$$T_{n-1}+n=T_n.$$

Tetrahedral numbers

In the same way that squares can be stacked to create cubes, we can stack the triangular numbers to create the tetrahedral numbers, as shown in Figure 4.

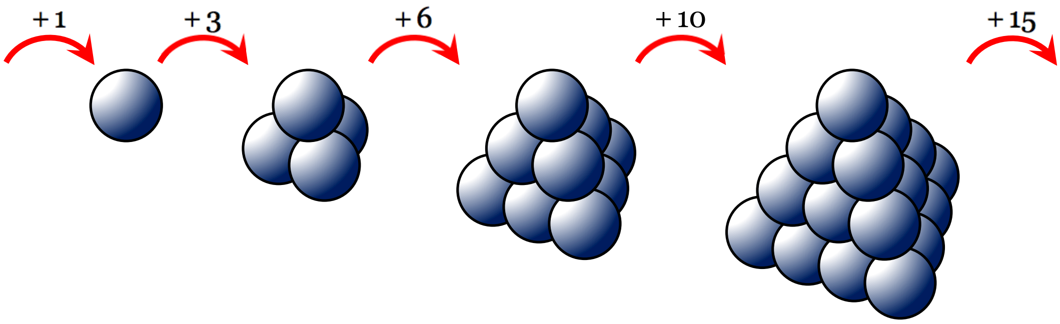


Figure 4. Tetrahedral numbers formed by adding triangular numbers.

The sequence begins

$$1, 4, 10, 20, 35\ldots$$

and the formula for the nth tetrahedral number is $H_n = \dfrac{n(n+1)(n+2)}{6}$ or ${}^{n+2}C_3$. The defining property is

$$H_{n-1} + T_n = H_n.$$

Hyper-polyhedral numbers

While we will not investigate this in any depth in this paper, it is possible to extend the ideas of two- and three-dimensional shapes to higher dimensions. For example, 'stacking n cubes of size n^3 in four-dimensional space' (whatever that might mean) results in fourth powers, with formula n^4, while 'stacking n tetrahedra of increasing size 1, 4, 10, 20…' results in what we might call the four-dimensional hyper-tetrahedral numbers, with formula ${}^{n+3}C_4$.

Fibonacci numbers

This famous sequence begins

$$1, 1, 2, 3, 5, 8, 13\ldots$$

where, after starting with 1 and 1, each successive term is the sum of the two before it. (Remarkably, there is a closed formula for the nth Fibonacci number, namely

$$F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}},$$

but we will not need to refer to this result.)

Powers of 2

The sequence begins

$$1, 2, 4, 8, 16\ldots$$

and the formula is 2^n. Each term is twice the one before.

To motivate the search for patterns in the times tables, we first summarise some well-known properties of Pascal's Triangle.

A brief survey of patterns in Pascal's Triangle

Pascal's Triangle is an infinite triangular array of numbers with the property that, starting with ones on the outer edge, every other number is the sum of the two above it, as illustrated by the highlighted entries in Figure 5. It is also intimately connected with combinations.

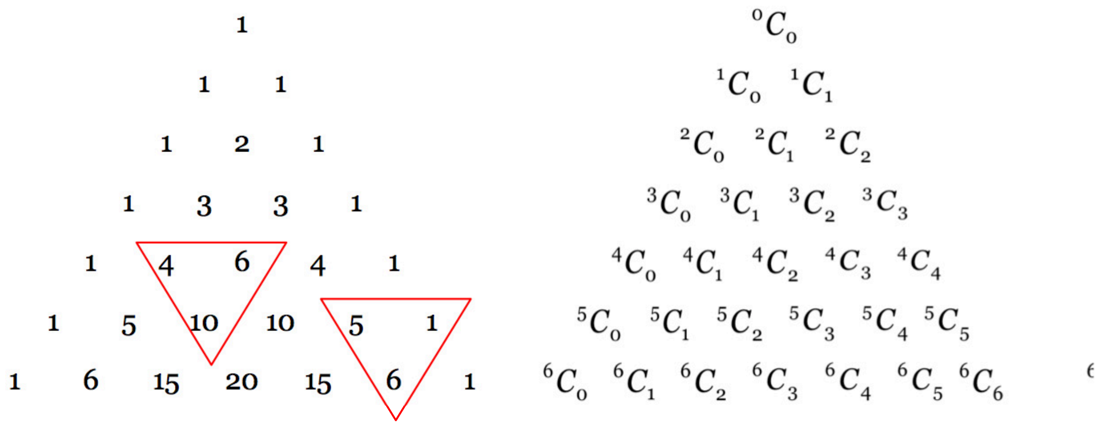


Figure 5. Additive property and combinations in Pascal's Triangle.

The properties of Pascal's Triangle which are illustrated in Figures 6–9 are well-known and no explanations are offered here (although some of them can be explained via the detail given in the previous section on interesting sequences). The patterns are noted here simply to motivate the search for similar patterns in the Times Tables. For more details the interested reader is directed to the *Wolfram Mathworld* page on Pascal's Triangle (http://mathworld.wolfram.com/PascalsTriangle.html or any number of other sites on the topic that can be found via a search engine).

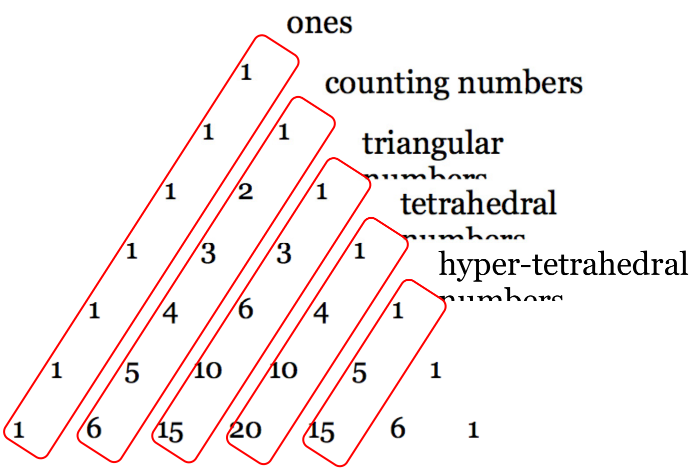


Figure 6. Number patterns in the main diagonals of Pascal's Triangle.

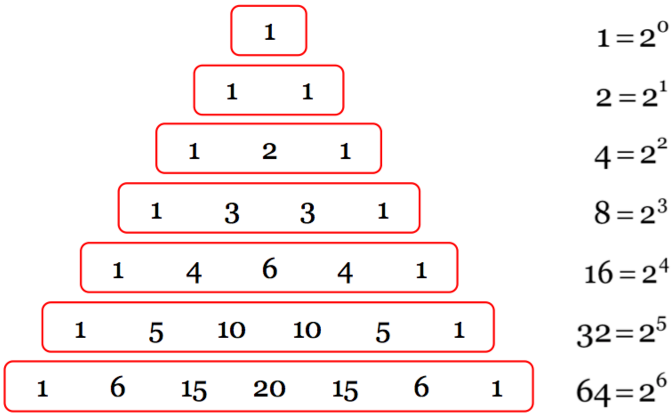


Figure 7. Powers of 2 in the totals of rows of Pascal’s Triangle.

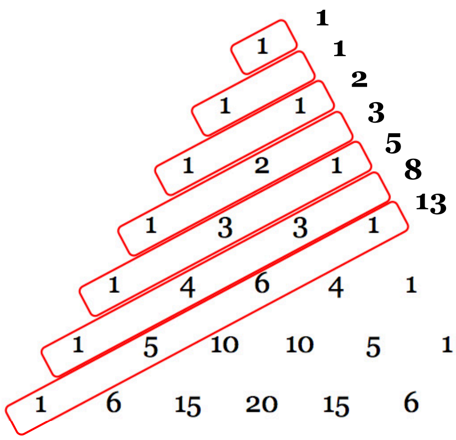


Figure 8. Fibonacci numbers in the totals of minor diagonals of Pascal’s Triangle.

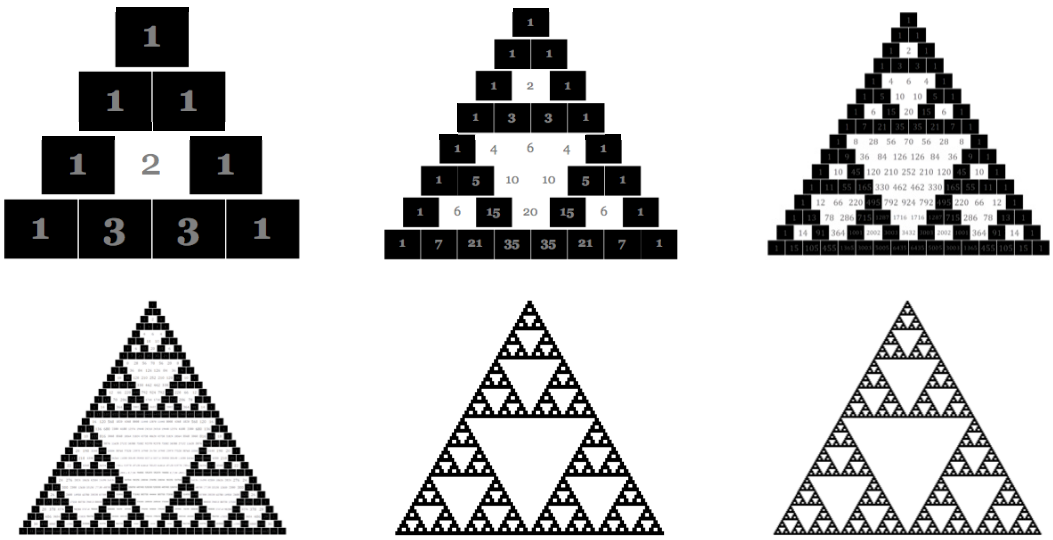


Figure 9. The Sierpinski Triangle fractal in the odd entries of Pascal’s Triangle, showing the first 4, 8, 16, 32, 64 and 128 rows at different levels of magnification.

Creating Pascal's Triangle in Microsoft Excel

The recursive nature of Pascal's Triangle makes it very easy to construct in Excel. The screenshots in Figure 9 show the simple formulae that can be entered in cells B1 and B2. To produce the full triangle, drag the bottom right corner of B2 as far as desired across to the right, then drag the bottom right corner of the selected cells of row 2 as far down as desired. This will produce lots of zeros on the 'outside' of the triangle, which can be hidden with conditional formatting, as shown in Figure 9. Since the size of the numbers involved grows rapidly, resizing the column widths, row heights and/or font sizes will be required to achieve a uniform looking triangle.

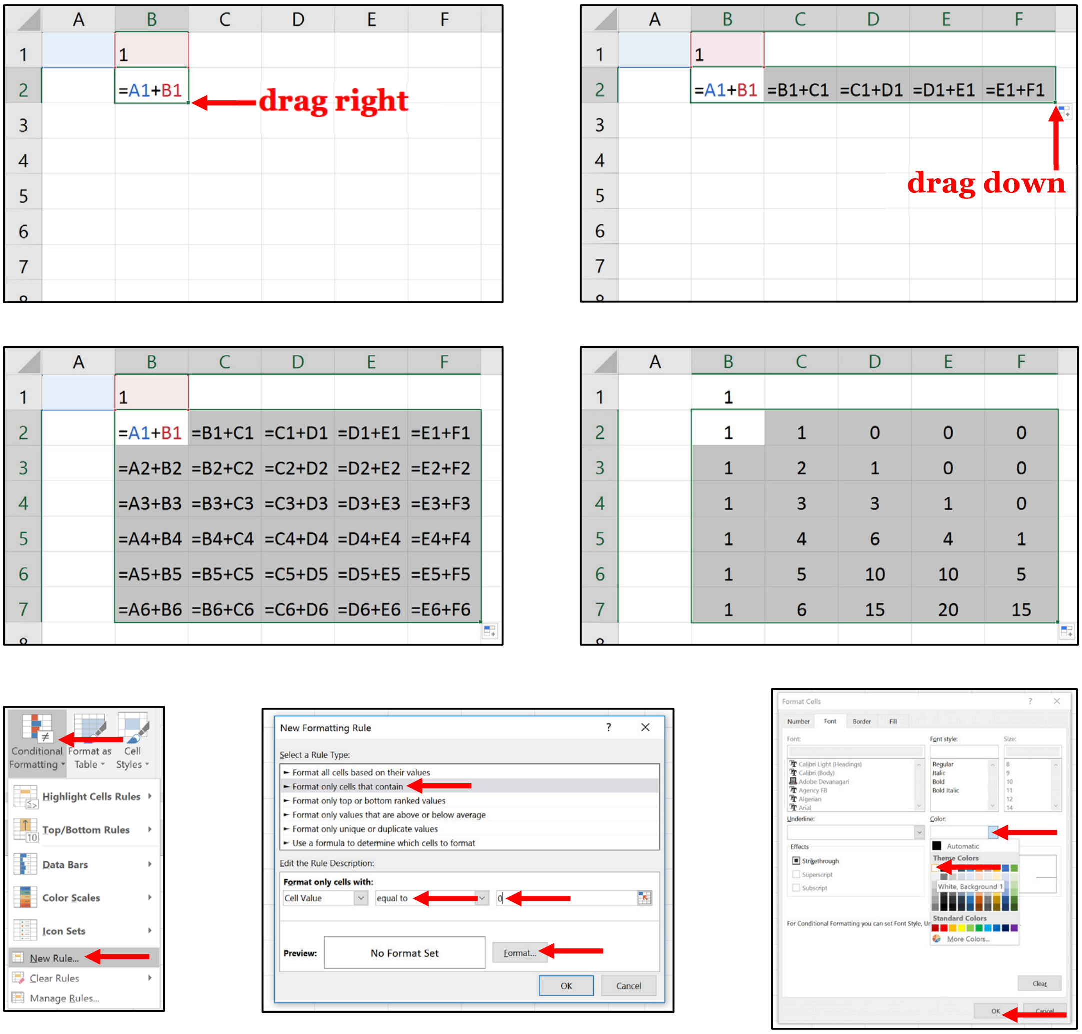


Figure 9. Excel instructions for creating a right-angled Pascal's Triangle, including conditional formatting to hide the extra zeros

The Pascal's Triangle outlined above is right-angled, rather than the more traditional equilateral variety. The accompanying spreadsheet Wetherell_C_2017_Pascal.xls also contains an equilateral version, which can be achieved by first merging cells in an alternating brick-like pattern. This said, the advantage of the right-angled version in Excel is that you do not need to decide the size of the triangle in advance since, should you require more entries for whatever reason, you can effectively continue to 'drag right' and 'drag down' without ever running out of room.

To create the Sierpinski Triangle, it is convenient to reduce every number in Pascal's Triangle to its remainder upon division by 2 using the MOD function. Thus, every even number is represented by 0 and every odd number is represented by 1. The advantage is that the numbers produced in the triangle do not grow rapidly in size, unlike the original triangle. The formulae and conditional formatting instructions are shown in Figure 10. Interesting variations can be achieved by reducing to remainders upon division by other numbers instead of 2; see Wetherell_C_2017_Pascal.xls.

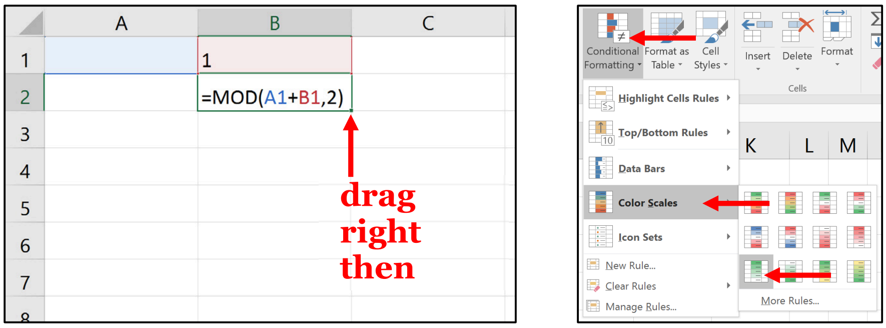


Figure 10. Excel instructions for creating a right-angled Sierpinski Triangle of 0s and 1s, including conditional formatting to colour them (select all cells with Ctrl-A or ⌘-A first).

Number patterns in the times tables

Finally, to the topic of investigation! This familiar grid of numbers has rows and columns both indexed by the counting numbers and the value in the i^{th} row and j^{th} column is the product i × j, as shown in Figure 11. Note that we will ignore the shaded row and column headings in later diagrams.

×	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	2	4	6	8	10	12	14
3	3	6	9	12	15	18	21
4	4	8	12	16	20	24	28
5	5	10	16	20	25	30	35
6	6	12	18	24	30	36	42
7	7	14	21	28	35	42	49

Figure 11. The times tables.

We begin with a search for some of the interesting number sequences discussed earlier.

Square numbers

Easy! As highlighted in Figure 12, the main diagonal contains the square numbers since when the row number and column number are both equal to n, say, then the entry is $n \times n = n^2$. Square numbers will be revisited shortly.

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	12	15	18	21
4	8	12	16	20	24	28
5	10	16	20	25	30	35
6	12	18	24	30	36	42
7	14	21	28	35	42	49

Figure 12. Square numbers in the times tables.

Triangular numbers

Suppose you are in the $(i,j)^{\text{th}}$ cell, that is in the i^{th} row and j^{th} column with value $i \times j$. Moving one entry to the right adds i to the value since, being in the i^{th} row, we are moving to the next number in the i-times tables. Similarly, moving one down adds j. It is possible to exploit these facts to find a sequence of cells which follow the triangular numbers starting at the 1 in the $(1,1)^{\text{th}}$ cell, as illustrated in Figure 13. Being in the first row means that moving right by two cells adds $2 \times 1 = 2$, resulting in the 3 in the $(1,3)^{\text{th}}$ cell. Now we are in the third column, so moving down by one row adds 3 to get the 6 in the $(2,3)^{\text{th}}$ cell. Now we are in the second row, so that repeating the move by two cells to the right adds $2 \times 2 = 4$ and leaves us in the fifth column. Repeating the move by one cell down adds 5 to the value. Hence, starting at 1, the overall effect of tracing this path is to add 2, then 3, then 4, then 5, … which results in the sequence of triangular numbers 1, 3, 6, 10… as shown. Of course, by the symmetry of the times tables the same pattern can be observed by first moving down two, then one to the right.

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	12	15	18	21
4	8	12	16	20	24	28
5	10	16	20	25	30	35
6	12	18	24	30	36	42
7	14	21	28	35	42	49

Figure 13. Triangular numbers in the times tables.

To formalise this argument we observe that the n^{th} combined move of two-to-the-right-followed-by-one-down takes us from the $(n,2n-1)^{\text{th}}$ cell to the $(n,2n+1)^{\text{th}}$ cell with an increase of $2n$ (twice to the right in the n^{th} row), and then to the $(n+1,2n+1)^{\text{th}}$ cell with an increases of $2n$+1 (one row down in the $(2n+1)^{\text{th}}$ column). This results in the pattern of increases by 2-then-3, 4-then-5, ... as claimed.

Alternatively, we can justify this pattern using the formula $T_n = \frac{1}{2}n(n+1)$ for the n^{th} triangular number. Following the description above, the value of the $(n,2n-1)^{\text{th}}$ cell is

$$\begin{aligned} n(2n-1) &= \tfrac{1}{2}(2n)(2n-1) \\ &= \tfrac{1}{2}(2n-1)((2n-1)+1)) \\ &= T_{2n-1} \end{aligned}$$

which generates the 1st, 3rd, 5th... triangular numbers, and the value of the $(n,2n+1)^{\text{th}}$ cell is

$$\begin{aligned} n(2n+1) &= \tfrac{1}{2}(2n)(2n+1) \\ &= T_{2n} \end{aligned}$$

which generates the 2nd, 4th, 6th... triangular numbers.

Consider next the zigzag-like sequence of cells traced by repeatedly moving down then right, as shown in Figure 14. For convenience, we have added an extra row of zeros for the ‘zero-times-tables’.

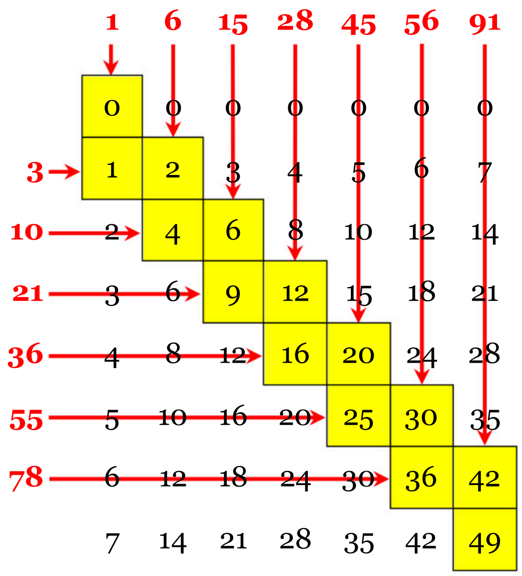


Figure 13. Triangular numbers in the sums of adjacent cells in a zigzag path.

Now calculate the sums of adjacent cells in this zigzag path, which produces the sequence of triangular numbers alternating between vertically and horizontally adjacent cells. This can be justified using similar techniques to the pattern above. This is left as an exercise.

Square and triangular numbers revisited

Consider the $n \times n$ block of cells in the top-left corner of the table. Figure 14 shows the first few cases, together with the sum of all entries in that block. Thus, it appears that the sum equals the square of the n^{th} triangular number!

1

1	2
2	4

1	2	3
2	4	6
3	6	9

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

1	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	16	20	25

1×1, sum $=1$ $=1^2$ $=T_1^2$

2×2, sum $=9$ $=3^2$ $=T_2^2$

3×3, sum $=36$ $=6^2$ $=T_3^2$

4×4, sum $=100$ $=10^2$ $=T_4^2$

5×5, sum $=225$ $=15^2$ $=T_5^2$

Figure 14. Square of a triangular number in the sum of a square block.

To illustrate why this pattern occurs, consider the 4×4 block and the square of the fourth triangular number, namely $T_4^2 = (1+2+3+4)^2$. Rather than simply evaluate this as 10^2 and compare it with the total, we instead expand the perfect square term by term, ensuring that every term in the first bracket eventually gets paired up with every term in the second:

$$
\begin{aligned}
T_4^2 &= (1+2+3+4) \times (1+2+3+4) \\
&= \quad 1 \times (1+2+3+4) \\
&\quad + 2 \times (1+2+3+4) \\
&\quad + 3 \times (1+2+3+4) \\
&\quad + 4 \times (1+2+3+4) \\
&= \quad 1\times1 + 1\times2 + 1\times3 + 1\times4 \\
&\quad + 2\times1 + 2\times2 + 2\times3 + 2\times4 \\
&\quad + 3\times1 + 3\times2 + 3\times3 + 3\times4 \\
&\quad + 4\times1 + 4\times2 + 4\times3 + 4\times4
\end{aligned}
$$

Hence it is apparent that the value obtained from squaring the fourth triangular number is precisely the sum of the entries in the 4×4 top-left block of the Times Tables. This can be generalised using sigma notation as follows, noting that the final sum is of all entries of the form $i \times j$ where i and j take all possible pairs of values between 1 and n:

$$
\begin{aligned}
T_n^2 &= T_n \times T_n \\
&= \left(\sum_{i=1}^{n} i\right) \times \left(\sum_{j=1}^{n} j\right) \\
&= \sum_{i=1}^{n}\left(i \times \sum_{j=1}^{n} j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} ij \\
&= \text{sum of entries in } n \times n \text{ block}
\end{aligned}
$$

Cube numbers

It is a reasonably well-known fact that the sum of consecutive cubes is the square of the corresponding triangular number, that is

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2.$$

The most convenient way to establish this is by induction. Being certainly true for $n = 1$, where both sides yield the value 1, we assume it is true for some arbitrary number of terms k, that is

$$
\begin{aligned}
1^3 + 2^3 + 3^3 + \cdots + k^3 &= T_k^2 \\
&= \left(\tfrac{1}{2}k(k+1)\right)^2 \\
&= \tfrac{1}{4}k^2(k+1)^2
\end{aligned}
$$

Then it needs to be verified that the addition of the next cube, namely $(k + 1)^3$, results in the square of the next triangular number, namely T_{k+1}^2:

$$
\begin{aligned}
(1^3 + 2^3 + 3^3 + \cdots + k^3) + (k+1)^3 &= \underbrace{\tfrac{1}{4}k^2(k+1)^2}_{\text{by assumption}} + (k+1)^3 \\
&= (k+1)^2\left(\tfrac{1}{4}k^2 + k + 1\right) \\
&= \tfrac{1}{4}(k+1)^2(k^2 + 4k + 4) \\
&= \tfrac{1}{4}(k+1)^2(k+2)^2 \\
&= \left(\tfrac{1}{2}(k+1)(k+2)\right)^2 \\
&= T_{k+1}^2
\end{aligned}
$$

This establishes the desired result.

Given the connection between T_n^2 and the $n \times n$ blocks described above, it should now be possible to find the cube numbers hidden in the times tables. Indeed, the induction argument above indicates exactly how: look for the difference between successive blocks in the top-left corner, or in other words an L-shaped path. This is illustrated in Figure 15.

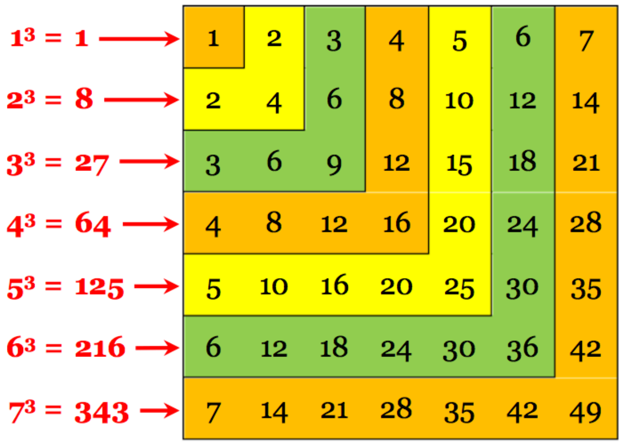


Figure 15. Cubes in the sum of L-shaped paths.

Tetrahedral numbers

Motivated by patterns in the main diagonals of Pascal's Triangle (see Figure 6), consider next the sum of entries illustrated in Figure 16.

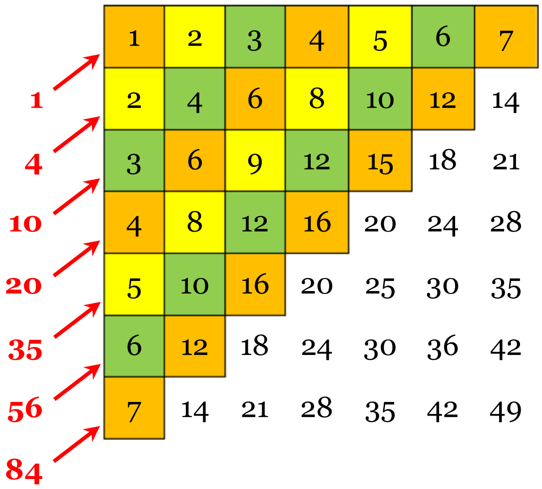


Figure 16. Tetrahedral numbers in the sum of major diagonals.

Recall that the tetrahedral numbers H_n are the sum of consecutive triangular numbers and therefore satisfy $H_{n-1} + T_n = H_n$, or equivalently

$$T_n = H_n - H_{n-1}.$$

Though we will not formalise the inductive aspect of the argument here, to establish that the n^{th} diagonal does sum to the n^{th} tetrahedral number, it is enough to show instead that the difference of the n^{th} and $(n-1)^{\text{th}}$ diagonal sums is the n^{th} triangular number.

For example, consider the 6th and 5th diagonals, with totals 56 and 35, respectively. From the entries of the times tables, the difference can be calculated as follows; note that a redundant term of 0×6 has been included for the 5th diagonal to illustrate the more general pattern:

$$
\begin{aligned}
&6^{\text{th}}\text{ diagonal} - 5^{\text{th}}\text{ diagonal}\\
&\quad = \quad (\ 6\times1 + 5\times2 + 4\times3 + 3\times4 + 2\times5 + 1\times6\ )\\
&\qquad\ -(\ 5\times1 + 4\times2 + 3\times3 + 2\times4 + 1\times5 + 0\times6\ )\\
&\quad = \quad 1 + 2 + 3 + 4 + 5 + 6\\
&\quad = T_6
\end{aligned}
$$

Using sigma notation again, the general argument can be written as follows; note that the change of limit from $n-1$ to n from the second to third lines is allowed because of a redundant $0 \times n$ term which can be added to the $(n-1)^{\text{th}}$ diagonal:

$$
\begin{aligned}
&n^{\text{th}}\text{ diagonal} - (n-1)^{\text{th}}\text{ diagonal}\\
&\quad = \sum_{i=1}^{n} i(n+1-i) - \sum_{i=1}^{n-1} i(n-i)\\
&\quad = \sum_{i=1}^{n} i(n+1-i-(n-i))\\
&\quad = \sum_{i=1}^{n} i\\
&\quad = T_n
\end{aligned}
$$

Other pyramids

We mention in passing that, inspired particularly by the 'minor' diagonals used in the construction of the Fibonacci numbers in Pascal's Triangle (see Figure 8), adding entries of other diagonals results in other pyramidal numbers. For example, the sums formed when moving diagonally 'two-right-one-up', as in Figure 17, are the square-pyramidal numbers 1, 5, 14, 30... which are the sums of consecutive squares, $1^2 + 2^2 + 3^2 + \ldots$

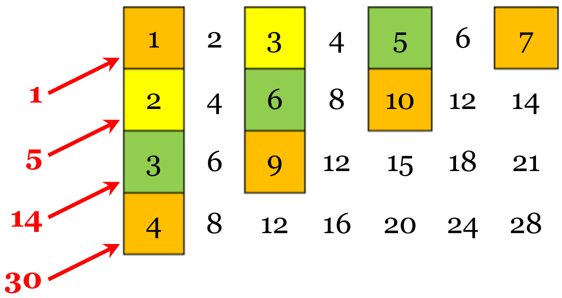


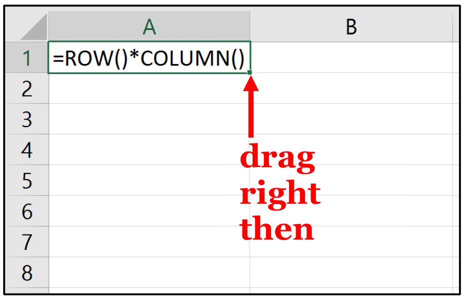
Figure 17. Square-pyramidal numbers in the sum of minor diagonals.

Colour patterns in the times tables

In this final section, we are motivated by the Sierpinski Triangle which emerges when colouring Pascal's Triangle. Are similarly intricate patterns hidden in the times tables?

Creating the times tables in Microsoft Excel

The ROW() and COLUMN() commands return the row number and column number of the current cell. Hence, starting from cell A1 to achieve the entry 1×1, the formula =ROW()*COLUMN() can be used throughout the spreadsheet to create the times tables of any desired size, as illustrated in Figure 18. Again, resizing of column widths, row heights and/or font sizes may be required to achieve a uniform grid.



	A	B	C	D	E	F	G	H
1	1	2	3	4	5	6	7	8
2	2	4	6	8	10	12	14	16
3	3	6	9	12	15	18	21	24
4	4	8	12	16	20	24	28	32
5	5	10	15	20	25	30	35	40
6	6	12	18	24	30	36	42	48
7	7	14	21	28	35	42	49	56
8	8	16	24	32	40	48	56	64

Figure 18. Excel instructions for creating the times tables.

The remainder of this paper will deal with variations on this basic spreadsheet, including using conditional formatting to colour cells and using the MOD function to achieve fractal-like patterns; see also Wetherell_C_2017_TimesTables.xls .

Important tip: when dealing with large spreadsheets, Excel copes best if you remove conditional formatting before you edit formulae, then reapply the desired formatting.

Colour-coding by length

As both i and j increase, so too does the product $i \times j$. As a first coarse investigation into this property, consider colour-coding the cells of the times tables according to the number of digits in the entry $i \times j$, as in Figure 19.

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
3	6	9	12	15	18	21	24	27	30	33	36	39	42	45
4	8	12	16	20	24	28	32	36	40	44	48	52	56	60
5	10	15	20	25	30	35	40	45	50	55	60	65	70	75
6	12	18	24	30	36	42	48	54	60	66	72	78	84	90
7	14	21	28	35	42	49	56	63	70	77	84	91	98	105
8	16	24	32	40	48	56	64	72	80	88	96	104	112	120
9	18	27	36	45	54	63	72	81	90	99	108	117	126	135
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150
11	22	33	44	55	66	77	88	99	110	121	132	143	154	165
12	24	36	48	60	72	84	96	108	120	132	144	156	168	180
13	26	39	52	65	78	91	104	117	130	143	156	169	182	195
14	28	42	56	70	84	98	112	126	140	154	168	182	196	210
15	30	45	60	75	90	105	120	135	150	165	180	195	210	225

Figure 19. Values in the times tables colour-coded by length.

The cell values have been preserved in Figure 19, but at the expense of changing these values the underlying pattern of colours can easily be achieved in Excel with conditional formatting, as follows:

- replace the standard =ROW()*COLUMN() formula in each cell by
- =LEN(ROW()*COLUMN())
- which calculates the length of the string of digits representing the number

- select all cells with Ctrl-A or ⌘-A and then use a colour scale via the conditional formatting menu (refer to Figure 10).

To achieve the same colouring effect while preserving the actual values in each cell, more complicated conditional formatting rules can be used via the 'Use a formula to determine which cells to format' option. However, simply applying a colour-scale based on the value itself, rather than its length, turns out to be more illuminating.

Colour-coding by number

Returning to the basic formula =ROW()*COLUMN(), apply a colour-scale to achieve the images in Figure 20. The first version shows the slightly pixelated effect in the 10×10 grid, while the second shows the very smooth transition of colours in the 200×200 grid (with values omitted).

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

Figure 19. Times tables colour-coded by values.

What is the nature of the curve that is emerging in Figures 18 and 19?

A familiar graph

In order to better understand what the colour-coding is telling us, we do three things:

- reverse the order of rows so that they are indexed in the order consistent with the convention for the y-axis (increasing up, decreasing down)
- extend both the rows and columns into the negative direction, in order to analyse the behaviour of the colour scheme in other quadrants of the Cartesian plane
- change the colour-scale settings to reduce the complexity of the image

Let us decide upon a domain and range of both [–100, 100], which will require a total of 201 rows and columns. To achieve the first two dot-points above, we replace the standard =ROW()*COLUMN() formula with

=(COLUMN()-101)*(101-ROW())

For example, the top-left cell A1, with regular cell reference (1,1), is now referenced by the coordinates (–100,100), since $x = 1 - 101 = -100$ and $y = 101 - 1 = 100$. Adjusting the original colour-scale slightly results in the diagram in Figure 20. Note that we see evidence of the naturally occurring coordinate axes in bright yellow, when either $x = 0$ or $y = 0$, and not-unexpected rotational symmetry which is explained by the behaviour of positive and negative values in the various quadrants.

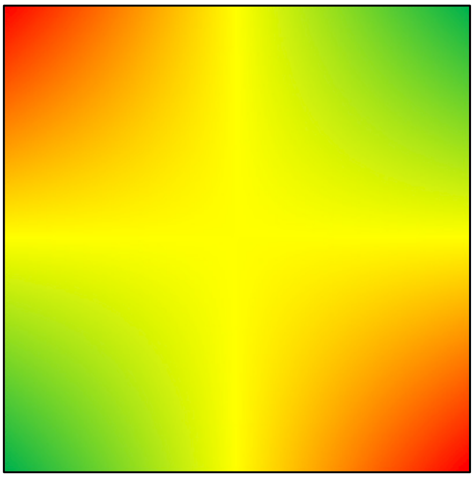


Figure 20. Times tables with flipped rows and negative values.

At this point it is worthwhile analysing the meaning of the colour-scale a little more deeply. The grid in Figure 20 consists of values ranging from –10 000 in the top-left and bottom-right corners, through to 10 000 in the other corners. The value in each cell therefore represents a certain percentile of the entire dataset and the cell's colour is then assigned from a chosen spectrum according to that same percentile. In these examples the spectrum is orange-yellow-green, so the lowest values are orange, values near the median (namely zero) are yellow, and the highest values are green. In Excel, the default colour-scale setting is to assign those percentiles uniformly; for example, a cell-value which is at the 20th percentile is assigned a colour which is 20% of the way along the chosen colour spectrum. However, these settings can be manipulated to, in effect, tune the sensitivity of the cell-values' colour-coding.

Figure 21 shows the instructions for editing the colour-scale settings in Excel. The desired effect is to produce a graph, complete with coordinate axes, which emphasises a much more restricted range of cell values. Specifically,

- the orange-yellow-green spectrum is replaced by white-blue-white
- the default colour transition cut-offs (namely the 0th, 50th and 100th percentiles) are adjusted to the 64th, 65th and 66th percentiles, respectively
- a new rule is applied to colour all cells containing zero black

First click in any non-empty cell and select the entire grid with Ctrl-A or ⌘-A. Then follow the instructions in Figure 21.

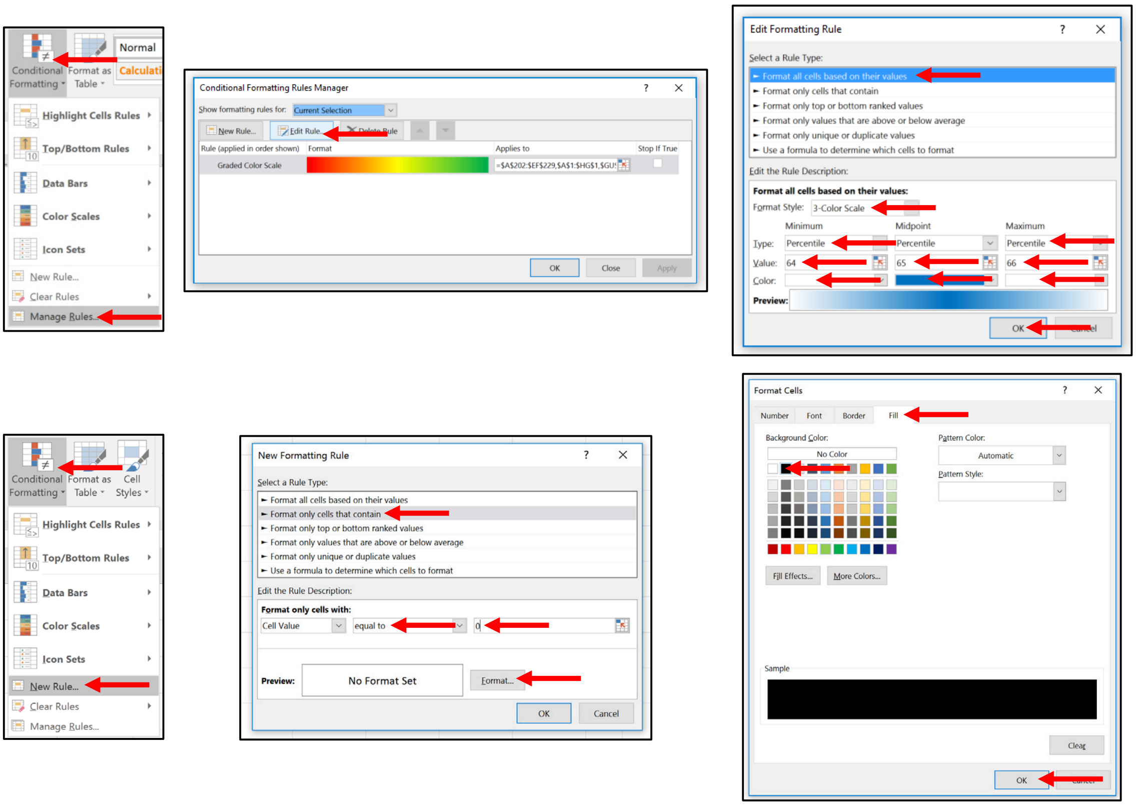


Figure 21. Excel instructions for editing the graph.

The resulting image is shown in Figure 22. It looks remarkably like a hyperbola! Adjusting the new 64-65-66 cut-offs to different values has the effect of dilating and/or reflecting the hyperbola; higher values dilate it away from the axes, values closer to 50 contract it towards the axes, and values less than 50 reflect it into the other quadrants.

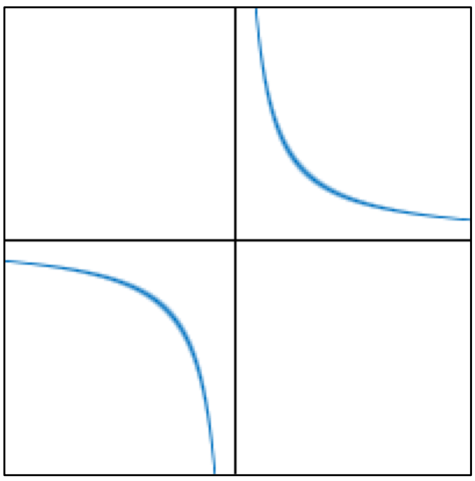


Figure 22. A hyperbola?

The explanation for this curve is surprisingly straightforward. For a given choice of percentile cut-offs, there is a small range of values in the dataset which will be coloured blue. For example, for a cut-off around the 65$^{\text{th}}$ percentile these values are around 884. In general, if this target value is called a then a cell with coordinates (x, y) is coloured blue precisely when $xy \approx a$, or equivalently when

$$y \approx \frac{a}{x}.$$

This is, approximately at least, the standard equation of a rectangular hyperbola with dilation factor a.

Fractal-like patterns

Our final goal is to explore the patterns which arise with variations on the ‘odds and evens’ theme. Though the mathematical discussion is kept to a minimum, the reader’s attention is drawn to the hyperbolic nature of many of the patterns, and particularly to those where hyperbolas appear simultaneously at several different scales; this is the essence of ‘fractal-like’ behaviour, even if these are not true fractals. *Excel* formulae are given in the captions for Figures 23–29. The colour-scale is white-red-purple (for the default 0th-50th-100th percentiles). See also Wetherell_C_2017_TimesTables.xls .

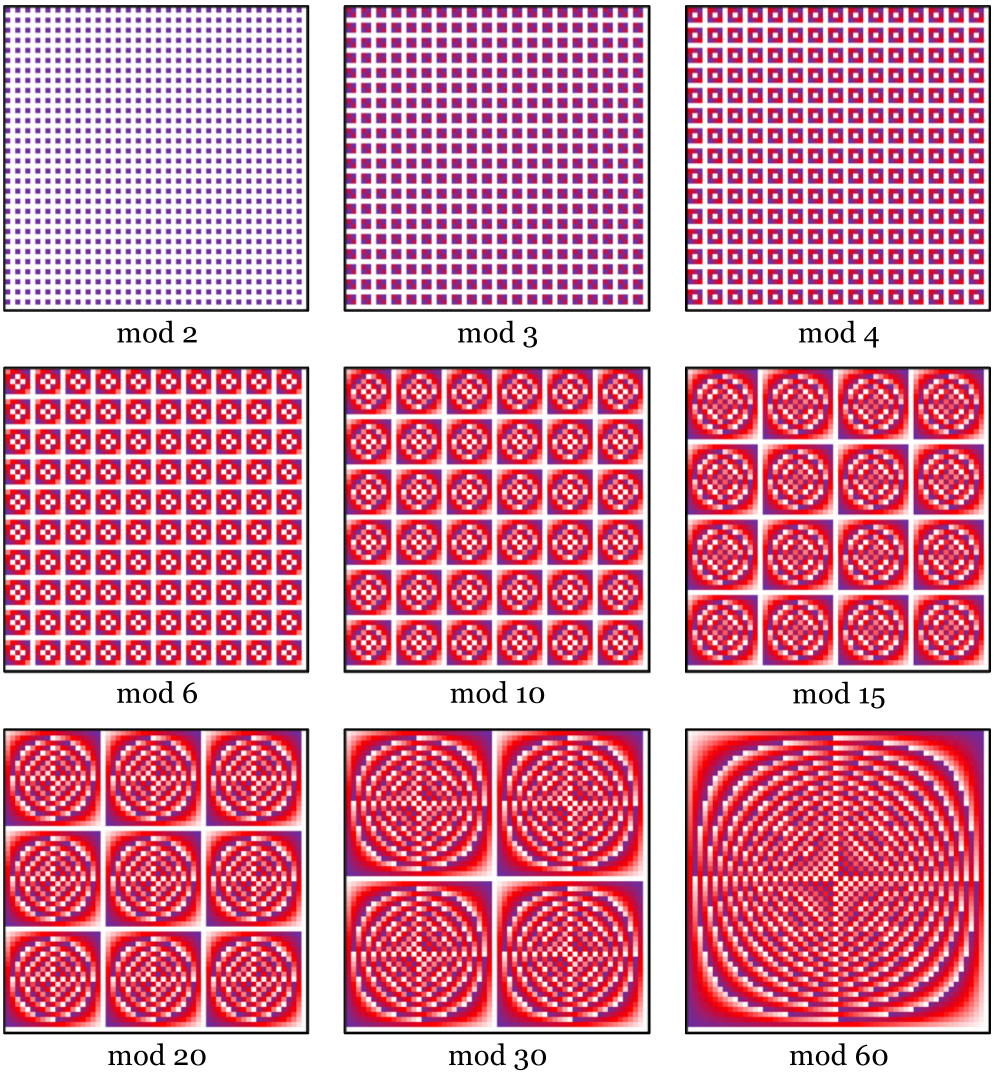


Figure 23. The 60×60 grid with entries reduced to remainders modulo various numbers, e.g., =MOD(ROW()*COLUMN(),6) for mod 6.

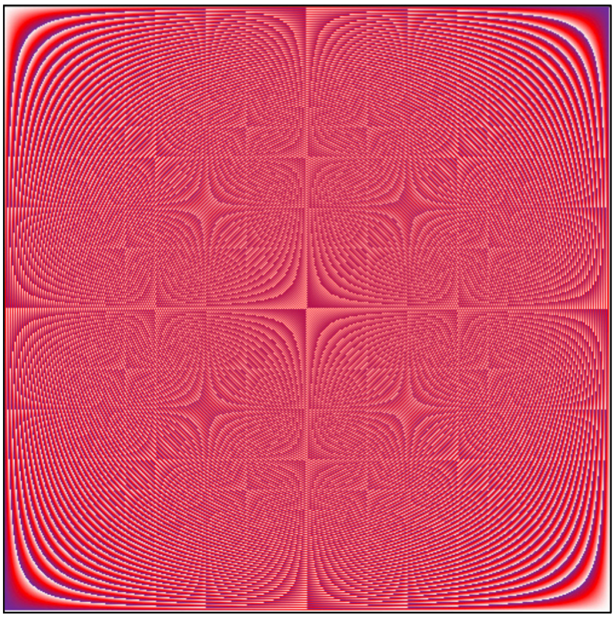


Figure 24. The 500×500 grid with entries reduced to remainders modulo 500, =MOD(ROW()*COLUMN(),500).

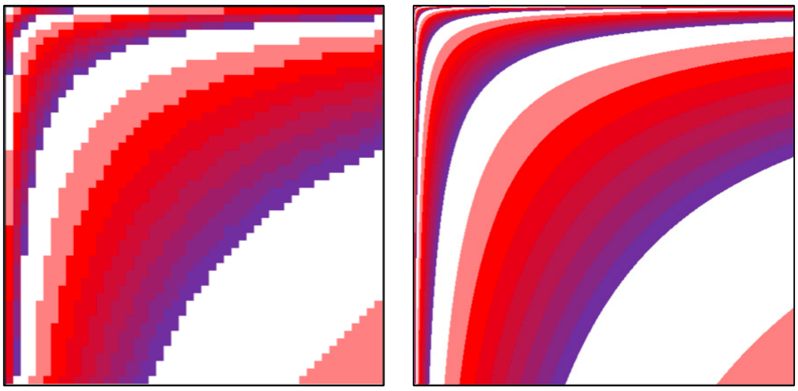


Figure 25. The 50×50 and 500×500 grids with entries colour-coded by their first digit, =VALUE(LEFT(ROW()*COLUMN(),1)).

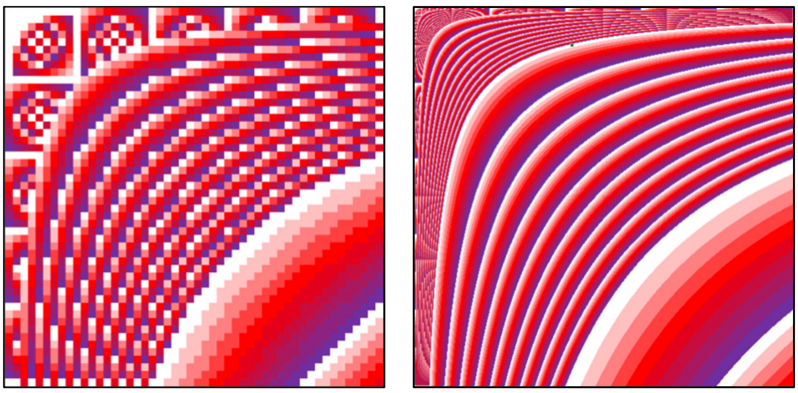


Figure 26. The 50×50 and 500×500 grids with entries colour-coded by their second digit, =VALUE(MID(ROW()*COLUMN()&"0",2,1)) (the extra zero is to avoid an error for 1-digit numbers).

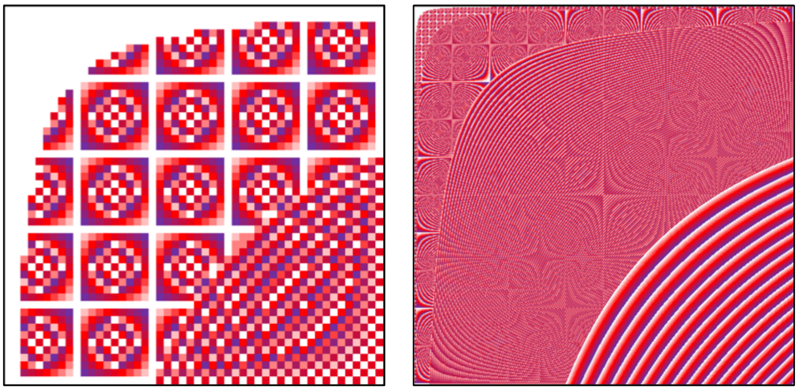


Figure 27. The 50×50 and 500×500 grids with entries colour-coded by their third digit, =VALUE(MID(ROW()*COLUMN()&"00",3,1)) (the extra zeros are to avoid an error for 1- and 2-digit numbers).

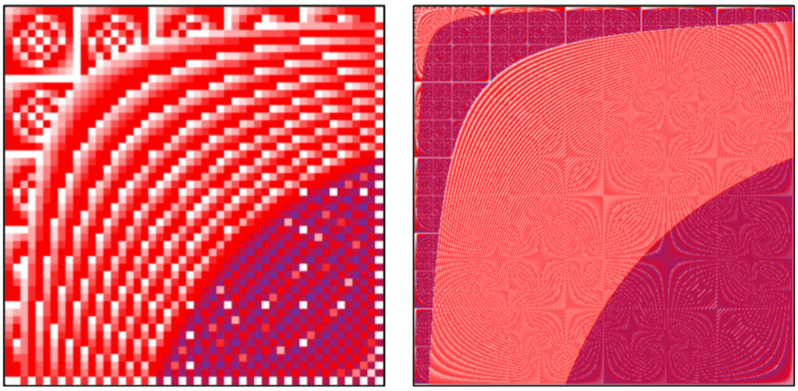


Figure 28. The 50×50 and 500×500 grids with entries colour-coded by the number formed from the middle digit or middle two digits, =VALUE(MID(ROW()*COLUMN(),INT(1+LEN(ROW()*COLUMN())/2),2-MOD(LEN(ROW()*COLUMN()),2))).

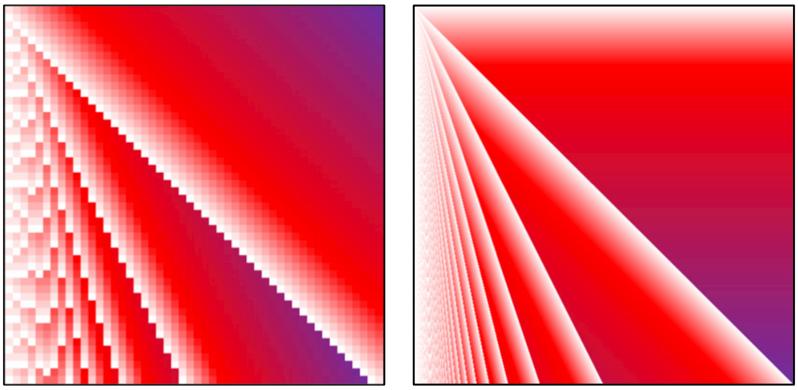


Figure 29. The 50×50 and 500×500 grids with entries reduced to remainders modulo their column number plus 1, =MOD(-ROW(),COLUMN()+1).

There is not much method to the madness here; any variation on =ROW()*COLUMN() which is vaguely complicated enough seems to give interesting results. If you are inspired to have a play with your own spreadsheets and find something new, interesting or even just confusing that you want to share, please email me at cjtwetherell@gmail.com.

PROFESSIONAL PAPERS

THE IMPACT OF NATURAL LANGUAGE ON PERFORMANCE IN WORD PROBLEMS AT LOWER SECONDARY LEVEL

KHEMDUTH SINGH ANGATEEAH

Mauritius Institute of Education

k.angateeah@mieonline.org

This paper examines the effect of using natural language on students' performance on word problems at Grade 8 level (13 years old) in Mauritius—an island with a multi-ethnic population. A quasi-experimental mode of inquiry was used involving 233 students. A control group was established in which English (L2) was used as the language of instruction, and another one, in which Mauritian Creole (L1) was used. Though the L1 group performed better, the difference was not statistically significant. Analysis by ability grouping (high, average and low) showed that only low ability students benefitted from the use of L1.

Language plays an important role in the teaching and learning of mathematics (Ríordáin & Donoghue, 2006). During the last thirty years, the body of research which has looked at the interface between language and mathematics proficiency has significantly increased (Durkin & Shire 1991; Pimm 1991). The problems associated with language and mathematics vary, based on local contexts, and studies in this area have looked at different aspects and have taken different orientations over time. For example, the issue of learning and teaching mathematics in a second language has been studied extensively in other countries such as the USA, Canada, South Africa, Australia, New Zealand, Spain and the UK.

However, there are conflicting views about the learning of mathematics in a second language at all levels of education. Some studies have found positive correlations with learning mathematics in a second language and academic achievement (Barwell, 2003), while other studies put forward concerns that such pupils underachieve in mathematics (Barton, Chan, King, Neville-Barton & Sneddon, 2005).

The cognitive load theory (CLT) suggests that the human cognitive system can be characterised as comprising of a relatively poor working memory (Miller, 1956), coupled with an effectively limitless long-term memory (Sweller & Chandler, 1994) intended to store a huge number of schemata. Tuovinen and Sweller (1999) observed that if the student has gained suitable automated schemas, cognitive load will be low, and sufficient working-memory resources are expected to be free. They pointed out that if the ideas must each be considered as a separate element in working memory, because no schema exists, cognitive load will be high. On the other hand, Ong, Liao and Alimun

(2009) highlighted that CLT can occur not only in performance tasks but in language as well. For instance, Bernardo (1999) found that students solved arithmetic word problems better when the problems were written in their first language (Filipino).

According to Heinze (2005), several processes are undertaken in comprehending a second language (e.g., finding meanings of ambiguous concepts) and in solving a problem (e.g., assembling thoughts, concepts and procedures). The distribution of cognitive resources involved in solving problems when written in a second language has two levels of processes occurring simultaneously. They are 'comprehending the second language' and 'solving the problem' which maximise one's cognitive resources (Ong et al., 2009). Thus, the cognitive resources become limited since both 'declarative knowledge in processing the language' and 'procedural knowledge in solving the problems' are used. The native language serves as an automatic process that enables the learner to perform a task without too much conscious awareness and demands little or no (cognitive) effort (Ong et al., 2009). These briefly explain how the use of first language may enhance students' ability to solve word problems. Consequently, students with low scores in problem-solving tests written in their second language do not necessarily have low problem-solving abilities; rather, their linguistic capabilities in that (second) language might be low (Ong et al., 2009).

Other studies have looked at mathematics ability of bilinguals and multilinguals (Barwell, 2003; Bose & Choudhury, 2010). Grosjean (1992) referred to bilingualism as the regular use of two or more languages. Two main perspectives of bilingualism are the monolingual view and the bilingual view. The former suggests that "the bilingual has two separate and isolated language competencies" (Grosjean, 1992, cited in Ong et al., 2009). This perspective holds that bilinguals are two monolinguals in one person (Grosjean, 1998). The monolingual view believes that bilinguals should be equally fluent in both the languages and use them at the same level, otherwise they are not bilingual. The monolingual view does not consider that the competency of one's first language may be affected if it comes in contact with one's second language.

On the other hand, the bilingual view proposes that bilinguals are not two monolinguals in one person but "an integrated whole which cannot be easily be decomposed into two separate parts" (Grosjean, 1992, cited in Ong et al., 2009). The two languages may be used separately or at the same time, depending on the call of different situations and are rarely equally fluent. Grosjean (1998) stated that in most cases, the bilingual uses the base language as his main language, which is considered to be the most active. The second language may be deactivated and activated, depending on whom the bilingual is communicating with.

Works by Ellerton and Clarkson (1996) and Setati (2003) support the view that the use of the natural language (L1) benefits learners of mathematics. It is not clear though whether these benefits will be for all learners or just a category of them. Bernardo and Calleja (2005) found that the use of natural language (Filipino) is advantageous to Filipino-English bilinguals in problem solving but such advantage was not observed with older students who develop problem-solving schema with time (Bernardo, 2001). Despite the considerable literature on the linguistic features of mathematical discourse in English (Clarkson, 1991; MacGregor & Moore, 1991), there has been limited research on the difficulties these cause for mathematics learners, particularly at the lower secondary level. For instance, Abedi (2001) is one of a limited body that has a focus on elementary mathematics.

In Mauritius, Creole is the natural language for the majority of learners, but English is the language of instruction. Many debates have taken place around the language policy in the Mauritian education system. There are voices for and against the use of natural language as the instructional language. In particular, the divide in beliefs on this issue between practitioners and policy makers is apparent. For example, it is not surprising to come across classes where Creole is the medium of instruction (teacher-led) whereas curriculum materials and assessment instruments (state prepared) are still devised in English. Due to limited research in this area, much of the debate around the issue of language policy has lacked scientific foundation and it might sound as if a mere switch to L1 will be a universal remedy of mathematics underachievement at school. The aim in this study is to establish whether, at Grade 8 level (13 years old), the

- use of Mauritian Creole improves achievement in solving word problems;
- use of Mauritian Creole is beneficial for all ability groups of students.

Method

This study is part of a larger study. Permission was granted by the Ministry of Education and Human Resources to access secondary schools and to collect data for the study. Subjects were ensured confidentiality of the data gathered. One thousand and thirty seven Grade 8 students, involving 29 classes from 13 schools, participated in a preliminary mathematics test used for sampling purposes. However, the data reported in this paper are based on only 12 classes (6 boys and 6 girls) involving 410 students (plus one additional class for piloting purposes). The preliminary mathematics test was based on 22 questions (17 questions from the Grade 7 syllabus together with five simple word problems) printed on four pages. The objective of the test was to categorise the students into ability groups. Prior to administration, the test was piloted with nine Grade 8 students from different schools and abilities, and adjustments made. A marking scheme was developed for consistent correction and the maximum score was 70 marks. Three sets of equivalent questionnaires were constructed for pre/post/retention tests. The items were multistep word problems adapted from past examination papers, research papers and Australian Mathematics Competition questions. An initial questionnaire of 16 questions was designed and piloted with 90 students (3 groups of 30 students with high, average and low ability). Following the piloting, the questionnaire was amended and 10 items were retained. After further piloting, the pre/post/retention tests were reviewed and only 9 items were retained with a maximum score of 36 marks. Twenty-three items were divided into two worksheets with 12 and 11 items respectively. The worksheets contained multistep word problems involving the four basic operators (+, −, ×, ÷).

The study was conducted during the first and second term of 2011. During the first term, the preliminary mathematics test was administered. Based on the scores, students were categorised into three ability groups as follows: High Achievers (marks ≥ 50); Average Achievers ($30 \leq$ marks < 50); Low Achievers (marks < 30). Each ability group was further divided into two subgroups and randomly assigned to the control and treatment groups. Each subgroup consisted of two classes (one boys and one girls).

Subjects from the six subgroups (12 intact classes) took a one-hour pre-test at the beginning of the second term. The pre-test was also used for selection of homogeneous groups, but the subjects were unaware of who forms part of the survey. In this way, performances were comparable for different treatments. The pre-test was marked based on a marking scheme devised by the researcher. Each group attended at least two training sessions totalling 160 minutes to work out the worksheets. Most of the sessions were conducted during activity periods which are scheduled twice a week.

- Control group (L2): Subjects were taught in English (L2) in the traditional manner. That is, the researcher worked out a problem as an example and asked subjects to solve other problems and finally correct the problems.
- Treatment (L1): Subjects were taught using Mauritian Creole (L1) as the language of instruction. The researcher proceeded as for the control group but used Creole as the language of instruction instead of English, while retaining the technical terms in English. All written materials were in English.

One-hour post and retention tests were administered between one to three days and seven weeks respectively after the training sessions. The marks were then entered into statistical software (SPSS) for analysis. T-tests were used to analyse the data. After eliminating subjects who were absent during any one of the sessions (pre/post/retention tests or training), only 233 students were retained. The preliminary mathematics test as a set was found to be very reliable with Cronbach alpha greater than 0.8. The pre, post and retention tests proved to be reliable with Cronbach alpha close to 0.7 for each.

Correction of both the preliminary mathematics test and the pre/post/retention tests was made according to marking schemes devised by an experienced teacher and the researcher respectively. Both have at least eleven years of teaching experience at the secondary level and have marked scripts for the National Cambridge Examination for at least eight years. All training sessions were conducted by the researcher in order to reduce variation due to different teaching styles.

Results

The post-test results indicate a gain in performance (see Table 1 and Figure 1) over the pre-test for both groups. Performance on the retention test was similar to that of the post-test.

Table 1. Mean scores and standard deviations for pre-/post-/retention tests.

Group		Pre-test		Post-test		Retention test	
	n	Mean	SD	Mean	SD	Mean	SD
L2	114	15.47	7.679	23.83	8.232	24.30	7.412
L1	119	15.56	6.489	25.33	6.994	24.71	6.446

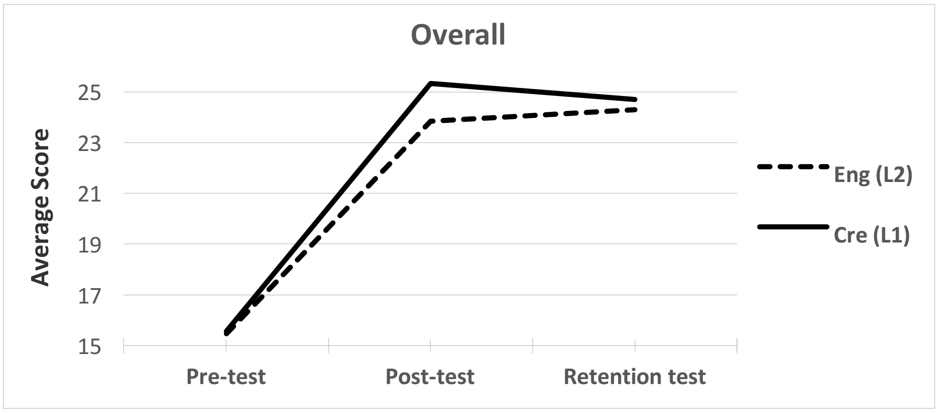


Figure 1. Trends in performance for pre-/post-/retention tests for overall sample.

In order to investigate any statistically significant difference in performance between the strategies, a *t*-test was carried out. No significant differences between the scores were observed in the pre-test $t(221) = -0.096$, post-test $t(221) = -0.490$ and retention test $t(223) = -0.447$ performance, $p > 0.05$. Since the performance of the two groups was indistinguishable in the pre-test as well, the choice of language was not found to impact on achievement levels of students on word tasks.

Scores by ability levels

Independent of the teaching strategies employed, all ability groups improved their performance from pre-test to post-test as shown in Table 2. From post-test to retention test, a slight fall in performance was observed for high achievers in general (Figure 2), and for low achievers who were taught using L1 (Figure 4). A slight rise in performance, from post-test to retention test, was noted for average achievers (Figure 3) and for low achievers (Figure 4) who were taught using L2.

Table 2. Performance of students in pre-/post-/retention tests, by ability groups and teaching strategy.

Ability	Teaching strategy	*n*	Pre-test Mean	SD	Post-test Mean	SD	Retention test Mean	SD
High	L2	36	23.47	3.443	32.08	3.901	31.28	4.676
	L1	33	21.94	4.220	31.70	3.746	30.36	4.336
Average	L2	40	14.23	5.201	22.55	5.991	23.38	5.569
	L1	55	15.13	5.368	22.93	6.495	23.15	5.616
Low	L2	38	9.21	6.010	17.37	6.619	18.66	5.781
	L1	31	9.55	3.510	22.81	6.306	21.45	5.999

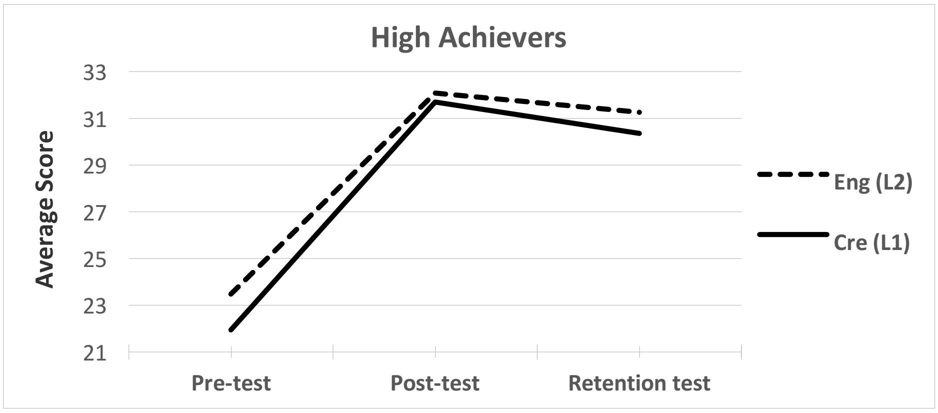


Figure 2. Trends in performance for pre-/post-/retention tests for high achievers.

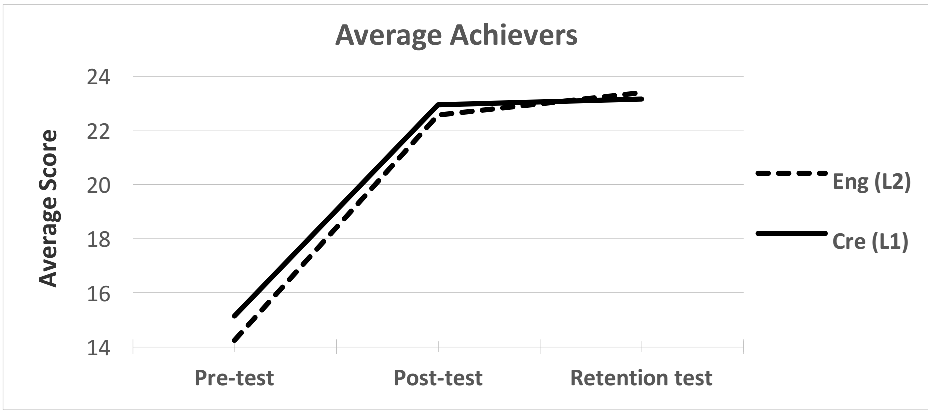


Figure 3. Trends in performance for pre-/post-/retention tests for average achievers.

Figure 4. Trends in performance for pre-/post-/retention tests for low achievers.

High Achievers

Among high achievers, the *t*-test confirmed that the use of L1 does not produce a statistically significant ($p>0$) gain in performance over the control group in the post- and retention tests (Table 3). Both strategies resulted in comparable mean performance (around 32 marks) and standard deviations for the post/retention tests.

Table 3. T-test for pre-, post-, retention tests for average achievers.

	Pre-test	Post-test	Retention test
t	1.659	0.419	0.840
df	67	67	67
sig (1-tail)	0.051	0.339	0.202

Average achievers

The *t*-test confirmed that performance of students in L1 group is not significantly higher than those for L2 group in post/retention tests (see Table 4). As for high achievers, the influence of language on performance is insignificant in this ability group.

Table 4. T-test for pre-, post-, retention tests for average achievers.

	Pre-test	Post-test	Retention test
t	−0.819	−0.289	0.197
df	93	93	93
sig (1-tail)	0.208	0.387	0.422

Low achievers

A different scenario was observed in this ability group. The *t*-test showed that the L1 group performed significantly better than the control group (see Table 5) in both the post- and retention tests. Further, the achievement of the L1 group in the low ability band was found to be similar to the performance of both L1 and L2 groups in the average ability groups for post-test. Such equilibration in performance is important given the statistically significant difference noted between these groups during pre-test.

Table 5. T-test for pre-, post- and retention tests for low achievers.

	Pre-test	Post-test	Retention test
t	−0.291	−3.467	−1.963
df	61.215	67	67
sig (1-tail)	0.386	0.005	0.027

Conclusion

The teaching of mathematics is done in many countries in a language other than the natural language. There have been a lot of debates about the impact of these practices on underachievement in mathematics and the possible benefits that a switch to the natural language could generate. This quasi-experimental study shows that natural language is a factor which can lead to improvement in mathematics achievement; however, the improvement obtained is not generalised and is a function of student ability. Students who are high or average achievers in mathematics do not appear to benefit from the use of the natural language as the language of instruction. It is essentially students who are in the low ability bands who benefit the most. In this group, the impact of using Mauritian Creole on performance is substantial, and students are found to equilibrate their performance with average achievers. The benefit of using L1 for low achievers was still visible after two months. This behaviour of natural language utilisation on performance on word problems is attributed to the fact that English as a medium of instruction mainly poses a problem to low ability students in Mauritius. It is therefore highly probable that intervention in Creole helps students overcome this language barrier and produces higher gains in achievement on word problems. These observations clearly show that low achievers face difficulties with English language and consequently performed better when instructions were in Creole. These behaviours are in line with the cognitive load theory (Ong et al., 2009) and the 'bilingual views' of bilinguals (Grosjean, 1992). The findings suggest that the use of natural language in the teaching of mathematics should not become a generalised policy, but its use should be permissible as a support for low ability students in particular.

References

Abedi, J. (2001). The language factor in mathematics tests. *Applied Measurement in Education, 14*(3), 219–335.

Barton, B., Chan, R., King, C., Neville-Barton, P. & Sneddon, J. (2005). EAL Undergraduates Learning Mathematics. *International Journal of Mathematical Education in Science and Technology, 36*(7), 721–729.

Barwell, R. (2003). Patterns of attention in the interaction of a primary school mathematics. *Educational Studies in Mathematics*, *53*, 35–59.

Bernardo, A. B. I. (1999). Overcoming obstacles to understanding and solving word problems in mathematics. *Educational Psychology, 19,* 149–163.

Bernardo, A. B. I. (2001). Principle explanation and strategic schema abstraction in problem solving. *Memory & Cognition, 29,* 627–633.

Bernardo, A. B. I. & Calleja, M. O. (2005). The effects of stating problems in bilingual students' first and second languages on solving mathematical word problems. *The Journal of Genetic Psychology, 116,* 117–128.

Bose, A. & Choudhury, M. (2010). Language negotiation in a multilingual mathematics classroom: An analysis. In L. Sparrow, B. Kissane & C. Hurst (Eds), *Shaping the future of mathematics education* (Proceeding of the 33rd Conference of the Mathematics Education Research Group of Australasia, pp. 93–100). Freematle, WA: MERGA.

Clarkson, P. (1991). *Bilingualism in mathematics learning*. Geelong: Deakin University Press.

Durkin, K. & Shire, B. (1991). *Language in mathematical education: Research and practice.* Milton Keynes: Open University Press, UK.

Ellerton, N. F. & Clarkson, P. C. (1996). Language factors in mathematics teaching. In A. J. Bishop, et al. (Eds), *International handbook of mathematics education* (pp. 987–1033). Netherlands: Kluwer.

Grosjean, F. (1992). Another view of bilingualism. In R. J. Harris (Ed.). *Cognitive processing in bilinguals* (pp.51–62). Amsterdam: Elsevier Science Publishers.

Grosjean, F. (1998). Studying bilinguals: Methodological and conceptual issues. *Bilingualism: Language and Cognition*, *1*, 131–149.

Heinze, K. (2005). *The language of math*. TESOL. San Antonio, TX.

MacGregor, M. & Moore, R. (Eds) (1991). *Teaching mathematics in the multicultural classroom: A resource for teachers and teacher educators*. Melbourne: Institute of Education, The University of Melbourne.

Miller, G. A. (1956). The magical numbers seven, plus or minus two: Some limits on our capacity for processing information. *The Psychological Review*, *63*(1), 81–97.

Ong, P., Liao, V. & Alimun, R. (2009). Moderating Language and Number of Mathematical operations in the Relationship between Problem Solving scores and learning strategies. *TESOL Journal,* 1, 58–78.

Pimm, D. (1991). Communicating mathematically. In K. Durkin & B. Shire (Eds), *Language in mathematical education: Research and practice* (pp. 17–24). Philadelphia: Open University Press.

Ríordáin, M. N. & Donoghue, J. O. (2006). Mathematics teaching in a second language in Ireland. Paper presented at the British Educational Research Association Annual Conference, University of Warwick, 6–9 September.

Setati, M. (2003). Researching mathematics education and language in multilingual South Africa. *The Mathematics Educator*, *12*(2), 6–20.

Sweller, J. & Chandler, P. (1994). Why some material is difficult to learn. *Cognition and instruction*, *12*(3), 185–233.

Tuovinen, J. E. & Sweller, J. (1999). A comparison of cognitive load associated with discovery learning and worked examples. *Journal of Educational Psychology*, *91*(2), 334–341.

INCOMPREHENSIBLE WORKINGS FOLLOWED BY 'CORRECT ANSWERS'

MEREDITH BEGG
University of Melbourne
meredith.begg@unimelb.edu.au

ROBYN PIERCE
University of Melbourne
r.pierce@unimelb.edu.au

ALASDAIR MCANDREW
Victoria University
alasdair.mcandrew@vu.edu.au

Lecturers and teachers lament over the difficulties their students have writing coherent mathematical statements. Students' habitual misuse of mathematical symbols, combined with an inability to explain what mathematical steps are being taken and why, is a concerning observation. At best this represents a lack of mathematical rigour and at worst, may undermine learning. This paper reports on findings to-date of a project which investigates the use of symbols in mathematics and related sciences, as students progress from secondary school to university, with a focus on implications for secondary school teaching and assessment.

Introduction

Being proficient in mathematical communication—that is, reading, writing and conversing mathematically—is a fundamental expectation of students throughout all levels of study in mathematics. This is reinforced by the Australian Curriculum for Senior Mathematics (ACARA, n.d.) which emphasises the importance of mathematical communication in the aims listed for every senior mathematical course, for example "Mathematical Methods aims to develop the students'… capacity to communicate in a concise and systematic manner using appropriate mathematical and statistical language". Mathematics is a highly symbolic language, which increases the complexities of communication; indeed, research has shown at secondary level that the conciseness and abstraction of symbols can be a barrier to learning (MacGregor & Stacey 1997; Pierce, Stacey & Bardini 2010).

Our project 'Secondary and university mathematics: Do they speak the same language?', focuses specifically on the symbolic aspect of mathematical language in early university mathematics compared with secondary school, and how this impacts on students' confidence and capability in mathematics (For more details of the project see Bardini, Pierce & Vincent 2015).

What practitioners are encountering

As part of our study we asked educators what observations they had made of their students' experiences with mathematical symbols and notation. These data were collected during face-to-face semi-structured interviews with six experienced Victorian senior secondary teachers (ST1-ST6) and twenty one first year university mathematics and/or statistics lecturers and tutors from four Australian universities (T1-T21). Analysis of transcripts of these interviews revealed multiple recurring themes, with almost all practitioners mentioning that their students have difficulties with communication in mathematics and statistics. Specifically, students were observed to struggle to: (i) write coherent mathematical statements and (ii) provide written reasoning in their workings.

> T2: They just don't write down what they're doing, they don't explain. It's just literally, they think they just need to write a page of equations with each [of] these funny little symbols joining everything together and they'll think they're done.
> T6: ...just thinking about symbols. I mean one of the sort of things that I find ...I really want students to do is like develop mathematical reasoning and communication skills and I find that often you see what is just a blind sort of statement of symbols one after the other and one of those things that I sort of try to get students to do is to basically write less symbols, write more English. Not necessarily less symbols but write more words to connect them....

When we asked what difficulties they could recall students experiencing with symbols in mathematics, a number of examples were repeatedly highlighted. For example, T2 noted that "implications and therefores and equals—a lot of students seem to either want to use implications symbols instead of equals signs when they are writing out a page of equations, or just not use equals signs at all". T5 highlighted that the "sum of symbol... it's not only a confidence thing, but also just limited sort of intuition between seeing something and knowing what it means at all".

Whilst many of the examples were provided by university educators, the majority of the notation difficulties mentioned related to that which students encounter during secondary schooling or earlier. A list of the most frequently mentioned symbols by these educators is summarised in Table 1.

Table 1. Most commonly mentioned symbols perceived to present challenges for students.

Symbol of concern	% of educators who mentioned this (n= 27)
Using '=' incorrectly/not at all	41%
Not understanding 'Σ' symbol	33%
Using '⟹' inappropriately	26%
Confusion with derivatives symbology, $\frac{dy}{dx}$ or $f'(x)$	26%
Inflexibility changing letters/symbols in well-known formats e.g., $y = mx + c$	26%
Inverse e.g., $\sin^{-1}(x)$ or $f^{-1}(x)$ misinterpreted as reciprocals	19%
Use of Greek letters in any mathematical setting	19%
Confusion over varied use of different brackets {}, [], (), e.g., matrices, order of operations, $\binom{n}{r}$	15%
Confusion using $\bar{x}$ and μ	15%
Set notation—reading or writing e.g., $\{x: x \in Z^+\}$	15%

Which symbol for what?

One factor which is considered a point of confusion for students with symbols, is the inconsistency of their usage.

> T15: I think it's probably impractical to expect the first years to be able to adapt very easily to the [variety of] notation and so I think what we should do…there should be some standardsed set of notation and then we can say OK this is what we are going to use and this is this is how it is…the weight of evidence from what I've seen, is that the students are just tripping over unnecessarily.

Multiple symbols can have the same meaning, for example $\tan^{-1}(x)$ and $\arctan(x)$ are both commonly used interchangeably in high-stakes examinations and textbooks, (see VCE Specialist Mathematics examinations (VCAA, 2016) and Specialist mathematics Units 3 & 4: Cambridge senior mathematics Australian curriculum/VCE (Evans, Cracknell, Astruc, Lipson & Jones, 2016)). To add further confusion the Specialist Mathematics examination formulae sheet (VCAA, n.d.) acknowledges the alternative notations with column headings $\sin^{-1}$(arcsin), $\cos^{-1}$(arccos) and $\tan^{-1}$(arctan).

Similarly, a symbol may be used for multiple purposes, across multiple subjects. The equality sign, which was highlighted by over 40% of the educators in our interviews, presents difficulties even for some high achieving students at university. One only needs to consider the multiple subtle differences in meaning for this symbol to see why it might be confusing: it is an equality in equations, an assignment in statements, an operator when calculating, and also in an ICT sense, in Excel all formulae are preceded with an '=', or in popular programming languages like Python, '=' is an assignment, whereas '==' means to check for equality. Vincent, Bardini, Pierce and Pearn (2015) noted that "instead of serving a relational role between two equivalent expressions, the equals sign has been misconstrued as a cue that an answer is required, that is, an operation must be performed" (p. 3). Whilst many teachers would expect that a symbol as fundamental as the '=' sign is well understood by their students by the time they reach secondary school or university, this is not necessarily the case, as is demonstrated in Figures 1 and 2.

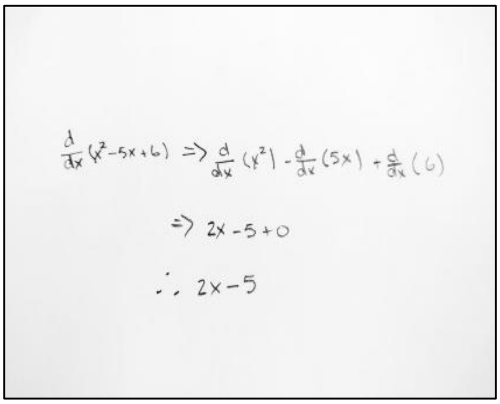


Figure 1. Student workings; not an '=' sign in sight.

Whilst students do need to come to terms with the variety of notation used in mathematics, a strong argument can be made for a scaffolded introduction to a concept using consistent notation, and explicit explanation of symbolic synonyms or conversely, multi-purposed symbols.

Across the universities and secondary schools where we spoke with educators, few mathematics departments have a formal, well-understood policy for the consistent introduction and use of symbols. A complexity of achieving such a standardised policy in the mathematics/statistics department was highlighted by one academic:

> T15: …we all have our own opinion on that. But yeah I definitely think it should be standardised and I don't think that there is a policy or if there is I don't know of it. And I think especially in this department…I think here you can really struggle I think to get everybody to agree. Because everybody grew up with their own way and mathematicians get very attached to their notation

To further complicate the task of teaching the meaning of symbols, students refer to written materials which may not be consistent either with classroom instruction, or even necessarily between sections of the same textbook. For example, ST1, a co-author of senior secondary textbooks, suggested that when textbooks are being updated, "they write small sections of new things, so you are going to have a whole exercise of questions that would have been there 10 years ago, and won't have been updated. So you're going to have mixtures of things [referring to symbols]". Additionally, online information which students may access to supplement their learning can also introduce different notation.

> T7: I will teach using letters a, b and c, but then if you go to say YouTube to watch a video and they might use alpha, beta and gamma or you know, m, n and p, and they sort of say 'well, hang on! It looks similar but you use a, b and c, and they're using m, n and p
> T19: I tell them, forget about it if you have learned in YouTube or somewhere…lots of methods, lots of ways to solve it, makes them more confused than if there is just one.

Internationally, different notation is used in mathematics for example ',' for a decimal point and $\overrightarrow{AB}$ rather than $\overrightarrow{AB}$. These students' understanding of mathematical notation needs to be specifically contemplated, with differences being explicitly highlighted to minimise the likelihood of confusion.

Are they really confused by notation?

> T9: Everyone says this happens [misuse of symbols], but you don't really know whether this is one out of a thousand, or if it's actually a real issue, and I can tell you it's a real issue. This is a significant proportion of our top students.

That students do genuinely find mathematical notation a point of difficulty is strongly supported by literature. Chirume (2013) concluded that "students fail to grasp mathematical concepts because they take the symbols themselves as the objects of mathematics rather than the ideas and processes which they represent". Our own early findings demonstrate some of these specific difficulties. Table 2 provides a summary of responses provided by 152 students in first year university, all of whom had successfully completed VCE Mathematical Methods (CAS) or equivalent (Bardini & Pierce, 2016). They were answering the question 'Explain the meaning of -1 in the following', as a written survey response (that is, not multiple-choice).

Table 2 demonstrates the significant proportion of first year university mathematics student respondents, who had experienced success in their senior secondary mathematics, but still held misconceptions in regard to the varying meanings of the '-1' in a 'template' of $\_\_^{-1}$. Whilst most understood the meaning of x^{-1}, over half of the students thought that $sin^{-1}(x)$ represented the reciprocal or $\frac{1}{\sin(x)}$ and over 17% thought similarly in terms of $f^{-1}(x)$. This does give educators reason for concern.

Table 2. 'Explain the meaning of –1 in the following'.

x^{-1}	No. Students	$sin^{-1}(x)$	No. Students	$f^{-1}(x)$, when $f(x) = 3x + 1$	No. Students
$\frac{1}{x}$	104	arcsin(x)	63	$f^{-1}(x) = \frac{(x-1)}{3}$	56
Reciprocal of x	18	$\frac{1}{\sin(x)}$ or cosec(x)	63	$\frac{1}{3x+1}$	27
Inverse or $\frac{1}{x}$	8	Reciprocal	14	Inverse of $f(x)$	43
To the power -1	9	Not $((\sin(x))^{-1}$	1	Derivative	4
Other response	10	Other response	4	Integral	4
Missing/no idea	3	Missing/no idea	7	Other response	11
n=152				Missing/no idea	7

What is in an answer?

> T9 ...so for them it's all about the answers because so much of their work is multiple choice or short answer and it's all about whether the answer is correct. 'I got 5, it doesn't really matter what I scrawled down on the piece of paper, I got 5, that was the correct answer', whereas what we want to do at university is not just make them get the correct answer, but explain how they got it, set it out properly and we're trying to teach them how to write mathematics, and that is a huge shift even for the top students.

The university lecturers we spoke with felt so strongly about this, that the majority of assignments given to students have a component of marks attributed exclusively to correct usage of mathematical notation, with the view that students will not value the feedback unless it has marks attributed to it. Further strategies are being used, such as providing to students documented guidelines for writing in mathematics with exemplar answers, in an attempt to upskill them in the early stages of their tertiary mathematics studies. Without the rigour in writing, it is considered (T6) that "these kinds of things are things that just sort of provide another impediment to their learning".

However, it is not just at university where students need to write coherent mathematics. By requiring rigour in the workings and explanations of students, we gain an insight into their misconceptions and depth of knowledge. Figure 2 shows a student's workings where they were asked to find the argument of the complex number $-1-\sqrt{3}i$. This was a first year university student, but could equally have been a student of VCE Specialist Mathematics with a question of this nature. This student has made multiple errors, from not labelling what they are doing, stating they are using tan not arctan and stating that one line of workings is equal to the next when it clearly is not. However, the final answer of $\frac{-2\pi}{3}$ is, in fact, correct. Had this question been examined via a multi-choice mechanism, the student would have been awarded full marks. However, when the teachers, lecturers and tutors we interviewed were presented with this, most indicated they would mark students down with varying severities, for workings which were plainly incorrect and all clearly provided feedback that the student needed to use mathematical notation more accurately. T9 noted,

"...this is completely incoherent and they've come out with some angles and it just doesn't make any sense... we would not be awarding marks for stuff that makes no sense and we can't even work out what their answer is, and notationally it's just rubbish". Some educators inferred that the correct answer implied an understanding of the concepts, whilst others such as ST4 thought that "It makes you wonder when you're looking at it, about the students' understanding of exactly what they are doing, whether they have memorised a step or whether they actually understand exactly what it is that they have worked out".

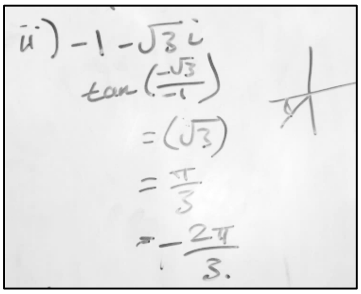


Figure 2. Multiple errors in these workings but a correct final line.

Implications for teaching and assessment

We must consider that some students in every classroom—even some of our most capable students—will bring with them misconceptions and misunderstandings regarding the use of symbols, including those symbols we would expect to be well understood from prior years of learning. One of the challenges facing educators is in the identification of these misconceptions, so that they can be directly challenged and unwound.

When reviewing student work, it is tempting to infer that a student properly understands the mathematics when the final result is a correct answer, even if there are a few 'careless' notational errors peppering the workings. However, if this were the case, those students should be able to readily produce correct workings and explanations when specifically asked to do so. What we have seen in practice, is that students do misunderstand how to read and use mathematical notation and that some students are completing workings by implementing a memorised process to achieve a final 'correct' result, with limited understanding of the underlying mathematics.

In the classroom setting, by carefully and explicitly introducing new notation and placing high value on students' correct use of mathematical notation from early mathematics instruction onwards, teachers will draw attention to clear mathematical communication. Strategies such as having students read each other's work, or asking them to correct deliberately constructed poor examples may alert them to the importance of their own clear and correct use of symbols and words to communicate mathematical thinking. Given implications for learning if students do not fully grasp the subtle nuances of the symbols and notation in mathematics, the opportunity for early intervention in this regard is one that cannot be missed.

Acknowledgements

The authors wish to thank Caroline Bardini for her development of this research project.

The authors also wish to thank the teachers, lecturers and tutors across multiple universities and schools who gave their time, knowledge and course materials for the benefit of this research.

This research has been funded by the Australian Research Council: DP150103315.

References

Australian Curriculum, Assessment and Reporting Authority (n.d.) *Australian Curriculum Senior Secondary Curriculum*. Retrieved 26/03/2017 from http://www.australiancurriculum.edu.au/seniorsecondary/mathematics/mathematical-methods/rationaleaims.

Bardini, C., Pierce, R. & Vincent, J. (2015). Contemplating symbolic literacy in first year mathematics students. In M. Marshman, V. Geiger & A. Bennison (Eds) *Mathematics education in the margins. Proceedings of the 38th annual conference of the Mathematics Education Research Group of Australasia* pp 77–84. Adelaide, Australia: MERGA.

Chirume, S. (2013). How does the use of mathematical symbols influence understanding of mathematical concepts by secondary school students? *International Journal of Social Science and Education*, *3*(1), 35–46.

Evans, M., Cracknell, N., Astruc, J., Lipson, K. & Jones, P. (2016*). Specialist mathematics. Units 3 & 4 : Cambridge senior mathematics Australian curriculum VCE*. Port Melbourne, VIC: Cambridge University Press.

MacGregor, M. & Stacey, K. (1997). Students' understanding of algebraic notation: 11–15. *Educational Studies in Mathematics, 33*(1), 1–19.

Pierce, R. & Bardini, C. (2016). Déjà vu in mathematics: what does it look like? *13th International Congress on Mathematical Education Hamburg, 24–31 July 2016*.

Pierce, R., Stacey, K. & Bardini, C. (2010) Linear functions: teaching strategies and students' conceptions associated with *y=mx+c*. *Pedagogies: An International Journal Special Issue: The teaching of Algebra, 5*(3), 202–205.

Victorian Curriculum and Assessment Authority (VCAA). (2016). *Special Mathematics Examination Paper 1*. Retrieved from: http://www.vcaa.vic.edu.au/Documents/exams/mathematics/2016/2016specmath1-w.pdf

Victorian Curriculum and Assessment Authority (VCAA). (n.d.). *Special Mathematics Examination Formulae Sheet*. Retrieved from: http://www.vcaa.vic.edu.au/Documents/exams/mathematics/specmaths-formula-w.pdf

Vincent, J., Bardini, C., Pierce, R. & Pearn, C. (2015). Misuse of the equals sign: An entrenched practice from early primary years to tertiary mathematics. *Australian Senior Mathematics Journal*, *29*(2), 31–39.

MAKING THE MOST OF MULTIPLE-CHOICE ITEMS

JOAN BURFITT
The University of Western Australia
joan.burfitt@research.uwa.edu.au

The advantages of using multiple-choice (MC) items in assessments of mathematical achievement are such that they will continue to be used. Periodically the ongoing concerns with the use of MC items and the effects of these concerns on the measures of student achievement are the subject of discussion and comment. It is possible, however, to identify the associated problems and address the concerns in new ways that can lead to improved measures without adding to the test demands on the students.

Introduction

One structural type of item used to assess mathematical achievement is the multiple-choice (MC) item. It typically contains a stem (introductory text) and a list of options from which the students choose the best answer to accurately complete the stem or answer the question posed in the stem. Selection of the correct option, the key, is typically awarded one mark and zero is given for all other selections and for missing responses. There are usually three or four incorrect options and these are often referred to as distractors. In another type of item, described as being a constructed response (CR), the student creates a response to the stimulus or question provided rather than select from the available options.

There are several reasons why MC items are more popular than other types of items in some testing situations. Students find them attractive because they know the answer is there and they do not need to find it. Furthermore, they know they can guess if the answer is not readily recognised and hence score points when the answer is unknown. Not having to show working appeals to many students though others prefer to write out their solutions and many teachers prefer to see this evidence of students' mathematical thinking. A further attraction for some students is the objective scoring process because the marker cannot be prejudiced by poor handwriting, bad grammar or personal bias.

There is anecdotal evidence of the increasing use of MC items in classroom tests. This increase may be occurring because more recent textbooks not only contain MC items for students, but also provide teachers' versions of the text which provide sets of topic tests containing MC items. The attraction of using MC items in large-scale assessments continues (e.g., NAPLAN) and there is evidence to suggest their use is becoming more extensive. Betts, Elder, Hartley and Trueman (2009) suggested that the increase could be attributed to the fact that a broader range of the curriculum can be covered with fewer items than would be needed if only CR items were used. Furthermore, MC items are easier to score and to administer to a large cohort of students with a less costly marking process and a shorter turnaround time for providing feedback.

Creation of MC items

Writing effective MC items is necessary if assessment data is to be valid and reliable. Haladyna, Downing and Rodriguez (2002) tested and validated an MC, item-writing guideline for classroom assessment and suggested their taxonomy might have uses for large-scale testing. Their recommendations included (a) placing the main idea in the stem rather than in the options, (b) keeping choices independent, (c) writing stems and options using positive language, (d) presenting choices vertically rather than horizontally, (e) ensuring all distractors are plausible, and (f) using typical errors of students to write these distractors. Haladyna et al. (2002) also recommended that writers should try to have options of approximately equal length and of similar grammatical structure to prevent clues being given to the key. They also recommended that writers avoid "all of the above" and take care when using "none of the above".

Connolly (2011) describes other factors that need to be considered when writing multiple-choice items. These include (a) content, (b) proficiency aspects of mathematical competence, (c) context and (d) literacy demands. The content and proficiency aspects should be mapped to the expectation of the students' curriculum. Contexts need to be chosen so that they are sufficiently familiar for the student so as not to confuse or distract them from the mathematical demands of the task (Cumming & Maxwell, 1999). Greenlees (2010) summarised this imperative more succinctly by describing the attempt to make questions more realistic as confounding. The students can be affected by too much unrelated information which they try to use to solve the problem rather than be specifically drawn to the relevant mathematics.

The common language of the items needs to be appropriate for the age of the test-takers and the mathematical aspects tested rather than the literacy ones. Abedi and Lord (2001) in their study of language in mathematics tests recommended that item creators (a) use active tense rather than passive tense, (b) remove infrequently used words, (c) separate conditional sentences into two sentences, (d) rewrite relative clauses, for example ask determine the number of, rather than how many components (e) change complex expressions to simple ones, and (f) use concrete descriptions rather than abstract ones, for example, replace the cost with the cost of the car.

Concerns with MC items

Even though there has been an increase in the quality and the popularity of MC items there are several issues of concern. One concern is the perceived bias for gender and the idea that males are more able than females to respond to MC items. The second concern relates to the level of thinking that can be assessed by MC items and the belief that only low-level cognition is addressed. The process by which a respondent selects a correct option is a third concern as the selection may be for the wrong reason or because the student has guessed. Either way, the score will be inflated and not an accurate reflection of the student's knowledge or achievement. It is also a concern when students are not given credit for selecting an option that is not correct but which reflects an understanding of the problem which is greater than that shown by the selection of any of the other distractors.

Gender bias

Garner and Engelhard (1999) found males significantly outperformed females in the MC items relating to proportional reasoning, geometry, number and data analysis whereas females significantly outperformed the males on MC items in algebra. However, research findings from Behuniak, Rogers and Dirir (1996), Betts et al. (2009), Bond et al. (2013), Bonner (2013), deMars (1998) and O'Neil and Brown (1998) show no significant differences in achievement between males and females on MC items. There is insufficient evidence to support this concern.

Levels of thinking

Addressing high levels of thinking in MC items has been described by McCurry (2008) as encouraging conceptual thinking and testing understanding as well as knowledge. This may include following an argument, making judgements, interpreting unfamiliar stimulus material, discriminating between concepts and analysing reasons which support conjectures.

One item from a previous investigation by the author required the students to interpret an unfamiliar situation and to understand the concept of 120% to work backwards to 100%. The item, reproduced below, was deemed to test higher order thinking and only 18% of over 130 students from Years 9 and 10 chose the correct answer.

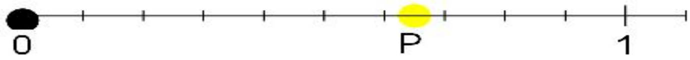
> The number of frogs in the creek in 2011 was 120% of what it was in 2010. If there were 60 frogs in 2011, then in 2010 the number of frogs must have been
> 1. 40 2. 48 3. 50 4. 72 5. 80

Higher-order thinking can also be facilitated in MC items (Haladyna et al., 2002) with the provision of visual material such as a graph or table to provide a context for the item and hence extend the thinking required to select the correct option. According to Zoumboulis (2015), item writers can be trained to write MC items that test high order thinking. This concern with the levels of thinking tested using MC items is best addressed through improving the quality of the MC items themselves.

Guessing

Guessing occurs when students have low ability relative to the item and it results in similar proportions of students selecting each option when the concept is unfamiliar. This can be seen in Figure 1 where the horizontal axis represents student ability (Person Location in logits) and the vertical axis represents the empirical probability of the student selecting a particular option. Option 5 was rejected by most students and at the lowest ability level, where the mean location is about -1.3, the proportion of students selecting each of the other options is about 0.3. This pattern is suggestive of guessing. The item (option 1. was excluded) from the author's previous study is reproduced below.

The best estimate for the value of the number at P (yellow dot) on this number line (on which 0 and 1 are labeled) is



2. 65%   3. 6.5%   4. 6.5   5. 7%

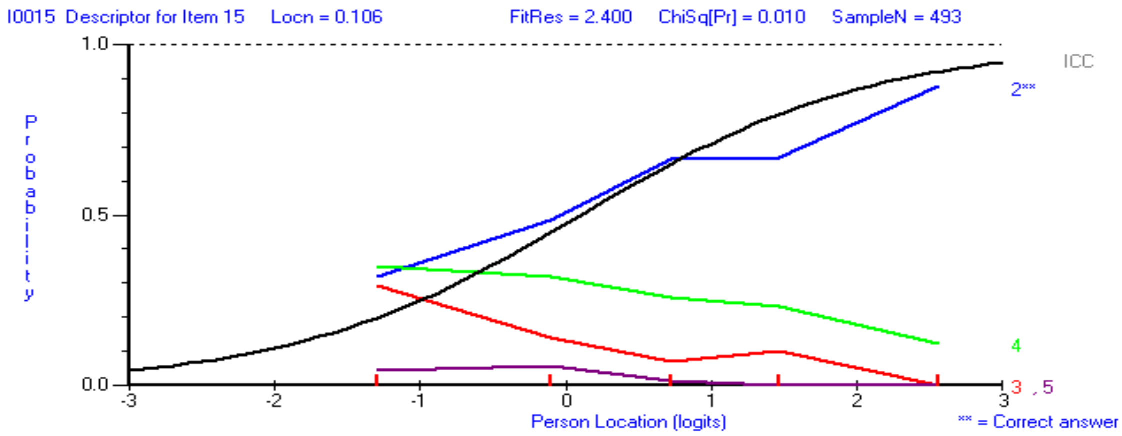


Figure 1. Distractor curves for Item 15.

The opportunity to guess is reported by Betts et al. (2009) as causing students' scores to be inflated thus decreasing test reliability and the validity of any measures or scales determined from test scores. Strategies to minimise the effects of successful guessing in MC items relate to scoring penalties, test design and post-hoc analysis. In the conventional and most popular method of scoring one mark is allocated to the selection of the key and zero marks allocated for any other selections. With this type of scoring it is advantageous to guess, yet students will still omit items. Other scoring methods involve penalties and marks are deducted for incorrect or missing responses. A common technique, known as *formula scoring* is summarised in the formula given below (Lindquist & Hoover, 2015, p. 16).

$S = R - \frac{W}{n-1}$ where S = corrected score, R = number of correct answers,
W = number of incorrect options chosen, n = total number options

Using different scoring techniques is at times associated with undesirable consequences. There may be extra demands placed on test-takers and these include longer times needed to complete tests, increases in taking risks when deciding to omit or complete items and the need to consider more strategies for test completion. All of these can influence test reliability by introducing unrelated factors which can increase the error in measuring the intended construct. There is no evidence to suggest that such penalties are used in large-scale tests within Australia.

Another strategy designed to reduce the level of guessing is to provide a test that is not too hard for the test-taker. An algorithm can be used to detect student success on a limited number of questions and according to those responses change the student's pathway through the test (ACARA, 2014). This process may be described as "computer adaptive testing" or "tailored testing": the test is tailored to the student's ability. In theory, students responding to items more commensurate with their own ability are less likely to guess. When determining the level of difficulty of the MC test items using Rasch Measurement Theory, allowances can be made to adjust the item difficulty according to the probabilities of the students guessing behaviours. According to Andrich, Marais and Humphry (2015), such adjustments can lead to the formation of more accurate scales of achievement for the test-takers.

Partial knowledge

A major criticism of using MC items to assess student understanding is that they are usually marked right or wrong and any partial knowledge that the student might have about the item's content is not rewarded with a score. In the author's previous investigation the existence of partial knowledge was indicated when the proportion of students selecting the incorrect option was higher than that predicted by the analytical model. This can be seen by the curve for Option 1 as shown in Figure 2. The item was as follows:

> The price of a maths text book has risen 25% to a new cost of \$100. The old price must have been:
> 1. \$75 2. \$80 3. \$100 4. \$125 5. None of the above

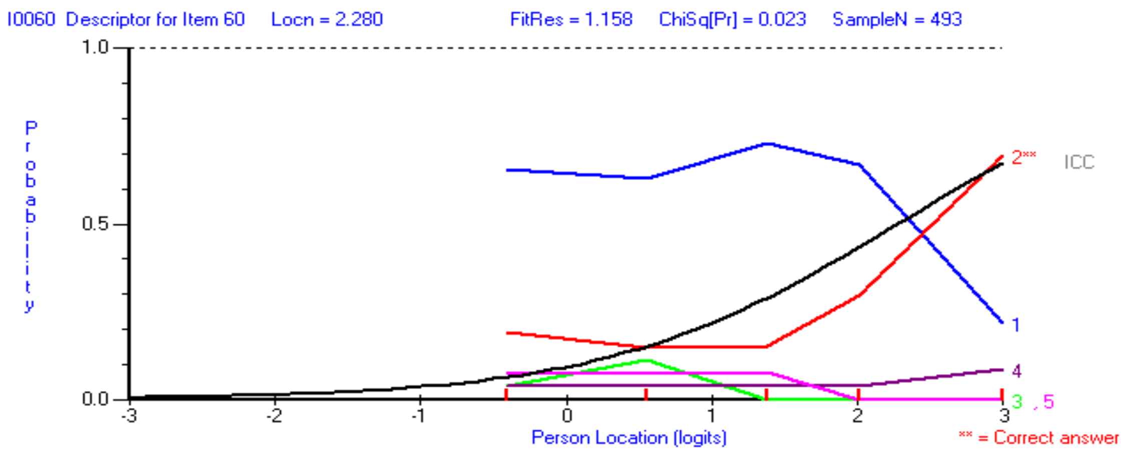


Figure 2. Distractor curves for Item 60.

As expected, very few students chose any of the last three options as they would have known that the answer had to be less than \$100. The proportion of students selecting \$75 is high throughout the low and average ability range and the associated thinking is that \$75 + 25% = \$100. This incorrect selection for this item indicated students were thinking of fixed or absolute change rather than proportional change. Rewarding partial knowledge can be achieved by scoring the elimination of distractors, by asking students to rate the options in order of preference, by asking students how confident they feel about the answer they have chosen or by scoring the distractors. Having partial knowledge of a concept can be considered as being on the way to developing a full understanding of the concept and it is hypothesised that MC items can be written to allow students to demonstrate their partial knowledge and be rewarded for such.

Current study

One of the aims of the current study is to demonstrate that it is possible to improve the amount of information about students that can be collected from MC items by giving credit for partial knowledge. Sixty MC items were developed using the guidelines described previously. The items were mostly written by the author and the remainder were adopted and adapted from research studies. For each item there was one distractor written to reflect any partial knowledge that the student might have, as well as the key. The other two distractors were designed to provide less information about student learning.

The context chosen for the study is Proportional Reasoning which has been described by Siemon, Bleckly and Neal (2011, p. 22) as a key concept for students in early secondary, "without which, students' progress in mathematics will be seriously impacted". Items were designed to test the skills and understanding necessary for the development of sound proportional reasoning, the ability to solve problems when the relationship between quantities or variables is proportional. The skills and understanding that Year 8 students cover in their study of the Western Australian curriculum include understanding and manipulating ratios, rates, fractions, decimals and percentages.

The test items were written at a level deemed to reflect the standard to be achieved by Year 8 students. There were three review processes used to check the items before the test was administered to the students. Firstly the items were analysed by five experienced teachers of Year 8 mathematics and the feedback on the difficulty of the item, the appropriateness of the language used and the mapping to the curriculum was provided. Secondly two teachers with expertise in assessment review analysed the items in respect of the distractors to be awarded partial knowledge. In the third review cognitive interviews were conducted with 10 Year 9 students with the aim of checking that the items were consistently interpreted as intended by the author. Following these reviews, some items were edited with diagrams added and language simplified.

Results

In November 2016 the test was given to over 1200 students and in the table of results for the first 10 items (Table 1), the percentages of students who are correct and who have selected the options designed to be awarded worth partial credit (PC option) are highlighted. For these ten items, the percentage of students selecting either the key or the PC option is between 62% and 90%. This is to be expected given that the test was done at the end of the year and students should have covered the test content

Key	Correct answer
PC	Option created for partial credit

OPTIONS

Item	1	2	3	4
1	48%	8%	40%	4%
2	24%	22%	38%	17%
3	13%	11%	10%	66%
4	16%	6%	73%	5%
5	65%	7%	13%	15%
6	17%	35%	28%	20%
7	24%	36%	26%	14%
8	8%	78%	12%	2%
9	56%	18%	7%	19%
10	12%	46%	28%	13%

Table 1. Proportions of students selecting each option.

For Item 1 the students (40%) selecting the PC option may have understood the relative change in size but have not processed the indirect nature of the proportion. Their response is arguably better than the one where the change is a factor of 2.

Item 1
If the number of people sharing the cost of building a Cat Refuge were to quadruple (multiply by 4), then the amount of money that each person needs to give will
a. reduce to a quarter of the original amount
b. reduce to a half of the original amount
c. increase to four times the original amount
d. increase to double the original amount

For the PC option in Item 4 it is assumed that the students thought that 100 km/h was equivalent to 1 km/min and hence used the speed incorrectly. Students might have determined that the answer had to be greater than 100 km but there is no other justification for the selection of the last option. The proportion of the responses, 73% for the key and 16% for the PC option, are justifiable.

Item 4
Milly drives her delivery truck from the farm to the depot at an average of 100 km per hour.
The journey takes 90 minutes. What is the distance from the farm to the depot?
a. 90 km
b. 100 km
c. 150 km
d. 175 km

Item 7 was the only item in which the PC option and not the key was selected by the highest proportion of the students. The selection of 4% to represent the percentage increase was almost as high as the selection of the key. This result is difficult to justify but a possible explanation involves the 'fourness' of the difference between 1.2 and 1.6.

Item 7
Oil production was forecast to be 1.2 million barrels per day.
Instead, it reached 1.6 million barrels per day.

This increase in what was forecast is closest to
a. 4%
b. 25%
c. 30%
d. 40%

For Item 9, the key was selected by 56% of the students and the PC option was selected by 18% but option d. was selected by 19%. It was not expected that the students would fail to see the two errors in this option, namely that $800 was not decreased by more than $100 and that $66 + $44 is not $100. The representation of 44% as $44 was identified as an additive error and a misconception that could represent a stage in the learning of percentage increase.

Item 9
The **correct** answer in a student's homework was $744.
The question could have been
a. Increase $600 by 24%
b. Increase $700 by 44%
c. Decrease $700 by $44
d. Decrease $800 by $166

Conclusion

Creating MC items in which distractors can be scored requires a qualitative approach as well as an analysis of the numerical results. To justify awarding partial credit to one of the distractors requires knowledge of the development of student learning, consideration of the conceptual understanding of the content, and an awareness of the errors often made and of the misconceptions commonly held. The numerical analysis can reveal unexpected misconceptions as well as provide evidence to support the existence of the students' partial knowledge. The confirmation or otherwise of the planned partial knowledge, or the recognition of it when not expected can provide the classroom teacher with valuable information for the identification of tasks that would support the development of greater conceptual understanding.

It is possible to apply different scores to the students' responses to the items and apply Rasch measurement theory to the results. This would allow a comparison of the measurement scales for both item difficulty and student achievement under different scoring methods. One such scoring method could include awarding some credit to the students who select the PC option but a greater credit for those selecting the key.

Acknowledgements

The support of the author's supervisors, Chapple Professor David Andrich and Professor Peter Merrotsy, in previewing this article has been invaluable. The author also wishes to acknowledge the support of the Australian Government through an Australian Government Research Training Program Scholarship.

References

Abedi, J. & Lord, C. (2001). The language factor in mathematics tests. *Applied Measurement in Education, 14*(3), 219–234.

Andrich, D., Marais, I. & Humphrey, S. (2015). Controlling guessing bias in the dichotomous Rasch model applied to a large-scale, vertically scaled testing program. *Educational and Psychological Measurement*. 1–24. http://dx.doi.org/10.1177/0013164415594202

Australian Curriculum, Assessment and Reporting Authority (ACARA), 2014. *Tailored design study 2013: Summary research report*. Sydney: ACARA.

Behuniak, P., Rogers, J. B. & Dirir, M. A. (1996). Item function characteristics and dimensionality for alternative response formats in mathematics. *Applied Measurement in Education, 9*(3), 257–275.

Betts, L. R., Elder, T. J., Hartley, J. & Trueman, M. (2009). Does correction for guessing reduce students' performance in multiple-choice examinations? Yes? No? Sometimes? *Assessment and Evaluation in Higher Education, 34*(1), 1–15. http://dx.doi/org/10.1080/02602930701773091

Bond, A. E., Bodger, O., Skibinski, D. O. F., Jones, D. H., Restall, C. J., Dudley, E. & van Keulen, G. (2013). Negatively-marked MCQ assessments that reward partial knowledge do not introduce gender bias yet increase student performance and satisfaction and reduce anxiety. *PLoS ONE 8*(2): e55956. http://dx.doi.org/10.1371/journal.pone.0055956

Bonner, S. M. (2013). Mathematics strategy use in solving test items in varied formats. *The Journal of Experimental Education, 81*(3), 409–428. http://dx.doi.org/10.1080/00220973.2012.727886

Connolly, N. (2011). Refining the NAPLAN numeracy construct. In J. Clarke, B. Kissane, J. Mousley, T. Spencer & S. Thornton (Eds), *Mathematics: Traditions and [New] Practices*. Proceedings of the AAMT–MERGA conference (Vol. 2, pp. 908–916). Alice Springs: AAMT–MERGA.

Cumming, J. J. & Maxwell, G. S. (1999). Contextualising authentic assessment. *Assessment in Education: Principles, Policy & Practice, 6*(2), 177–194. http://dx.doi.org/10.1080/09695949992865

deMars, C. E. (1998). Gender differences in mathematics and science on a high school proficiency exam: The role of response format. *Applied Measurement in Education, 11*(3), 279–299.

Garner, M. & Engelhard, G. (1999). Gender differences in performance on multiple-choice and constructed response mathematics items. *Applied Measurement in Education 12*(1), 29–51.

Greenlees, J. (2010). The terminology of mathematics asessment. In L. Sparrow, B. Kissane & C. Hurst (Eds), *Shaping the Future of Mathematics Education*. Proceedings of the 33rd annual conference of the Mathematics Education Research Group of Australasia (Vol. 1, pp. 218–224). Fremantle: MERGA

Haladyna, T. M., Downing, S. M. & Rodriguez, M. C. (2002). A review of multiple-choice item-writing guidelines for classroom assessment. *Applied Measurement in Education, 15*(3), 309–333. http://dx.doi/org/10.1207/S15324818AME1503_5

Lindquist, E. F. & Hoover, H. D. (2015). Some notes on corrections for guessing and related problems. *Educational Measurement: Issues and Practice 34*(2), 15–19.

McCurry, D. (2008). A high degree of difficulty: Assessing higher-order thinking in cross-curricular tests with multiple choice items. Paper presented at the Conference of the International Association for Educational Assessment, Cambridge 2008 http://www.iaea.info/documents/paper_2b7130e3.pdf

O'Neil, H. F. & Brown, R. S. (1998). Differential effects of question formats in math assessment on metacognition and affect. *Applied Measurement in Education 11*(4), 331–351.

Siemon, D, Bleckly, J. & Neal, D. (2012). Working with the big ideas in number and the Australian curriculum: Mathematics. In B. Atweh, M. Goos, R. Jorgensen & D. Siemon (Eds). *Engaging the Australian curriculum mathematics: Perspectives from the field*. Online publication: Mathematics Education Research Group of Australasia. Retrieved from http://www.merga.net.au

Zoumboulis, S. (2015). Multiple choice: Multiple pathways for learning. Session presented at the 25th Biennial Conference of the Australian Association of Mathematics Teachers.

TECHNOLOGY IN EXTERNAL EXAMINATIONS IN QUEENSLAND

THEO CLARK

Queensland Curriculum and Assessment Authority

theo.clark@qcaa.qld.edu.au

After more than 40 years of 100% school-based assessment, Queensland will introduce the new Queensland Certificate of Education (QCE) system commencing in 2019, which includes a common external assessment of senior subjects. This paper outlines the research, consultation and rationale that underpins the position on technology allowed in the external examinations for Mathematical Methods and Specialist Mathematics in Queensland, specifically, a supervised examination in two papers: technology-free and technology-active (handheld non-CAS graphics calculator).

Background

New assessment requirements

Queensland is one of the few educational jurisdictions in the world with a system of 100% school-based assessment in senior secondary schooling, with the majority having some form of external assessment of senior subjects (McCurry, 2013). In 2019 this changes, with the Queensland Government introducing the new Queensland Certificate of Education (QCE) system (starting with students entering Year 11). In the new system, subject results will be based on a student's achievement in three school-based assessments and one external assessment set and marked by the Queensland Curriculum and Assessment Authority (QCAA). External assessment results will contribute 25% towards a student's result in most subjects. In mathematics and science subjects, it will generally contribute 50% (Department of Education and Training, 2016).

This paper outlines issues related to the use of technology in the external assessment for Mathematical Methods and Specialist Mathematics,[1] and the research, consultation and rationale that underpins the final position for these new Queensland syllabuses.

Brief literature review

Pedagogical approaches that use the power of technologies in a variety of forms and contexts, that assist students to make the bridges between real-life modelling, symbolic

1 These are newly developed Queensland senior secondary mathematics subjects, which are developed from the equivalent senior secondary Australian Curriculum.

and visual representations of mathematical phenomena, are encouraged in the mathematics teaching profession. Technology use can and does involve multiple forms, nevertheless, calculators are the technology most commonly used in mathematics lessons at all levels (Norton & O'Connor, 2016).

Reliance on calculators is often raised as a concern, with some considering their overuse to be a problematic trend; arguing students do not gain fluency with basic facts and processes because they rely on calculators for simple numerical calculations (Norton & O'Connor, 2016). Hattie (2009) examined the research on the effectiveness of calculator use in learning. Overall, the presence of calculators in mathematics has a low positive effect. Their use is supported for computation, practice, checking work and aiding conceptual understanding—when used purposefully in teaching and learning.

A stronger positive effect is found when examining the use of graphics[2] calculators. Ellington (2006) conducted a meta-analysis on forty-two studies comparing the learning and achievement for students with access to graphics calculators to students who did not have access to graphics calculators. The conclusion from the meta-analysis was unambiguous in its support for their use, stating that "graphing calculators should be an integral part of the study of mathematics" and "[t]here were no circumstances under which the students taught without calculators performed better than the students with access to calculators" (Ellington, 2006, p. 24). A more recent review, published by the National Council of Teachers of Mathematics, reached the same conclusion. The research consistently shows that the use of calculators in mathematics does not lead to negative outcomes in development of skills and procedural fluency, and has a positive impact on understanding concepts and student disposition (Ronau et al., 2011).[3]

Jurisdictional scan

An environmental scan of the external assessment requirements of Australian and selected international jurisdictions was undertaken by reviewing documentation provided on the relevant jurisdictional websites. The majority have two sections or papers for their external assessment, often with one technology-free and one technology-active. The scan is summarised in table 1 for the equivalent of Mathematics B (Mathematical Methods).

[2] Note that graphics, graphing and graphics display are often used interchangeably, when referring to large screen calculators that allow for graphs, tables, matrices, statistical displays and other complex data and mathematical operations. This paper refers to graphics calculators unless in quotations.

[3] Based on a synthesis of nearly 200 research studies, dating from 1976 to 2009.

Table 1. External assessment in other jurisdictions.

Jurisdiction	External assessment information
Victoria	66% external assessment – examination Paper 1 (technology-free): 22% • 60 minutes • No technology or notes of any kind permitted Paper 2 (technology-active): 44% • 120 minutes • One bound reference allowed
New South Wales	50% external assessment – examination • 180 minutes • Section 1 Multi-choice (10 marks) • Section 2 Short-response questions (90 marks)
South Australia	30% external assessment – examination Examination (technology-active) • 120 minutes • Two unfolded A4 sheets (four sides) of handwritten notes
Western Australia	50% external assessment – examination Paper 1 (technology-free): 35% • 50 minutes Paper 2 (technology-active): 65% • 100 minutes • Notes on two unfolded sheets of A4 paper
International Baccalaureate (IB)	80% external assessment – examination Paper 1 (technology-free): 40% • 90 minutes (Mathematics SL) Paper 2 (technology-active): 40% • 90 minutes (Mathematics SL)
Hong Kong	100% external assessment Paper 1: 65% • 135 minutes • Conventional questions Paper 2: 35% • 75 minutes • Multiple-choice questions

A scan of the technology allowed in external assessment in Australian and selected international jurisdictions, by reviewing information provided on the relevant jurisdictional websites or contacting the jurisdiction directly, was also undertaken. The majority require some form of graphics calculator.[4] The scan is summarised in table 2.

[4] As noted by Kissane, McConney and Ho (2015), there is no clear consensus internationally about the type of technology allowed in external examinations, and the picture of what is allowed between and within nations is complex and changing. However, the general trend is for the inclusion of more technology, not less.

Table 2. Functionality of calculators allowed in external assessment in other systems.

Jurisdiction	**Scientific calculator**	**Graphics calculator (non-CAS)**	**Graphics calculator (CAS)**
Austria	Y	Y	Y
China	N	N	N
Denmark	Y	Y	Y
Finland	Y	Y	Y
Germany[5]	Y	Y	Y
Hong Kong	Y	N	N
India	N	N	N
International Baccalaureate	Y	Y	N
Israel	Y	N	N
Netherlands	Y	Y	N
New South Wales	Y	N	N
New Zealand[6]	Y	Y	N
Northern Territory	Y	Y	N
Norway	Y	Y	Y
Singapore	Y	Y	N
South Australia	Y	Y	N
Switzerland	Y	Y	Y
Tasmania	Y	Y	Y
United Kingdom	Y	Y	N
United States	Y	Y	Y
Victoria	Y	Y	Y
Western Australia	Y	Y	Y

Consultation

Consultation forum

Key stakeholders—teachers, a mathematics education academic and the Victorian Curriculum and Assessment Authority (VCAA) Mathematics manager—met on 23 February 2016 to discuss calculator (and other digital technology) functionality within external assessment contexts. To set the scene, the VCAA Mathematics manager outlined Victoria's journey (see Leigh-Lancaster, 2010), noting they only moved to using computer algebra system (CAS) calculators after extensive trials and decades of experience in implementing state-wide external examinations. Participants then provided their views on four possible options for technology in external examinations: graphics calculator with CAS functionality, graphics calculator without CAS functionality, scientific calculator only, or other options such as BYO devices or school supplied laptops.

A variety of issues and positions were raised. A common theme was schools' concerns for the cost of purchasing or upgrading to a different type of technology than

[5] Summarising a jurisdiction such as Germany is, overall, not completely possible. Similar to Australia, Germany has states which all have their own independent senior curriculum and assessment (Kissane, McConney & Ho, 2015).
[6] Graphics calculators (CAS) are allowed in higher lever statistics and calculus courses.

that which they were currently using,[7] as well as the cost and time associated with teacher professional development to learn something new. The incoming change to curriculum and assessment will account for a significant proportion of school budgets and teacher professional learning, planning and preparation time, as it is.

When participants were asked if they could accept a compromise, different to their first preference, scientific calculator only was the most popular second choice (but no-one's first choice). The major concern raised about using a scientific calculator only, is it essentially endorses the primacy of a technology that many schools have moved beyond. However, it was pointed out by the university academic and other participants that the significant majority of first-year university mathematics, physical science and engineering courses, only allow scientific calculators in their examinations. They use more powerful or professional software such as *MATLAB* or *R* in 'laboratory' components.

Survey of current technology used in senior secondary mathematics in Queensland

During the syllabus development and consultation process in 2016, QCAA surveyed Mathematics Heads of Department[8] about the technology they allowed in their school-based examinations for Prevocational Mathematics and Mathematics A, B and C.[9] The results are provided below.

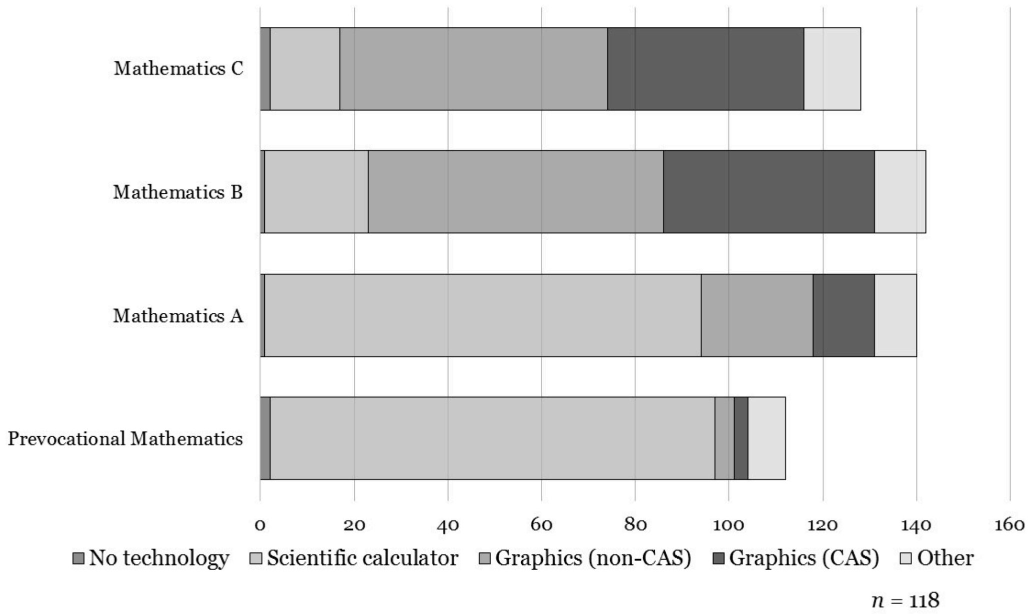


Figure 1. Technology use in mathematics examinations.

---

7 Kissane, McConney and Ho (2015, p. iv) investigated this and found that when adjusting for CPI changes graphics calculators (CAS and non-CAS) are no more expensive that scientific calculators in the 1970s or graphics calculators in the 1990s. Further, calculator manufacturers also offer emulators for Windows, Mac OS X, Android and iOS at prices ranging from free to $50.

8 Via an email sent to the QCAA's primary contact for senior secondary at all schools (the 'school moderator').

9 These subjects are broadly aligned with, and will be respectively replaced by Essential Mathematics, General Mathematics, Mathematical Methods and Specialist Mathematics.

118 responses were recorded, which represents more than 20% of the secondary schools in Queensland.[10] For Mathematics B and C, the significant majority use graphics calculators (both non-CAS and CAS, with more using non-CAS). This is consistent with a previous survey of technology use in mathematics classes in Queensland schools (Goos & Bennison, 2008, p. 120).

CAS specialist consultation

QCAA invited CAS specialists, including those with expertise in Texas Instruments, Casio and HP calculators, to provide expert advice on the advantages CAS functionality can bring to students.

One specialist was asked to detail the advantages CAS provided over a non-CAS graphics calculator for each question in the 2016 external assessment for Mathematics B that QCAA was trialling with Year 11s. They found that CAS functionality could, overall, give advantage in some questions due to function notation, speed, exact numerical arithmetic functionality, trial and error testing by ‘button pressing’ and solving trigonometric equations exactly. This advantage would most likely benefit high achieving students who know how to use CAS calculators effectively, which would bring into question the assessment’s validity (Pantzare, 2012). They did note that the use of cues in the paper minimised the CAS advantage as it specifically requested an algebraic approach.

As part of the consultation the other specialists worked through the VCAA Mathematical Methods (CAS) external assessment paper, demonstrating the approach to solving questions using their CAS model of calculator. Even more so than with the QCAA trial paper for Mathematics B (Year 11), for the VCAA paper (Year 12), CAS provided significant extra functionality beyond a graphics calculator (non-CAS), specifically when manipulating mathematical expressions symbolically and evaluating functions with symbolic arguments.

In the experience of the specialists, many of the reasons teachers and students prefer the CAS models have little, if anything, to do with CAS functionality. It is the other features of the models, such as bigger and higher quality touch screens, data logging applications, and more aesthetically pleasing industrial design, that teachers and students prefer. As with most technology, the latest models are easier to use irrespective of CAS functionality.

The specialists viewed their CAS models as excellent teaching tools, but acknowledged the challenge of writing examination questions for them. They also noted that in their experience university lecturers tend not to like CAS calculators (or indeed, any graphics calculator) and that this is unlikely to change.

Out of this process came the following recommendations for the development of future external assessment:

- QCAA should develop a two-tiered exam system, that is, a technology-free paper and a technology-active paper.
- A technology-active paper should focus on ‘non-traditional questions’—modelling and problem solving, and model creation.

[10] The sum totals for each subject are not 118 and vary from subject to subject, as respondents were able to select more than one response for each technology type.

- QCAA should scrutinise the external assessment with a variety of devices, checking the differences between functionality to minimise the risk of inequality.

Discussion

Based on consideration of the research and consultation outlined above, the final position reached by QCAA, the syllabus Expert Writing Teams (EWT) and Mathematics Learning Area Reference Group (LARG)[11] for the external assessment in Mathematical Methods and Specialist Mathematics is a supervised examination in two papers: technology-free (Paper 1) and technology-active (Paper 2), with access to an approved handheld graphics calculator (non-CAS) for Paper 2. The following section briefly discusses the rationale for this position.

Proposal for two papers

As noted previously, it is normal practice for high school students to use calculators in their mathematics studies. There is evidence that calculator use supports conceptual understanding and improves student motivation. The combination of this research evidence and current practice makes the use of some form of graphics calculator technology in mathematics teaching and learning essential. Given the external assessment will be worth 50% of the subject result, the style of assessment and assessment conditions will have significant influence on classroom teaching and learning. If this technology is a necessary part of teaching and learning in senior mathematics, it necessarily must be a part of the external assessment. However, as is recognised in the literature, and raised by academics and industry bodies, students' foundational knowledge and skills are seen to be declining, and an over-reliance on technology is viewed by some to be a contributing factor.

Therefore, there are two key but paradoxical issues to consider when it comes to the use of technology in mathematics:

1. technology is an essential part of mathematics learning in the 21st century;
2. students lack foundational knowledge and skills in mathematics, and an over-reliance on technology may be partially to blame.

The external assessment practices, in the clear majority of jurisdictions reviewed, manage this dichotomy through the use of two papers, one with technology and one without. In this way technology is recognised for its ability to enhance conceptual understanding, but an over-reliance on technology is mitigated.

The focus of these two papers, as outlined in the following diagram, will be different and serve two equally valued educational aspects of mathematics. The technology-active paper will have a greater focus on conceptual understanding and the application of mathematics in context. The technology-free paper will have a greater focus on foundational knowledge and procedural skills.

[11] Syllabuses were developed by EWTs, and the suite of syllabuses in a learning area was overseen by a LARG, each consisting of a QCAA officer, practising teachers and academics. The Mathematics EWTs and LARG were consulted about the use of technology throughout the development of the syllabuses.

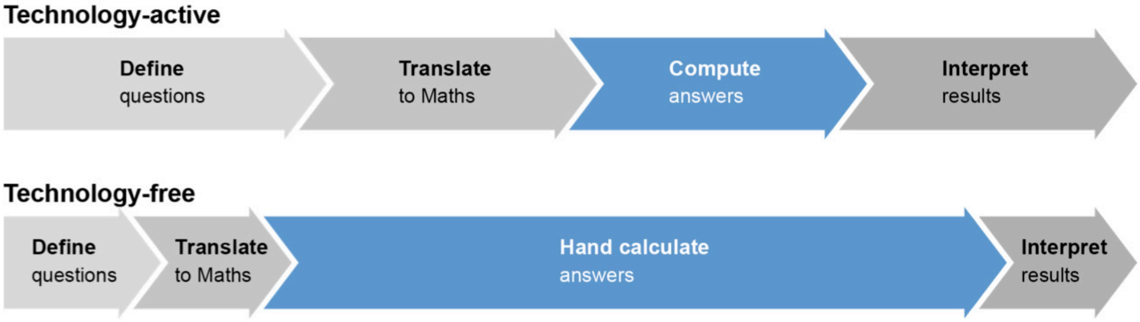


Figure 2. Focus of technology-active compared to technology-free[12].

Not all the technology-active questions will require students to use technology. They will be expected to make decisions about when and how to use the technology to respond to a question. There will be questions for which the use of technology is essential, questions for which it is unnecessary, and questions for which either option is appropriate. Some questions in the technology-free paper will require numerical calculations and some will not, such as defining mathematical terms, some geometric proofs and transformations of graphs.

Graphics calculator (non-CAS) only

The next consideration is the functionality of the technology that will be allowed in the external assessment. Four broad options for technology use in examinations were considered:

1. handheld scientific calculator only,
2. handheld graphics calculator (non-CAS),
3. handheld graphics calculator (CAS), and
4. other devices such as iPads and laptops with software such as *GeoGebra*, *MATLAB*, *Mathematica* or the calculator emulators such as those provided by Texas Instruments, Casio and HP.

Kissane (2000) discusses technology change in school mathematics curriculum and assessment and asks us to consider the speed at which different technology is implemented. He specifically cites the need to ensure adequate professional development for teachers—that it is unrealistic to expect too much change and that too much change will likely be unproductive. The Western Australian School Curriculum and Standards Authority recently published a commissioned research report on the use of technology in mathematics education and the related use of hand-held calculators with or without CAS in assessment. The report notes that teachers are supportive of technology and developing their effective use of it, but have frequently been without adequate support and saddled with unrealistic expectations (Kissane, McConney & Ho, 2015).

For Queensland teachers, the change in curriculum with the new mathematics syllabuses and the associated new assessment requirements are significant. It is essential, therefore, that where possible extraneous changes are kept to a minimum. In the current syllabuses for Mathematics B and Mathematics C, a graphics calculator (or

[12] Adapted from Computerbasedmath.org 2016, Computer-Based Math for Policymakers, http://computerbasedmath.org/apply/policymakers.html

equivalent) is the minimum requirement for technology use (Queensland Studies Authority, 2008, p. 9). The survey of Mathematics Heads of Department (Figure 1), as well as a number of other informal surveys of teachers at QCAA meetings, indicate that the majority use handheld graphics calculators (non-CAS) in their Mathematics B and C classes.[13]

All Mathematics B and C teachers are expected to have competency with the core functionality of graphics calculators, but not with the extra functionality of CAS. When compared to any other option, the introduction of CAS calculators through external assessment would impose a 'hidden curriculum' on significantly more senior mathematics teachers in Queensland. As such, option 2, graphics calculator without CAS functionality, is the proposed option for Mathematical Methods and Specialist Mathematics.[14]

Thinking about the future

The proposed technology-active paper will initially be limited to handheld graphics calculators (non-CAS). Given the high-stakes nature of the external assessment, it is essential that QCAA develop an examination that is fair and equitable in its design and administration.

At this stage of curriculum development and implementation in senior secondary mathematics in Queensland, moving to CAS and/or other software/hardware would be 'a bridge too far' for many schools and teachers. However, the technology allowed in the technology-active paper should be revised periodically. Irrespective of the technology initially approved for use in the external assessment, a regular review process will be undertaken to monitor and evaluate technology developments.

Acknowledgements

This paper was based on and adapted from an earlier version that was published on the QCAA website during consultation on syllabus drafts. The author acknowledges Emeritus Professor Barry Kissane for feedback on the earlier paper, Dr David Leigh-Lancaster (VCAA) for ongoing consultation and enthusiasm about the use of technology in mathematics, the teachers and calculator experts who gave up their time to consult with QCAA, and the QCAA senior mathematics curriculum team, in particular Sue Jones Luan Phan.

References

Department of Education and Training. (2016). *Queensland senior assessment and tertiary entrance systems*. Retrieved 12 May 2016 from http://deta.qld.gov.au/initiatives/senior-assessment-and-tertiary-entrance.html

Ellington, A. (2006). The effects of non-CAS graphing calculators on student achievement and attitude levels in mathematics: A meta-analysis. *School Science and Mathematics*, 106(1), 16–26. http://doi.org/10.1111/j.1949-8594.2006.tb18067.x

Goos, M. & Bennison, A. (2008). Surveying the technology landscape: Teachers' use of technology in secondary mathematics classrooms. *Mathematics Education Research Journal*, 20(3), 102–130. http://doi.org/10.1007/BF03217532

[13] There are also a significant number (but not a majority) that are using handheld graphics calculators (CAS), or alternative graphics technology.

[14] Essential Mathematics and General Mathematics specify scientific calculator only.

Hattie, J. (2009). *Visible learning: A synthesis of over 800 meta-analyses relating to achievement*. London: Routledge.

Kissane, B. (2000). New calculator technologies and examinations. *Proceedings of the Fifth Asian Technology Conference in Mathematics*, 365–374.

Kissane, B., McConney, A. & Ho, K. F. (2015). *Review of the use of technology in mathematics education*. WA: School Curriculum and Standards Authority. Retrieved from http://www.scsa.wa.edu.au/__data/assets/pdf_file/0020/152633/SCSA-Final-Report-30-Nov-2015_Review-of-the-use-of-technology-in-Mathematics-education.PDF

Leigh-Lancaster, D. (2010). The case of technology in senior secondary mathematics: Curriculum and assessment congruence? *ACER research conference 2010*, 43–46. Retrieved from http://research.acer.edu.au/research_conference/RC2010/16august/12/

McCurry, D. (2013). *Overview of senior assessment and tertiary entrance in Australia and other countries*. Retrieved from http://research.acer.edu.au/cgi/viewcontent.cgi?article=1004&context=qld_review

Norton, S. & O'Connor, B. R. (2016). *Literature review for senior syllabus revisions: Mathematics*. Brisbane: Queensland Curriculum and Assessment Authority.

Pantzare, A. L. (2012). Students' use of CAS calculators – effects on the trustworthiness and fairness of mathematics assessments. *International Journal of Mathematical Education in Science and Technology*, 43(7), 843–861. http://doi.org/10.1080/0020739X.2012.662289

Queensland Studies Authority. (2008). *Mathematics B 2008 senior syllabus*. Brisbane: Queensland Studies Authority. Retrieved from https://www.qcaa.qld.edu.au/downloads/senior/snr_maths_b_08_syll.pdf

Ronau, R., Rakes, C., Bush, S., Driskell, S., Niess, M. & Pugalee, D. (2011). Using calculators for teaching and learning mathematics. *National Council of Teachers of Mathematics – Technology Research Brief*. National Council of Teachers of Mathematics. Retrieved from http://www.nctm.org/Research-and-Advocacy/Research-Brief-and-Clips/Calculator-Use

HOW MATHEMATICS TEACHERS CAN EXPLAIN MULTIPLICATION AND DIVISION IN THE MANNER OF RENÉ DESCARTES AND ISAAC NEWTON

JONATHAN J. CRABTREE

research@jonathancrabtree.com

If "the purpose of life is to contribute in some way to making things better", how might we make mathematics better?[15] Teachers often explain multiplication and division with repeated addition and subtraction. Yet such approaches do not extend beyond the positive integers. By contrast, the ideas of René Descartes and Isaac Newton on multiplication and division can be extended from the naturals to the reals. So, I reveal how, if they were alive today, they might explain multiplication and division visually in ways seldom seen in western mathematics curriculums.

Background

The 'Cartesian plane', named after Descartes, has both a horizontal x-axis and a vertical y-axis, that intersect at zero. The plane thus has four quadrants, the first of which is often used for an area model of multiplication. For example, a rectangle drawn with a base of 8 and a height of 3 will cover 24 'square units' on the Cartesian plane, as shown as Figure 1. A rectangle with base of 3 and height of 8 also covers 24 square units.

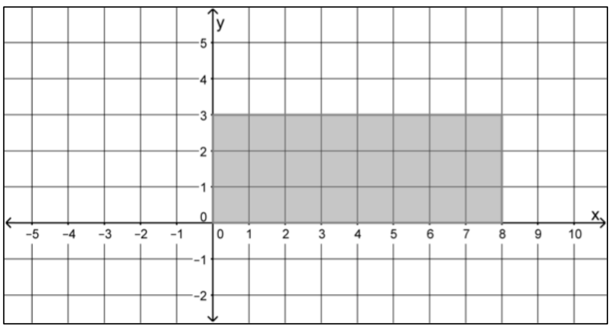


Figure 1. An area model depicting 8 × 3 on the Cartesian plane.

The 'Cartesian product', also named after Descartes, consists of a product set formed from two or more other sets. For example, a child has a set of 8 shirts, each a different colour and a set of 3 skirts, each a different colour. Altogether there are 24 different colour combinations of 'shirt and skirt' that can be worn.

[15] Quote attributed to American Senator, Robert F. Kennedy.

Modern mathematics began with two great advances from the 1600s. The first was analytic geometry, primarily attributed to Descartes, while the second was calculus, attributed in priority to Isaac Newton and publication to Gottfried Leibniz. However, neither the Cartesian Plane nor the Cartesian Product has anything to do with the original writings of René Descartes on multiplication.

Importantly, Isaac Newton read Descartes' 1637 *La Géométrie* and 1644 *Principia Philosophiae* (Principles of Philosophy). After this, Newton developed calculus and formulated the laws of motion and universal gravitation, later published in his 1687 *Philosophiæ Naturalis Principia Mathematica* (Mathematical Principles of Natural Philosophy). [16] Having climbed such scientific heights seldom seen before, Newton went on to write *Arithmetica Universalis* (Universal Arithmetic) published in 1707.

So, having built a reputation as a mathematical and scientific genius, did Newton draw upon repeated addition, equal groups, arrays, or area models to explain multiplication? Or, for that matter, did Descartes? No.

Repeated addition and dimension

Today, some might think little about how strange it is that the 'multiplication' of two one-dimensional lines produces two-dimensional area. Yet, if you were to stack an infinite number of horizontal lines 1 metre long side-by-side, (rather than end-on-end), the breadth of that stack of lines would be zero. That is because, as Euclid defined in *Elements* around 300 BCE, *a line is a breadthless length*.[17] So, the repeated addition model, applied to lines, works in only one dimension, when adjoined end-on-end. Similarly, an area has length and breadth only, so if you were to stack an infinite number of areas 1-metre square, the height of that area would also be zero.

Lines cannot be added to make area and areas cannot be added to make volume, so why pretend they can? This pretence is evident from the widespread use of the area model of multiplication (via the repeated addition of same-size areas) and the *magical* length × width calculation. Similarly, a two-dimensional area multiplied by a perpendicular line gives us 'length × width × height' which *magically* converts the two-dimensional area into a three-dimensional volume. If people have not been confused by this, it is perhaps evident they have not thought about this. Euclid thought about this, which is why he carefully wrote about squares *on* a line, and not squares *of* a line. The point may be subtle [18], (as well as zero magnitude in all dimensions), yet it is important. Whether you say multiplication is repeated addition or not, an infinite number of points repeatedly added will never make a line, nor lines areas, nor areas volumes.

Descartes' lost logic

If Descartes were alive today, he might be surprised to see so many students being led to believe multiplication is *only* repeated addition. We can explain how 2×3 is equal to $2 \times (0 + 1 + 1 + 1)$ and $0 + 2 + 2 + 2$. Yet later on with 2×-3, we get $2 \times (0 - 1 - 1 - 1)$ and $0 - 2 - 2 - 2$, so multiplication is repeated subtraction! So, multiplication is much *more* than repeated addition. By adopting the insights of Descartes and Newton, a meaning can be given to multiplication and division as applied to the real numbers

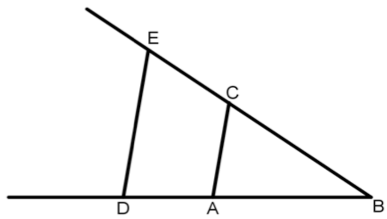
[16] The term 'natural philosophy' evolved into physical sciences and physics.
[17] Euclid's *Elements*, Book I, Definition 2.
[18] A point has zero magnitude, or as Euclid wrote in *Elements*, Book I Definition 1, 'a point is that which has no part'.

generally. Yet, such ideas are uncommon today, because a London haberdasher, Henry Billingsley, changed Euclid's (proportional) multiplication definition into an illogical repeated addition algorithm (Crabtree, 2016).

The first heading in Descartes' *La Géométrie* was *Problems the construction of which require only straight lines and circles*. Descartes' first diagram depicted the multiplication of line segments via similar triangles. The diagram was not new, as it was taken from Euclid's *Elements*. [19] Euclid defined a number as a multitude of units, and thus, for Euclid, the unit was not a number. The innovation of Descartes, almost 2000 years later, was to make one of the three given straight lines a unit (with length 1) while the other two straight lines were the two lines (numbers) to be multiplied. Translated from the French, we read:

> ...in geometry, to find required lines it is merely necessary to add or subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication)...

Descartes' original multiplication diagram and explanation is shown as Figure 2.



> For example, let AB be taken as unity, and let it be required to multiply BD by BC, then I have only to join the points A and C, and draw DE parallel to CA; and BE is the product of this Multiplication.

Figure 2. The diagram Descartes used to explain multiplication.

We can update Descartes' diagram for our 'Cartesian Plane' because the angle at B in the triangle is irrelevant, and also works as a right angle as shown in Figure 3.

[19] The diagram Descartes tweaked was from Euclid's *Elements*, Book VI, Definition 12, *To find a fourth proportional to three given straight lines.*

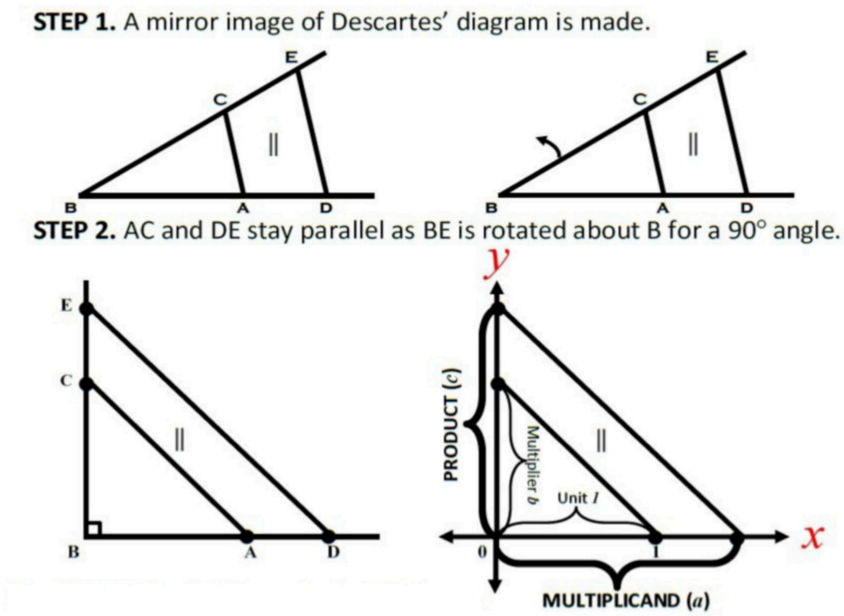


Figure 3. How to create similar triangles to reveal *a* multiplied by *b* = *c*.

Thinking outside the square

Given Descartes depicted multiplication with triangles, we first test whether or not an area model can emerge, not from unit squares, but from unit triangles. The short answer is yes, as shown in Figure 4.

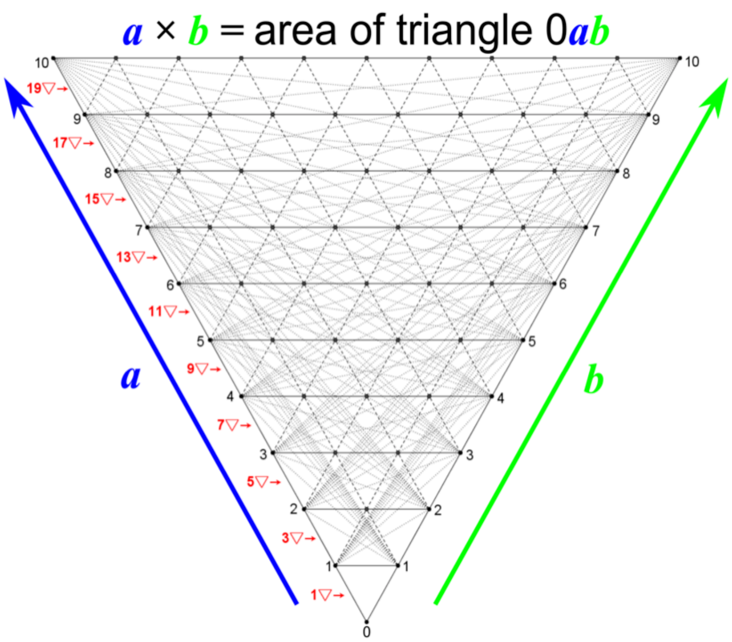
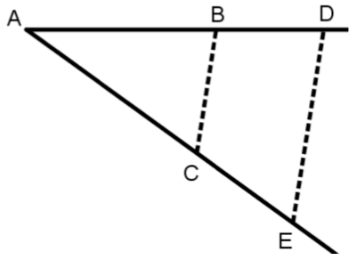


Figure 4. An area model for triangular units. Interactive applet at www.jonathancrabtree.com/mathematics/the-multiplication-triangle.

The above 10 × 10 multiplication table contains 100 triangles just as the standard 10 × 10 table contains 100 squares. With, for example, 5 × 5, the triangle contained by the points 0, 5 and 5 contains 25 triangular units. (Our standard table with products in squares is better pedagogically, as areas in the 'real world' are quoted in square units.)

From Descartes to Newton

In 1707, Newton followed Descartes with a similar explanation of multiplication, translated from the Latin in *Arithmetica Universalis*, shown below as Figure 5.



> If you were to multiply any two Lines, AC and AD, by one another, take AB for Unity, and draw BC, and parallel to it DE, and AE will be the product of this multiplication, because it [AE] is to AD as AC, [is] to AB Unity.

Figure 5. The diagram Newton used to explain multiplication.

Area model versus DesCartesian model

Because multiplication is a proportional concept, in all arithmetical equations, as the *unit* is to the *multiplier*, the *multiplicand* is to the *product*. With the simple example of two multiplied by three, written 2 × 3, as 1 varies to make 3, so 2 varies to make 6. Such proportional covariation (PCV) failed to emerge, either via the area model of multiplication or repeated addition model (Crabtree, 2016). As a line segment of 1 is to a line segment of 3, a line segment of two must be to a line segment of 6. To say a line of 1 is to a line of 3 as a line of 2 is to a rectangle of 6 is nonsense. Yet, with the *DesCartesian multiplication model*, we have a multiplication model that preserves proportional relationships, as is evident in Figure 6.

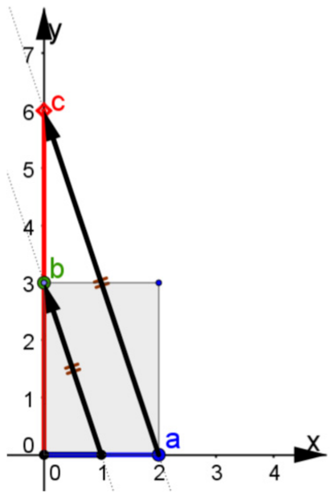


Figure 6. The standard area model for 2 × 3 alongside Descartes' proportional approach, which reveals 'as 1 is to 3, so 2 is to 6'. To multiply 2 by 3, a line is drawn from the unit 1 to the multiplier *b*, which is 3. Then a second line, parallel to the first drawn, is drawn from the multiplicand *a*, to produce the product *c*.

We can imagine ‘two square units stacked three times’ in the above, and also demonstrate commutativity of multiplication as shown in Figure 7, where we might imagine ‘three square units stacked two times’.

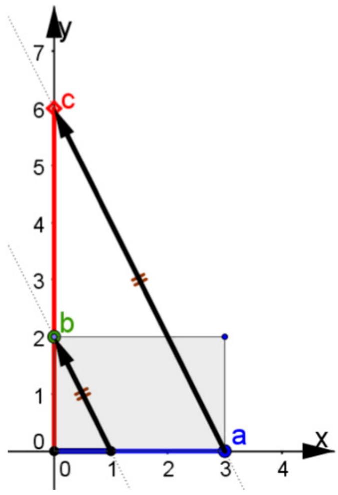


Figure 7. The standard area model for 3 × 2 alongside Descartes’ proportional approach which, via similar triangles, reveals ‘as 1 is to 2, so 3 is to 6’.

From the diagrams of Descartes and Newton, we have straight lines going up and down (albeit at an angle) and horizontal lines going left and right. Importantly (for what we are about to develop) with the following comments, Newton introduced the notion of positive and negative line segments that encompassed irrationals.

> In Geometry, if a line drawn any certain way be reckon’d for affirmative, then a line drawn the contrary way may be taken for negative: As if AB be drawn to the right; and BC to the left; and AB be reckon’d affirmative, then BC will be negative…

and

> Multiplication is also made use of in Fractions and Surds, to find a new Quantity in the same Ratio (whatever it be) to the Multiplicand, as the Multiplier has to Unity.

By the 1680s, Newton had drawn curves in all four quadrants, consistent with our understanding of the Cartesian Plane. These were published in 1704 as an appendix to *Opticks*. However, with the mathematics community focussed on what was to become algebraic geometry, the DesCartesian multiplication model, as applicable to the reals, appears to have been overlooked. Thus, primary mathematics teachers focus on the first quadrant of the Cartesian plane for the simple reason an area cannot be ‘less than zero’. Yet, such difficulties dissolve with the DesCartesian multiplication model. For example, beyond the first quadrant, the combinations of ±2 × ±3 are shown in Figure 8.

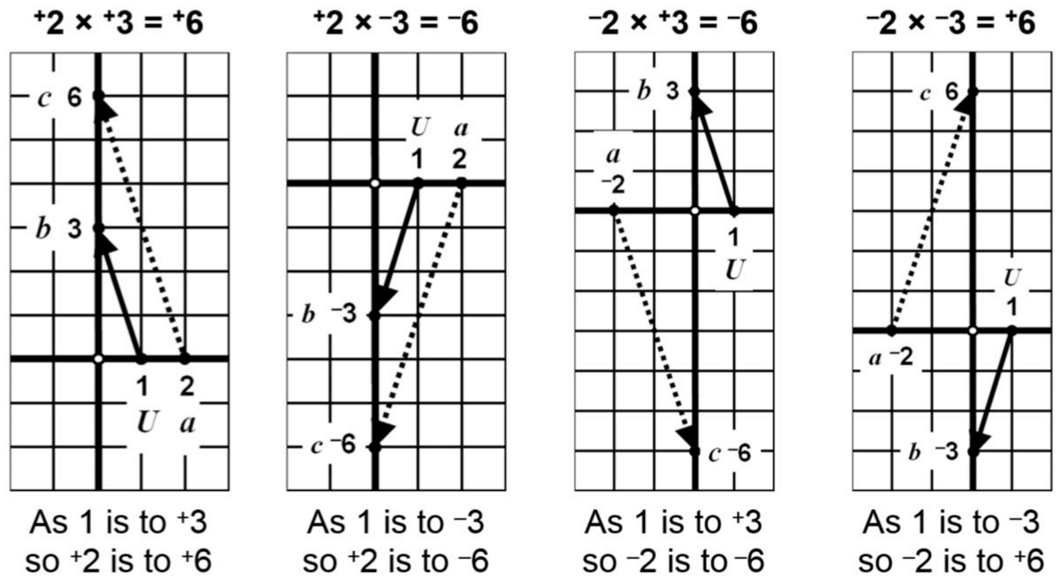


Figure 8. Combinations of ±2 × ±3 depicted with 'DesCartesian Multiplication', where 1 is the unit, b is the multiplier, a is the multiplicand and c is the product. Interactive applet at www.jonathancrabtree.com/mathematics/what-is-descartesian-multiplication

Regardless of the sign, in all cases: as 1 is to b, so a is to c. What we also see, is how the multiplication of similarly signed factors results in a positive product, while differently signed factors result in a negative product.

DesCartesian division

After Descartes explained multiplication, he used the same diagram (Figure 2) to explain division. "If it be required to divide BE by BD, I join E and D, and draw AC parallel to DE; then BC is the result of the division".

Thus, for division all we need do is 'invert' the multiplier, so instead of it encapsulating the ratio *1* to b, it becomes the divisor encapsulating the ratio b to *1*. In multiplication, whatever we did to the unit to make the multiplier, we do to the multiplicand to make the product. Unsurprisingly, (given division is the inverse operation of multiplication), in division whatever we did to the divisor to make the unit, we do to the dividend to make the quotient. As usual, a picture is worth a thousand words, so Figure 9 depicts 9 ÷ 3. Again, consistent with proportional covariation (PCV) however 3 is varied to make 1, so 9 is varied to make the quotient.

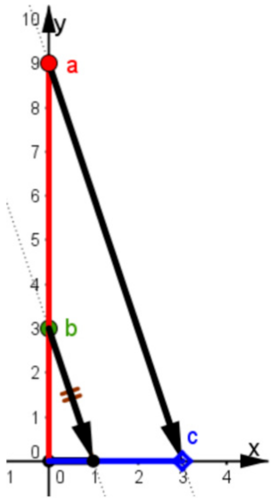


Figure 9. 'DesCartesian division' depicting 9 ÷ 3. To divide 9 by 3, a line is drawn from the divisor *b*, to the unit 1. Then a second line, parallel to the first drawn, is drawn from the dividend *a*, to produce the quotient *c*.

The DesCartesian approach simplifies the 'impossible'

Division has both a 'repeated subtraction' (quotitive) model and an 'equal shares' (partitive) model. Yet, without citing sign laws, if mathematics teachers rely only on these models, they simply *cannot* explain how to solve 9 ÷ −3. There are no negative threes in positive nine and you cannot divide nine into negative three groups. Yet, with the DesCartesian division model, there is little difficulty, as shown below in Figure 10.

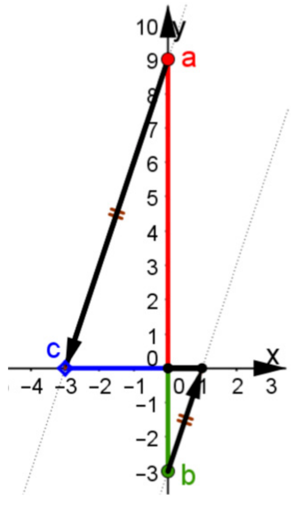


Figure 10. DesCartesian division depicting 9 ÷ −3.

The DesCartesian diagram depicting −9 ÷ −3 is shown in Figure 11. Put simply, as −3 is to 1, so −9 is to 3. In accordance with the laws of sign, our negative dividend divided by a negative divisor produces a positive quotient. To vary −3 and make 1, we take one of three equal parts of −3, which is −1, and change its sign to make 1. Having done that, we take one of three equal parts of −9, to get −3 and change its sign to make 3.

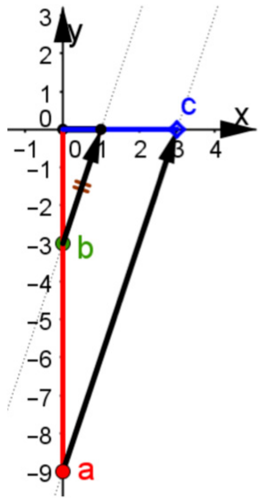


Figure 11. DesCartesian division depicting −9 ÷ −3. Interactive applet at www.jonathancrabtree.com/mathematics/what-is-descartesian-division

Final thoughts

Because we can make triangles between 0 and any two other points, one on each axis, the DesCartesian model for multiplication and division applies to the set of real numbers. These approaches to multiplication and division are in fact, applications of proportional covariation (PCV). From this long overdue historical evolution of arithmetical ideas, with further research and development by the mathematics education community, together, we might implement new approaches and unlock further useful and powerful ideas for teaching mathematics.

References

Crabtree, J. (2016, Dec). The lost logic of elementary mathematics and the haberdasher who kidnapped Kaizen. In W. Widjaja, E. Yook-Kin Loong & L. A. Bragg (Eds), 2016 *Mathematical Association of Victoria's annual conference proceedings* (pp. 98–106). Melbourne: MAV. Retrieved from www.mav.vic.edu.au/files/2016/2016_MAV_conference_proceedings.pdf

Crabtree, J. (2016). A new model of multiplication via Euclid. *Vinculum, 53*(2), 16.

Descartes, R. (1637). *La Géométrie in Discours de la Méthode. I.* Maire: [Holland] Leyde.

Descartes, R., Smith, D. E. & Latham, M. L. (1925). *The Geometry of René Descartes: With a Facsimile of the First Edition, 1637*. Chicago: Open Court.

Descartes, R. (1644). *Renati Des-Cartes Principia Philosophiæ*. Apud Ludovicum Elzevirium: Amstelodami.

Heath, T. (1908). *Thirteen Books of Euclid's Elements*. Cambridge, UK: Cambridge University Press.

Newton, I. (1687). P*hilosophiae Naturalis Principia Mathematica*. Londini: Jussu Societatis Regiae ac Typis Josephi Streater.

Newton, I. (1704). *Opticks*. London: printed for Sam. Smith, and Benj. Walford.

Newton, I. (1720). *Universal Arithmetic*, Translated by J. Raphson, London.

Newton, I., Machin, J. & Motte, A. (1729). *The Mathematical Principles of Natural Philosophy*. London: [UK] Printed for B. Motte.

GENERALISATION THROUGH NOTICING STRUCTURE IN ALGEBRAIC REASONING

LORRAINE DAY

The University of Notre Dame Australia

lorraine.day@nd.edu.au

"Mathematicians see generalising as lying at the very heart of mathematics" (Mason, Graham & Johnston-Wilder, 2005, p. 283). The Australian Curriculum: Mathematics develops number and algebra together as they complement each other. Developing number and algebra together provides opportunities for searching for patterns, conjecturing and generalising mathematical relationships. Further, it allows the focus to be on the process of mathematics and noticing the structure of arithmetic and our number system, rather than the product of arriving at a correct answer.

Algebraic reasoning

Algebraic reasoning underpins all mathematical thinking, as it allows us to explore the structure of mathematics. It pervades all of mathematics and is about describing patterns of relationships, generalising mathematical ideas and identifying mathematical structures (Ontario Ministry of Education, 2013; Van der Walle, Karp & Bay-Williams, 2010). Kaput and Blanton (2005) defined "Algebraic reasoning [as] a process in which students generalse mathematical ideas from a particular set of instances, establish those through the discourse of argumentation and express them in increasingly formal and age appropriate ways." (p. 99)

Focusing on algebraic reasoning alters the study of number and operations from a focus on finding numerical answers to arithmetic problems, or a product approach, to providing opportunities for discovering patterns, conjecturing and generalising mathematical relationships, a process approach (Schoenfeld, 1987; Siemon, Beswick, Brady, Clark, Faragher & Warren, 2015). It is the patterns that provide insights into the structure of mathematics. Noticing the structure of arithmetic forms the foundation of algebraic understanding. Continual development on recognising pattern and structure has been seen to have a positive influence on overall mathematical achievement and builds a stronger foundation for algebraic reasoning (Mulligan, Mitchelmore & Prescott, 2006). A deep understanding of numbers, operations and the relationships between them is necessary for the development of number and algebra sense and an acute sense of number, along with an appreciation of pattern and relationships are necessary requirements for deep mathematical understanding (Siemon et al., 2015).

The big ideas of algebraic reasoning

It can be seen that patterns are at the core of algebraic reasoning. Searching for patterns is a process natural to people (Mason et al., 2005; Siemon et al., 2015). The study of patterns in schools generally begins with repeating patterns, moving onto growing patterns and investigating and employing patterns in the number system. The study of patterns is necessary prior to the development of functional thinking, which focuses on the relationships between two or more varying quantities. Another critical idea in algebraic reasoning is the notion of equivalence. Equivalence is usually represented by an equal sign, an important, but poorly understood, symbol in mathematics. In order to comprehend the notion of equivalence, students need to understand that the equal sign represents a balance on either side, rather than meaning find the answer. Generalisation encompasses all of these big ideas, as we can generalise about pattern, about equivalence and about function and generalisation lies at the heart of algebraic reasoning (see Figure 1).

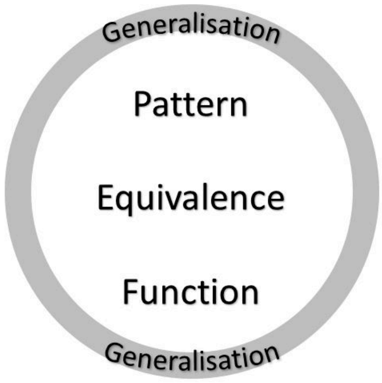


Figure 1. Big ideas of algebra.

Reframing Mathematical Futures II (RMFII) is a three-year project funded by the Australian Government Department of Education and Training under the auspices of the Australian Mathematics and Science Partnership Programme (AMSPP). The project is working with industry partners and practitioners in each State and Territory and the Australian Association of Mathematics Teachers (AAMT) to build a sustainable, evidence-based, integrated learning and teaching resource to support the development of mathematical reasoning in Years 7 to 10. The data collected in the algebraic reasoning component of the RMFII Project has reinforced pattern and function, equivalence and generalisation as the big ideas of algebraic reasoning (Day, Stephens & Horne, in press).

Generalisation

The process of generalisation is about noticing structure. Mason et al. (2005) stated that even very young children can generalise and specialise when they first come to school. While generalising is natural, students need time to notice that they have this sense of generality and they need opportunities to practise, strengthen and extend this natural ability to generalise. Asking students questions about what they notice, whether they can see any patterns and how they are making sense of the mathematics is important.

Often, in order to try to make sense of the mathematics, students will specialise. Specialising may take the form of trying several numbers to see what is happening in a problem. This is a natural approach to mathematical thinking (Mason et al., 2005; Siemon et al. 2015) and it helps students in sense-making while collecting data about a problem. Sense-making is easier if the problems are set in meaningful contexts, as the context allows students to relate what they are seeing back to a specific context. Familiar contexts can be presented using concrete materials, with diagrams and with numbers. Using a concrete-representational-abstract (CRA) approach with students has been shown to be effective (Mudaly & Naidoo, 2015; Sousa, 2008; Witzel, Mercer & Miller, 2003)

Once students are alerted to the idea that patterns are important and they begin to notice patterns, they are in the thinking process of generalisation (Siemon et al., 2015). Questioning students about what they notice, what changes and what stays the same, is important for students to start recognising the structure of the patterns. Eventually students will become attuned to the fact that what changes are variables and the things that stay the same are constants. Other questions that should be routinely asked of students are whether what they have identified always works and in all cases and for all operations (Cooper & Warren, 2008; Kaput, 1999). Much of algebraic reasoning is about searching for, describing, generalising and justifying patterns (Steen, 1988).

Providing students with the time to think, form and try to articulate generalisations to themselves before sharing with a small group or the whole class is essential (Mason et al., 2005) if students are to be confident in articulating ideas. It should be noted that students often do not attend to the same things as their teachers. They see things in different ways and it is the role of the teacher to listen carefully to student explanations about how they 'see' a problem and acknowledge and celebrate articulations and demonstrations that are correct, but not necessarily the same way the teacher 'sees' a problem. Experience with generalising in different contexts may lead to multiple expressions of the same thing (Mason et al., 2005). This, in turn, can often lead to an investigation of equivalent expressions.

Tasks that assist students to notice structure and generalise

Many problems that are designed as arithmetic problems for young students can be extended into generalisations. The following question (Figure 2) was designed for a Year 2 class as a problem-solving question.

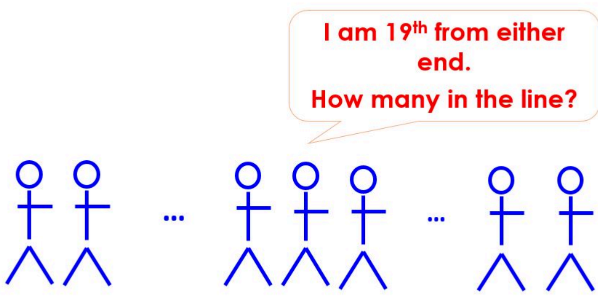


Figure 2. Line up question (adapted from Lovitt & Williams, 2015).

Rather than just seeking the answer to this question, although seeking solutions is useful, there is an opportunity to take this question further. After students have had the opportunity to solve this problem, they can be asked to explain how they 'saw' the problem and draw pictures to represent what their visualisations were. An example of the three ways most students visualise this problem are included in Figure 3. This is an important step for students to see important ideas emerge: that different students visualise problems in different ways, that there are several ways to arrive at a correct answer and that there are multiple ways to write equivalent expressions for the same problem.

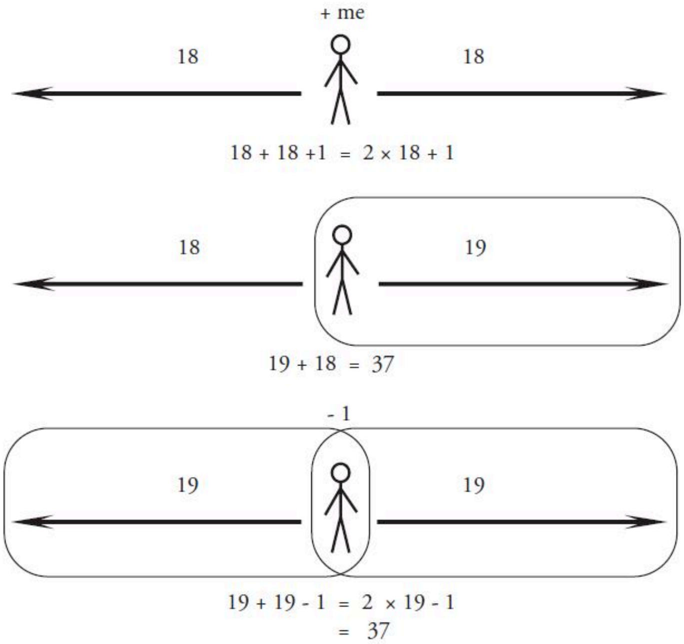


Figure 3. Line up visualisation pictures.

Students may then be asked to use the visualisation that makes sense to them to work out how many students would be in the line if they were 100th from either end and explain in words and/or pictures the process they went through to work out the answer. If students are able to do this successfully they may be asked to generalise the situation in words, pictures and/or symbols (depending on their readiness for symbolic work).

The three generalisations from the visualisations, in order, are:

$$t = 2(n - 1) + 1$$
$$t = n - 1 + n$$
$$t = 2n - 1$$

Interestingly, the third visualisation, which provides the generalisation in the simplest form is the one that the fewest students nominate as their preferred visualisation. Overwhelmingly students 'see' this problem as the first visualisation. That suggests that we should allow students, at least initially, to generalise problems as they visualise them and not always insist that algebraic expressions are in their simplest form, as the expressions need to make sense to the students. Eventually students will be expected to be able to move flexibly between equivalent expressions and express them in their simplest form, but we may rush students to this stage without recognising that they need to first make sense of the problem within its context.

Several arithmetic worded problems can be modified to encourage students to notice the structure of the mathematics. For example, a typical textbook question may read

> Abbey is 140 cm tall. Ben is 4 cm taller than Abbey and Abbey is 6 cm shorter than Charlie. How tall are Ben and Charlie?

By taking out the initial piece of information that Abbey is 140 cm tall, the question may be changed to encourage students to notice the structure and relationships contained in the question:

> Ben is 4 cm taller than Abbey. Abbey is 6 cm shorter than Charlie. Draw a picture showing Abbey, Ben and Charlie's heights. Explain what the 4 and the 6 represent. Try to express these height comparisons in other ways. (Adapted from Carraher, Brizuela & Schliemann, 2000)

By removing the information of how tall Abbey is, the students are forced to consider the height comparisons rather than just perform two computations. This can be taken even further with the introduction of the n-number line (Figure 4), which pays attention to the structure of number lines.

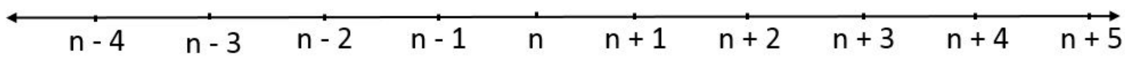


Figure 4. n-number line.

Now the question can be asked if Abbey is n cm tall, position Ben and Charlie's heights on the n-number line. What about if Ben were n cm tall, where would Abbey and Charlie be positioned? If Charlie were n cm tall, where would Abbey and Ben be positioned?

Visual growing patterns are another good way of helping students to notice structure. A great deal of mathematics can be mined from even simple growing patterns. One idea is to use triangles that are used in the construction of bridges and other structures. The simplest of these is known as a Warren truss and is pictured in Figure 5.

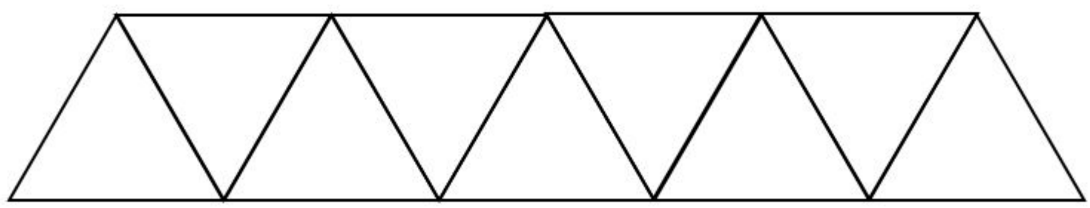


Figure 5. Warren truss.

Students should be encouraged to build this model using popsticks so that they can manipulate the model and show the structure they are considering by physically moving pieces of the model. Questions such as "What changes and what stays the same?" will encourage students to notice structure and there will be different ways in which the students see the structure. After some initial questions about how many popsticks would be needed to make bridges of certain lengths, students could be asked to find how many popsticks would be needed to build a bridge that contains 100 triangles as part of its Warren truss. Students should be able to demonstrate on their model how they worked it out. This is a good time for students who 'see' the problem differently to share their reasoning with other members of the class. In this way students hear of other visualisations and notice different structures. From this point students could be asked to generalise their result in words, pictures or symbols. A range of generalisations that have been observed in classrooms are included in pictorial and symbolic form in Figure 6.

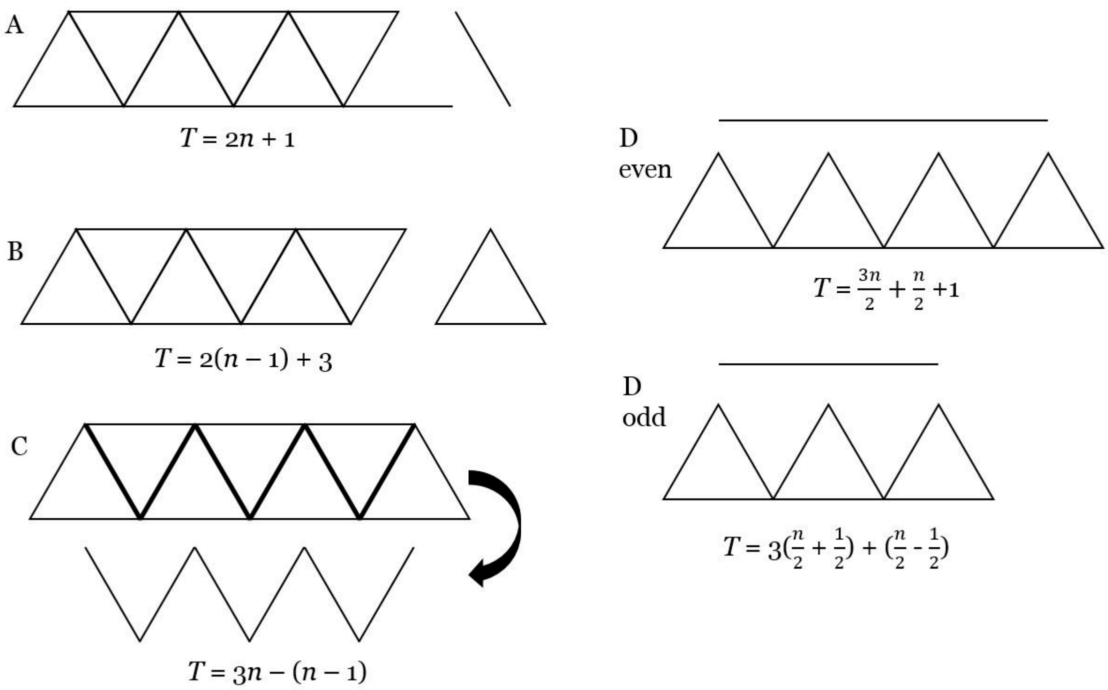


Figure 6. Warren truss bridge generalisations.

Another type of task that is suitable for students noticing structure is the investigation of number structures. For example students might investigate the sums of odd and even numbers, the property of commutativity, or the sums of consecutive numbers (Driscoll, 1999, Lovitt & Williams, 2015). One activity that assists students to recognise structure uses the story of the 18th century mathematician Carl Frederich Gauss being given the task by his teacher to add up all the numbers from one to 100. Students can use concrete materials to represent a simpler version of this problem by looking at how they might add the numbers from one to ten (see Figure 7).

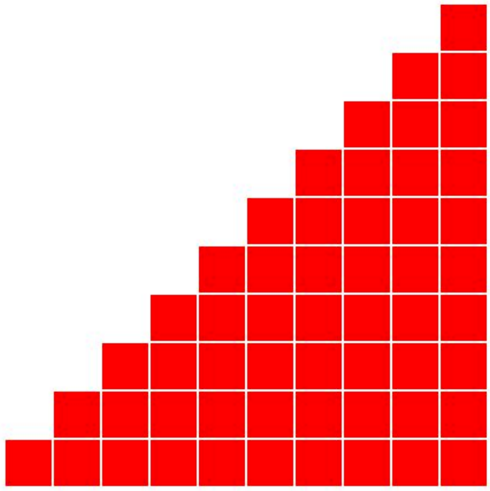


Figure 7. Concrete representation of adding the numbers from one to ten.

Through small group discussions about how these numbers could be combined in some way to make the addition easier there are generally three strategies that emerge in classrooms. These strategies represent three different visualisations that students see, and all are valid ways to solve this problem. Some students think that number combinations to ten are easy to work with, so they group their concrete materials in tens (see Figure 8). Other students notice that if they add the lowest number and the highest number and keep doing this with pairs of numbers that they have five equal groups of 11 (see Figure 9). Occasionally students will use the knowledge that arrays are useful representations and combine their model with a neighbouring group's model to form an array which is double the number that is required (see Figure 10).

When different groups share their strategies with the class it can be seen that those who visualised the problem in groups of ten found the solution $T = (5 \times 10) + 5 = 55$, those who saw the problem in groups of 11 found the solution $T = 5 \times 11 = 55$ and those who formed a rectangular array found the solution $T = (11 \times 10)/2 = 55$. Once students have identified their preferred method for summing the numbers from one to ten they can apply a similar strategy to the original challenge of summing the numbers from one to 100. These types of problems allow students to focus on the structure of the numbers, rather than just adding numbers together.

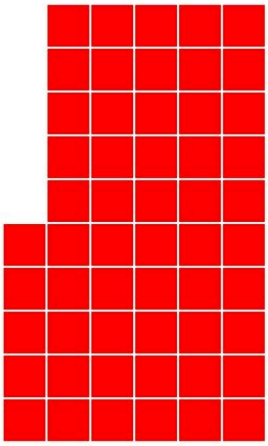


Figure 8. Grouping in tens.

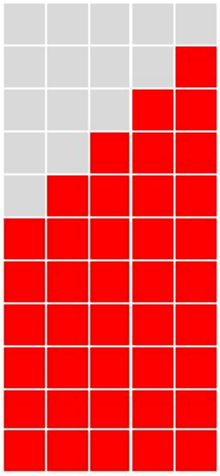


Figure 9. Grouping in 11s.

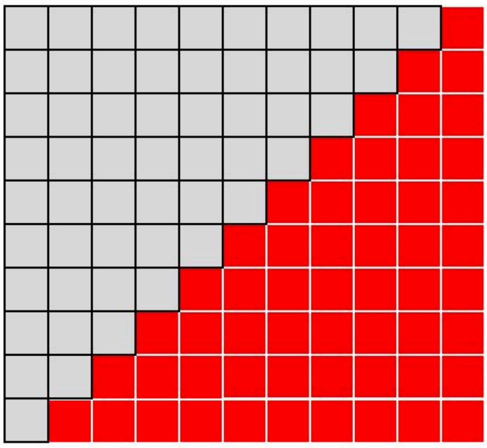


Figure 10. Forming a rectangular array.

Teachers can follow an activity like this by setting further challenges such as finding strategies for adding all the even numbers from one to 100, or all the numbers on a zero to 99 chart, or all the numbers on a multiplication chart. The major challenge could be to work out a strategy to find the total of all the numbers from one to n, the generalisation of the problem.

Conclusion

Teachers may use the development of number and algebra together as a powerful tool towards algebraic reasoning through the process of generalisation by providing students with a variety of ways in which to notice structure. Whereas arithmetic thinking tends to be about a product, finding the correct answer, algebraic thinking and reasoning is about the process of noticing pattern and structure in a variety of contexts. The noticing of structure assists students to make sense of the mathematics rather than just applying operations on numbers without necessarily understanding why they are doing so. Understanding how our number system is structured greatly helps students to reason mathematically.

References

Carraher, D., Bizuela, B. & Schliemann, A. (2000). Bringing out the algebraic character of arithmetic: Instantiating variables in addition and subtraction. In T. Nakahara & M. Koyama (Eds), *Proceedings of the 24th conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 145–152). Hiroshima, Japan: PME.

Cooper, T. & Warren, E. (2008). Generalising mathematical structures in Years 3–4: A case study of equivalence of expression. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano & A. Sepulveda (Eds), *Proceedings of the 32nd conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 369–376). Morelia, Mexico: PME.

Day, L., Stephens, M. & Horne, M. (in press). Developing learning progressions to support mathematical reasoning – Algebraic reasoning. In *Proceedings of the 40th annual conference of the Mathematics Education Research Group of Australasia*. Melbourne: MERGA.

Driscoll, M. (1999). *Fostering algebraic thinking*. Portsmouth, NH: Heinemann.

Kaput, J. (1999). Teaching and learning a new algebra. In E. Fennema & T. Romberg (Eds), *Mathematics classrooms that promote understandings*. Mahwah, NJ: Lawrence Erlbaum Associates.

Kaput, J. & Blanton, M. (2005). Algebrafying the elementary mathematics experience in a teacher-centered, systemic way. In T. A. Romberg, T. P. Carpenter & F. Dremock (Eds), *Understanding mathematics and science matters*. Mahwah, NJ: Lawrence Erlbaum Associates.

Lovitt, C. & Williams, D. (2015). *Maths300*. Retrieved from www.maths300.com. Mooroolbark, Vic.: The Task Centre Collective Pty Ltd.

Mason, J., Graham, A. & Johnston-Wilder, S. (2005). *Developing thinking in algebra*. London: The Open University.

Mudaly, V. & Naidoo, J. (2015).The concrete-representational-abstract sequence of instruction in mathematics classrooms, *Perspectives in Education, 33*(1), 42–56.

Mulligan, J., Mitchelmore, M. & Prescott, A. (2006). Integrating concepts and processes in early mathematics: The Australian pattern and structure mathematics awareness project (PASMAP). In J. Novotna, H. Moraova, M. Kratka & N. Stehlikova (Eds), *Proceedings of the 30th annual conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 209–216). Prague: PME.

Ontario Ministry of Education. (2013). *Paying attention to algebraic reasoning*. Toronto, ON: Queen's Printer for Ontario.

Schoenfeld, A. (1987). Cognitive science and mathematics education: An overview. In A. Schoenfeld (Ed.) *Cognitive science and mathematics education* (pp. 1–32). Hillsdale, NJ: Lawrence Erlbaum Associates.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R. & Warren, E. (2015). *Teaching mathematics: Foundations to middle years* (2nd ed.). Melbourne: Oxford University Press.

Sousa, D. (2008). *How the brain learns mathematics*. CA: Corwin Press.

Steen, L. (1988). The science of patterns. *Science, 240*, 611–616.

Van De Walle, J., Karp, K. & Bay-Williams, J. (2010). *Elementary and middle school mathematics: Teaching developmentally* (7th ed.). Boston, MA: Pearson Education.

Witzel, B., Mercer, C. & Miller, S. (2003). Teaching algebra to students with learning difficulties: An investigation of an explicit instruction model. *Learning Disabilities Research and Practice, 18*(2), 121–131.

TEACHING PLACE VALUE: CONCEPT DEVELOPMENT, BIG IDEAS AND ACTIVITIES

JUDY HARTNETT
Making Maths Reason-able
judyhartnett@bigpond.com

Primary school teachers teach at least eight if not nine subjects (including religious education in some schools). The ability to develop in-depth content knowledge and pedagogical content knowledge in each of the topics, in each of these subjects is challenging. This paper discusses the development and beginning implementation of a detailed sequence of concepts compiled by the author with the intention of supporting teachers to support students to understand place value, one of many topics in the Mathematics curriculum.

Introduction

Working as a mathematics education consultant in schools both as a system-based advisor and private consultant I am regularly asked to support teachers to work with students who do not understand place value. As reported by other researchers including Rogers (2013), I too am regularly dismayed at the superficial understanding so many primary students exhibit in relation to place value. I have also observed repeated amazement by teachers when I demonstrate lessons, or discuss the 'big ideas' relating to place value in comments such as 'I have never thought about it that way'.

Primary school teachers teach a number of subjects and while they continue to learn and access professional development it is not surprising that they do not have an in-depth understanding of the many concepts that underpin mathematical topics such as place value. This paper outlines the development of a list of concepts I hope will support teachers to develop their knowledge of this topic to help them plan and present lessons to assist their students to understand the concepts.

Teacher knowledges

Research has recognised that teachers require a range of types of knowledge. Shulman (1986) described seven types of teacher knowledge: knowledge of content, general pedagogical knowledge, curriculum knowledge, pedagogical content knowledge, knowledge of students, knowledge of educational contexts and knowledge of educational ends, purposes and values. Based on the work of Shulman and colleagues (e.g., Shulman & Grossman, 1988), Borko and Putnam (1995) proposed a model of teacher knowledge organised around three domains of knowledge: general pedagogical knowledge, subject matter knowledge and pedagogical content knowledge. Table 1 outlines these domains and their components.

Table 1. Domains and components of the knowledge base of teaching (Borko & Putnam, 1995)

Domains	Components
General pedagogical knowledge	Learning environments and instructional strategies Classroom management Knowledge of learners and learning
Subject matter knowledge	Knowledge of content and substantive structures Syntactic structures
Pedagogical content knowledge	Overarching conception of teaching a subject Knowledge of instructional strategies and representations Knowledge of students' understandings and potential misunderstandings Knowledge of curriculum and curricular materials

Teachers I had been supporting appeared to lack mathematical subject matter knowledge, and also welcomed suggestions about pedagogical content knowledge. Teachers have access to Australian Curriculum documents and often commercial maths programs and resources. These documents provide descriptions, glossaries and activities but lack mathematical depth to help the teachers develop their subject matter knowledge. A sequential list of big ideas and concepts that under-pin Mathematics could be a valuable support for teachers. Table 2 provides an example of how several concepts can underpin one Australian Curriculum description. I started to draft such lists for mathematics topics in a number of areas I had received requests for assistance about, including place value.

Big ideas

Place value is not a single concept. Schmittau and Vagliardo (2006) used concept mapping to describe place value as a complex system. Price (1998) described how the development of connected memory structures or schema can assist students to understand complex numeration concepts like place value. He described how assisting students to develop powerful schemas to understand the complexities of place value was attractive to mathematics educators but commented that for many students such schemas had not developed. My anecdotal observations lead me to suspect that many of the teachers I was supporting may not have developed a rich schema for a deep understanding of our base-ten number system themselves. This lack of subject knowledge would likely inhibit their ability to design learning activities and identify common misconceptions in their students.

Many researchers have identified concepts that need to be understood for students to become *place value experts* (Ross 1989, Rogers, 2012). Rogers (2012) conducted a comprehensive search of literature on the topic and focussed on work of Rubin and Russell (1992) and Ross (2002) to identify seven components of place value. Rogers noted that there is no developmental order implied in the list:

- **Count**: Counting forwards and backwards in place value parts (e.g., 45, 55, 65 is counting using the unit ten). Bridging forwards and backwards over place value segments (e.g., 995 and one more ten requires bridging forwards over hundreds to thousands).
- **Make/represent:** Make, represent or identify the value of a number using a range of materials or models—these may be proportional, non-proportional, canonical and noncanonical.
- **Name/record:** Read and write a number in words and figures (e.g., 75 is written as 'seventy-five'). Identify the value of digits in a number (e.g., the value of 3 in 345 is 3 hundreds). Rounding numbers to the nearest place value part (e.g., round 2456 to the nearest thousand).
- **Rename:** Recognise and complete partitions and regrouping of numbers. (e.g., 1260 has 126 tens).
- **Compare/order:** Compare numbers to determine which is larger or smaller and place them in descending or ascending order.
- **Calculate:** Apply knowledge of place value when completing calculations (e.g., 45 by 10 is 45 tens)
- **Estimate:** Use knowledge of magnitude of numbers when estimating (e.g., estimate how many oranges fill a classroom: 10? 100? 100 000?)

Other authors have identified concepts relating to place value which theoretically fit within the components above. Examples include the *odometer principle* (YuMi Deadly Centre, 2014) which states that in any place-value position, numbers count the same as in the ones place, counting forwards from 0 to 9 and then back to 0 with the digit to the left increased by 1; the *recursive HTO pattern* (Siemon, Beswick, Brady, Clark, Farragher & Warren, 2011); *composite units / super-unitising and sub-unitising* (Siemon et al., 2011, Baturo, 1998); and the *recursive multiplicative relationship* between the places where "10 of these is 1 of those" (Siemon et al., p. 302). Place value is a complex mathematical topic with a multitude of big ideas and connected concepts to be understood. A list of these big ideas of place value would need to include all these big ideas presented in a way that made sense to teachers.

Conceptual development

When learning mathematics, students progress through topics and associated concepts which get increasingly more sophisticated. This educative progression has been identified as back as far as Piaget (1952). Carpenter and Fennema (1991) noted that teachers need an understanding of the stages students move through when developing concepts and procedures. The Queensland University of Technology (QUT) YuMi Deadly Centre (2014) believes that it is "essential for teachers to know what mathematics precedes, relates to and follows what they are teaching" (p. 2). Recent studies in Australia have used this idea to identify learning progressions; e.g., *Growth Points* (Clarke et al., 2002) and *Learning Assessment Frameworks* (Siemon, Izard, Breed & Virgona 2006).

The idea of a list of the progression of concepts for place value as a teaching tool appealed to me as a way of supporting teachers who lacked subject matter knowledge of this topic. I began to draft a conceptual development list for place value based on ideas and concepts gathered over many years working with teachers, researching in mathematics education and working with other researchers. My intention was to write the list in teacher-friendly words rather than academic speak and for the list to progress from early concepts through to more advanced concepts. The statements are intended to be what we would like students to *understand* about place value rather than what we would like them to *do*. My intention was for this list to be a reference document for teachers. Teachers could then plan learning activities to help the students achieve these understandings. Below is an early section of the progression which currently is over three pages long and still being drafted.

- A base-ten number system uses only 10 symbols to represent any number large or small.
- The Hindu-Arabic number system uses 10 digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
- hen counting forwards, the digits in a place are named in order, from 0 to 9 and then back to 0 in that place. When the digit in a place returns to 0 the place to the left increases by 1 (odometer principle) e.g., 17, 18, 19, 20, 21Numbers beyond ten can be considered as 1 ten and some extras…
- Teen numbers consist of a full 1 ten and extra ones, but not enough extra ones to make another ten.
- Multiples of ten consist of several groups of ten and are written to show the number of tens with zero ones e.g., 4 tens = 40.
- 2-digit numbers consist of a number of tens and a number of ones with the tens recorded in the place to the left of the ones e.g., 45 is 4 tens and 5 ones.
- 2-digit numbers are ordered according to the number of tens and then the number of ones.
- Adding a multiple of ten to a number will increase the number of tens by 1 but not change the number of ones e.g., 45+30 = 75.
- There are 10 tens in 1 hundred.
- The place of a digit in a number indicates its value e.g., 4 in the tens place is worth 4 tens.
- The value of a digit is determined by multiplying its face value by the value assigned to its place in the number.
- Zeros are used to show when a number has none of a particular place value e.g., 30 is 3 tens and 0 ones; 405 has 4 hundreds, 0 tens and 5 ones.
- Zeros are used a place holders to maintain the place value structure of a number e.g., 304 is a 3-digit number that consists of 3 hundreds, 0 tens and 4 ones.

The concept development list extends through whole number place value concepts to concepts for decimal place value while aiming to include all big ideas including the structure of our number system, reading, writing and ordering numbers, the relationship between the places, the role of the decimal point and use of zero etc. Below is part of the conceptual development sequence as it extends into decimals

- The structure of the Hindu-Arabic number system extends to the right to allow us to show parts of whole numbers using place value
- The structure of the place value system remains constant—moving numbers to different positions in the place value chart will change the value of the digits. The place value chart structure including the decimal point does not move
- Multiplying and dividing numbers by powers of ten will move the numbers in relation to the place value chart and will change the value of digits
- The groups of three (HTO) structure continues to the right of the ones place although it is rarely used to describe decimals e.g., tenths, hundredths, one thousandths, ten thousandths etc.
- The groups of three (HTO) structure is reversed in the decimals because the first place is 1÷10 ($\frac{1}{10}$) which is tenths

WHOLE NUMBERS					PARTS OF WHOLES					
thousands		ones			parts		thousandths			
T	O	H	T	O	T	H	O	T		
					$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1000}$	$\frac{1}{10\,000}$		
				1	1÷10	1÷10÷10	1÷10÷10÷10	etc.		

- Mixed numbers are fractions that have a whole component and a part component. With fractions the number of parts (denominator) can be any number e.g., $2\frac{1}{10}$
- The decimal point is used to mark the ones place so numbers can be read and interpreted using place value e.g., 4.56 is 4 wholes, 5 tenths and 6 hundredths or 4 wholes and 56 hundredths
- The decimal point marks the separation of the whole component of a number and the part component e.g., 4.5 is 4 ones (whole) and 5 tenths (part)
- Zeros are used as place holders in decimals numbers to show there are no digits of a particular value in a number e.g., 4.06 is 4 ones, 0 tenths and 6 hundredths
- Zeros placed to the right of decimals do not change the value to the number as they do not add any further places of value e.g., 5.6 is the same as 5.60

Activities

The teaching of number concepts requires the student to abstract the learning from examples and activities provided by the teacher. Number concepts cannot be perceived directly with the physical senses. They can be represented or symbolised but the meaning must be abstracted by the learner (Price, 1998). The activities that teachers prepare need to provide examples and stimulus for students to abstract the concepts. The choice of resources can support student understanding. There has been discussion in the literature about the benefits of particular resources for teaching number and in particular teaching place value.

Resources for teaching place value

The use of hands-on materials is widely recognised as beneficial to the development of students' conceptual understanding in mathematics. Price (1998) reported that 96% of teachers he surveyed believed that materials benefitted children's learning and that curriculum documents recommended the use of materials. However, he commented that the use of materials to support place value learning needs careful planning and that teachers cannot assume that students are making sense of the representations the same way the teacher is.

Place value blocks, also known as multi-base arithmetic blocks (MAB), are the most common classroom hands-on material used to support the learning of place value concepts (Price, 1998). The structure of these blocks models the base-ten number system as the size of each block is proportional to the value it represents. However, the effectiveness of these blocks for teaching place value concepts has been questioned by several authors including Booker, Bond, Sparrow & Swan (2010); Fuson, (1990); Siemon et al., (2011); Miura and Okamato (2003); Price (1998); and Anna Rogers (2009). Using MAB to model place value concepts requires the students to see the relationship between the different blocks; that is, they require an understanding of area to see that the hundred is equivalent to 10 tens and volume to see the thousand block as 10 hundreds. I have observed many students counting to check how many segments there are in a ten block not trusting or knowing that it is being used to represent 1 ten. I have also observed students who believe the thousand block to be equivalent to 6 hundred as they see the faces as 100 but fail to recognise there would be more cubes 'inside' the block.

Another resource that can be used to represent the multiplicative relationship inherent in place value and to represent the idea that '10 of these is 1 of those" described by Siemon et al. (2011) is ten frames. I have used ten frames successfully to teach early place value concepts and my experience demonstrates that the multiplicative nature of place value is clearer for students to comprehend with this resource than with MAB. Ten frames are a frame that has space for ten objects. As the frame is always visible, the relationship to ten can be reinforced whether the frame is full or not. When the frame is full there is no need to count to find the quantity of objects. Siemon et al. (2011) describe how ten frames can help students gain a sense of ten. Double ten frames provide a clear representation of teen numbers as 1 ten and extras (see Figure 1).

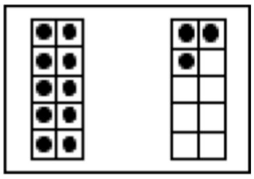


Figure 1. Double ten frames representing the teen number 13 as 1 ten and 3 ones.

Multiple ten frames can be used to represent two-digit numbers where the total quantity is visible as well as the multiples of ten place value structure. Figure 2 shows ten frames used to represent the number 45 in three different ways (canonical and non-canonical partitioning) and how a digit only focus showing a 4 and a 5 is clearly not representing the quantity. This material does get cumbersome when representing three-digit numbers although the multiplicative relationship of 1 hundred as being the same as 10 tens can be represented clearly. Additionally, students can benefit from this representation as a support of their understanding of the multiplicative structure of our number system that further abstract concepts can be built from (see Figure 3).

Figure 2. Using ten frames to represent 45 in many ways
and how a digit focus is clearly not 45 dots.

Figure 3. Representing numbers beyond 100 using ten-frames.

Other materials used to represent place value concepts include unifix cubes and bundling sticks which allow the building and un-building of tens. With both of these materials the ten is not immediately visible requiring counting to check if there are 10 cubes or 10 sticks in a ten. Use of a range of representations and materials in a teacher's program is to be encouraged in conjunction with knowledge of the concept the activity or lesson intends to teach. If the concept is for students to understand that 10 ones is 1 ten then bundling sticks would be a valuable resource. If the concept is to move from counting all to identifying ten without counting then ten frames would be valuable.

Implementation

The conceptual development sequence described in this paper along with draft sequences for other mathematics topics have been shared with four schools in Queensland. Teachers are using the document to support their planning of mathematics activities. Some teachers are using the conceptual statements as *learning intentions* as part of a *visible learning* focus (Hattie, 2012). Others are using the statements as checklist items to guide observational formative assessment. The use of ten frames to develop early place value concepts, in particular the multiplicative 10x relationship between adjacent places appears to be helping students to understand how place value works ahead of discussions about the structure of the whole number system and beyond. Table 2 shows how one school has been using the concept statements to plan and record lessons connected with the Australian Curriculum.

Other teachers have begun using the conceptual statements to differentiate learning experiences by helping them to identify prior or later concepts or concepts that individuals or small groups of students might be missing.

Table 2. Example of a possible lesson heading using the concept list.

Lesson focus	The place of a digit determines its value
Australian Curriculum	Year 2 ACMNA027 Recognise, model, represent and order numbers to at least 1000
Concepts	The most important place in our number system is the ones place. As long as we know which digit is in the ones place we can read any number, large or small. The value of a digit is determined by its place. Without a place value chart the value of a digit is not known unless the ones place can be identified. If there is only one digit the digit is assumed to be the ones place e.g., a 4 is assumed to be 4 ones. Recording a digit in a particular place does not represent the number with the same value until all the places to the ones place have been filled e.g., 4 in the hundreds place is not the number 4 hundred. It needs zeros in the empty tens and ones places to make the number 400.
Activities	Give each student a 3-digit number place value chart with headings H, T and O). Ask: What do you think the H, T and 0 are short for? (Hundreds, Tens and Ones) What do you think we are going to put into the columns? (Many will say numbers. Lead them to realise that they will use digits) What is a digit? (Accept a range of answers and ask student for more information… discuss this important structural aspect of our number system – Digits are used to make numbers. They are the building blocks of numbers. Compare to letters for words… letters make words, digits make numbers. We can have 1 letter words, and we can have 1-digit numbers) How many digits are there in our number system? (Ten 0-9) etc.

Looking ahead

The development and implementation of these concept lists is a work in progress. Some schools are trialling them and using them in a variety of ways. Anecdotal observations indicate a positive initial response and that teachers are developing mathematics subject matter knowledge and pedagogical content knowledge. These lists are providing personalised professional development and empowering the teachers to plan lessons to focus on these concepts as they can plan for what they want their students to understand rather than what they want them to do. There is potential for more formal data collection and research in the use of these conceptual development statements which I hope to gather and report on in the future.

References

Australian Curriculum, Assessment and Reporting Authority [ACARA]. (2016). *Foundation to Year 10 Curriculum: Mathematics*. Retrieved from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Baturo, A. (1998). The implication of multiplicative structure for students' understanding of decimal number numeration. In F. Biddulph & K. Carr (Eds), *Proceedings of the 20th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 90–97). Rotorua, NZ: MERGA.

Booker, G., Bond, D., Sparrow, L. & Swan, P. (2010). *Teaching primary mathematics* (4th ed.). Frenchs Forest: Pearson Australia.

Borko, H. & Putnam, R. T. (1995). Expanding a teacher's knowledge base: A cognitive psychological perspective on professional development. In T. R. Guskey & M. Huberman

(Eds), *Professional development in education: New paradigms and practices* (pp. 35–65). New York: Teachers College Press.

Carpenter, T. & Fennema, E. (1991). Research and cognitively guided instruction. In E. Fennema, T. Carpenter & S. Lamon (Eds), *Integrating research on teaching and learning mathematics* (pp. 1–16). Albany, GA: State University of New York Press.

Clarke, D., Cheeseman, J., Gervasoni, A., Gronn, D., Horne, M., McDonough, A., Montgomery, P., Roche, A., Sullivan, P., Clarke, B. & Rowley, G. (2002). *Early numeracy research project final report*. Melbourne: Mathematics Teaching and Learning Centre, Australian Catholic University.

Fuson, K. C. (1990). Issue in place-value and multi-digit addition and subtraction learning and teaching. *Journal for Research in Mathematics Education, 28*(2), 130–162.

Hattie, J. (2012). *Visible learning for teachers: Maximising impact on learning*. Oxon: Routledge.

Miura, I. T. & Okamoto, Y. (2003). Language supports for mathematics understanding and performance. In A. J. Baroody & A. Dowker, (Eds), *The development of arithmetic concepts and skills*. New Jersey: Lawrance Erlbaum Associates.

Piaget, J. (1952). *The child's conception of number*. New York: W. W. Norton.

Price, P. (1998). Year three students place value misconceptions: Another look at MAB. *Teaching mathematics in new times* (Annual Conference of the Mathematics Education Research Group of Australasia). Retrieved from http://www.merga.net.au/publications/counter.php?pub= pub_conf&id= 1803

Rogers, A. (2012). Steps in developing a quality whole number place value assessment for years 3–6: Unmasking the "experts". In 35th Annual Conference of the Mathematics Education Research Group of Australasia (pp. 1–8). MERGA, Inc.

Rogers, Angela (2013). Entering the 'new frontier' of mathematics assessment: Designing and trialling the PVAT-O (Online). In V. Steinle, L. Ball & C. Bardini (Eds). *Mathematics Education: Yesterday, today and tomorrow. Proceedings of the 36th annual conference of the Mathematics Education Research Group of Australasia*. Adelaide: MERGA.

Rogers, Anna. (2009). Place Value: Working with children at risk in Year 3. In C. Hurst, M. Kemp, B. Kissane, L. Sparrow & T. Spencer (Eds). *Mathematics: It's mine. Proceedings of the 22nd AAMT Biennial Conference*. Adelaide: Australian Association of Mathematics Teachers.

Ross, S. (1989). Parts, wholes and place value: a developmental view. *The Arithmetic Teacher, 36*(6), 47–51.

Ross, S. (2002). Place value: problem solving and written assessment. *Teaching Children Mathematics, 8*(7), 419–423.

Rubin, A. & Russell, S. (1992). Children's developing concepts of landmarks in the number system. In W. Geeslin & K. Graham (Eds), *Proceedings of the 16th Annual Conference of the International group for the Psychology of Mathematics Education* (Vol 3, pp. 136–140). Durham, NH: PME

Schmittau, J. & Vagliardo, J. J. (2006). Using concept mapping in the development of the concept of positional system. In *Concept maps: Theory, methodology, technology. Proceedings of the second international conference on concept mapping*. San José, Costa Rica: Universidad de Costa Rica (pp. 590–597).

Shulman, L. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4–14.

Shulman, L. & Grossman, P. (1988). *Knowledge growth in teaching: A final report to the Spencer Foundation*. Stanford, CA: School of Education, Stanford University.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R. & Warren, E. (2011). *Teaching mathematics: Foundations to middle years*. South Melbourne: Oxford.

Siemon, D., Izard, J., Breed, M. & Virgona, J. (2006). The derivation of a learning assessment framework for multiplicative thinking. In J. Novotná, H. Moraová, M. Krátká & N. Stehlíková (Eds), *30th Conference of the International Group for the Psychology of Mathematics Education* (pp. 113–120). Prague: PME.

YuMi Deadly Centre (2014). *YuMi deadly maths: Number. Prep to Year 9*. Kelvin Grove: Queensland University of Technology.

PROOF BY MATHEMATICAL INDUCTION: PROFESSIONAL PRACTICE FOR SECONDARY TEACHERS

GREGORY HINE

The University of Notre Dame Australia

gregory.hine@nd.edu.au

Mathematical induction is a proof technique that can be applied to establish the veracity of mathematical statements. This professional practice paper offers insight into mathematical induction as it pertains to the Australian Curriculum: Mathematics (ACMSM065, ACMSM066) and implications for how secondary teachers might approach this technique with students. In particular, literature on proof—and specifically, mathematical induction—will be presented, and several worked examples will outline the key steps involved in solving problems. After various teaching and learning caveats have been explored, the paper will conclude with some mathematical induction example problems that can be used in the secondary classroom.

Introduction

A significant amount of mathematics involves the examination of patterns. Many of these patterns are concerned with generalisations about sequences and series. Mathematical induction is a method of proof argument that is based in recursion, and it is used for proving conjectures which claim that a certain statement is true for integer values of some variable. One idea that has been used to illustrate this method is to imagine a number of dominoes lined up in a row (Peressini et al., 1998). These authors suggest that for each integer $k \geq 1$, if the kth domino falls over then it will cause the $(k + 1)^{st}$ domino to fall over as well. Furthermore, it could be argued specifically that if the first domino is pushed over, then all remaining dominoes would also fall.

If we suppose that for each positive integer n, $S(n)$ is a statement written in terms of n, then the principle of mathematical induction can be explained generally in two steps:

1. If $S(1)$ is true, and
2. for all integers $k \geq 1$, the assumption that $S(k)$ is true implies that $S(k + 1)$ is true,

then $S(n)$ is true for all positive integers n.

In other words, we commence the proof method through a verification of Step 1 (the *initial step*), or by pushing over the first domino. Then, we assume that $S(k)$ is true for a particular but arbitrarily chosen integer $k \geq 1$, known as the inductive assumption. In Step 2 (the *base induction step*) we show that the supposition that $S(k)$ is true implies that $S(k + 1)$ is true. Compared with the domino line-up, Step 2 corresponds to the assumption that if the k^{th} domino falls then so will the $(k + 1)^{st}$ domino.

The importance of proof in mathematics education

Mathematical proof involves following a logical way to explain a hypothesis and to offer a cogent explanation of how deductive reasoning has been used to reach a conclusion. (Hanna, 1995; Tall, 1998). During the proving process, proofs require us to create "a sequence of steps, where each step follows logically from an earlier part of the proof where the last line is the statement being proved" (Garnier & Taylor, 2010, p. 50). The concept of proof is considered to be central to the discipline of mathematics, and because of this centrality, scholars have argued that proof should feature prominently in mathematics education (Ball et al., 2002; Baştürk, 2010; Siemon et al., 2015). Specifically, proof is recognised as an essential tool for promoting mathematical understanding in students (Ball et al., 2002; Reid, 2011) and for providing educators with insight about how students learn mathematics (Wilkerson-Jerde & Wilensky, 2011). Güler (2016) proposed that proof is important in mathematics education for various reasons, in that it: improves skills in problem solving, persuasive argumentation, reasoning, creativity and mathematical thinking. Moreover, proof forms the basis of mathematics, enables mathematical communication to transpire, and prevents rote learning of information.

Mathematical induction

Mathematical induction is considered one of the most powerful tools for proving statements in discrete mathematics (Ashkenazi & Itzkovitch, 2014). While there is endless scope for the types of problems mathematical induction can be applied to, three popular 'types' of problems are used by teachers when teaching this type of mathematical proof to secondary students. These problem types include: General series, divisibility and implication. Each of these types will now be presented as a worked example.

General series

Let us propose that we are interested in finding a general statement to explain the sum of n consecutive odd integers starting at 1. If we tabulate our findings for the first 10 natural or counting numbers, and their partial sums, we have:

Table 1. Counting numbers and their sums, $1 \leq n \leq 10$.

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
T_n	1	3	5	7	9	11	13	15	17	19
S_n	1	4	9	16	25	36	49	64	81	100

It should be noted that the row T_n represents the nth odd integer, and the row S_n is the sum of the first n odd integers. One interesting pattern that can be observed is that the last row of the table, S_n, shows all integers n^2 for $n \geq 1$. A cursory comparison between the three rows reveals that the sum of the first n odd numbers appears to be the square of n. In making this statement, we have arrived at a conjecture—which is the first step in working towards a theorem—but we may not even know if the statement is true! The following worked example provides a precise mathematical statement of the result we are trying to prove.

Worked example 1: General series

Prove by mathematical induction that for all integers $n \geq 1$

$$S(n): \; 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

Worked solution

1. Initial step: We need to show that the conjecture is true for a small value of n, e.g., $n = 1$. Substituting this value into the series we have:

$$1 = 1^2$$

which is clearly true
∴ we have shown that $S(1)$ is true

2. Inductive Step: Here we assume the statement (inductive hypothesis)

$$S(k): \; 1 + 3 + 5 + \cdots + (2k - 1) = k^2 \tag{1}$$

is true for a fixed but arbitrary value of $k \geq 1$ and verify that the statement

$$S(k+1): \; 1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2 \tag{2}$$

Looking back at (1), we can see that the series $1 + 3 + 5 + \cdots + (2k - 1)$ exists in (2). We therefore substitute k^2 into (2) for $1 + 3 + 5 + \cdots + (2k - 1)$, and algebraically rewrite the Left Hand Side (LHS) until it equals the Right Hand Side (RHS).

$$\begin{aligned} \text{LHS} &= 1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\ &= 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) \\ &= k^2 + (2k + 1) \\ &= (k + 1)^2 \\ &= \text{RHS} \end{aligned}$$

Conclusion: Because we have verified the initial and inductive steps we can conclude by induction that the statement

$$\mathrm{S(n)}: \; 1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

is true for all integers $n \geq 1$.

Worked example 2: Divisibility

Prove by mathematical induction that for all integers $n \geq 1$

$$S(n): \ 3^{2n} - 1 \text{ is divisible by } 8.$$

Worked solution
1. Initial step: We need to show that the statement $S(1)$ is true. Substituting $n = 1$ into the expression gives us:

$$3^{2(1)} - 1 = 3^2 - 1 = 9 - 1 = 8$$

which is clearly divisible by 8.
Therefore, $S(1)$ is true.

2. Inductive step: We assume that the statement (inductive hypothesis)

$$3^{2k} - 1 \text{ is divisible by } 8 \tag{1}$$

is true for a fixed and arbitrary value *of* $k \geq 1$. We must verify that the statement

$$S(k+1): \ 3^{2(k+1)} - 1 \text{ is divisible by } 8$$

is true. Now, we manipulate the expression $3^{2(k+1)} - 1$ using algebraic rules until it becomes divisible by 8.

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k} \times 3^2 - 1 \\ &= 3^{2k}(9) - 1 \\ &= 3^{2k}(8+1) - 1 \\ &= 8 \times 3^{2k} + 3^{2k} - 1 \qquad (2) \end{aligned}$$

Now because from (1) we have assumed that $3^{2k} - 1$ is divisible by 8, there are two terms which are divisible by 8—one proven through clear algebra, and the other via an assumption from the inductive step. As such, both terms of (2) are divisible by 8 and therefore so is their sum. In other words, $S(k+1)$ is true.

Worked example 3: Inequalities

Using mathematical induction, prove that for all integers $n \geq 3$

$$S(n): \ 2^n > 2n + 1$$

Worked solution
1. Initial step: We need to show that the statement $S(3)$ is true. Substituting $n=3$ into this expression gives:

$$\begin{aligned} 2^3 &> 2(3) + 1 \\ 8 &> 7 \end{aligned}$$

which is clearly true.

Therefore, $S(3)$ is true.

2. Inductive step: We assume that the statement (inductive hypothesis)

$$S(k): \; 2^k > 2k + 1 \tag{1}$$

is true for a fixed and arbitrary value of k ≥ 3. We must verify the statement

$$S(k+1): \; 2^{k+1} > 2(k+1) + 1 \tag{2}$$

We now manipulate both sides of (1) to transform it into (2). In other words, the inductive statement will be manipulated algebraically so the values of n = k have been transformed into n = k + 1. Once we have done this, by implication we will have shown that the statement will remain true for all values of k and the very next value after k. Ideally, the 'finished product' will look like:

$$2^{k+1} > 2(k+1) + 1$$

Some annotations have been included on the RHS of the inequality to assist in following the steps in working out.

$2^k \times 2 > 2(2k+1)$	Multiply both sides by 2
$2^{k+1} > 4k + 2$	Simplify
$2^{k+1} > 2k + 2k + 2$	Re-express the RHS terms
$2^{k+1} > 2k + 2 + 2\text{k}$	Rearrange the RHS terms
$2^{k+1} > 2(k+1) + 2\text{k}$	Factorise the first two terms

Now, as the original problem stated, $n \geq 3$ which implies that the LHS of the original statement $2n + 1 > 1$. In particular, if we substitute $n = 3$ into the LHS we obtain a value of 7, which is clearly greater than 1. As such we can create a concatenated inequality statement:

$$2^{k+1} > 2(k+1) + 2\text{k} > 2(k+1) + 1$$
$$\therefore 2^{k+1} > 2(k+1) + 1$$

In this way, the inductive step $S(k)$ has implied $S(k+1)$ is true.

Some caveats associated with mathematical induction

A review of literature on mathematical induction reveals that this method is difficult to teach for a variety of reasons (Ashkenazi & Itzkovitch, 2014; Harel, 2002; Stylianides et al., 2007). To commence, Ashkenazi and Itzkovitch (2014) contended that although secondary school and university students can successfully apply this proof method to statements of the kind they are accustomed to, they do not understand the correctness of the proof. Put another way, these authors suggest that most students learn how to use the method mechanically; such learning does not foster a deep understanding of the correctness of the method and ultimately contributes to a failure to solve problems of a different style (Ashkenazi & Itzkovitch, 2014). Echoing the contention of these authors contention, both Harel (2002) and Stylianides et al. (2007) asserted that undergraduate university students often display both a fragile knowledge on mathematical induction and a propensity to follow the steps without understanding what they are doing. In his analysis of students' attempts at mathematical induction, Harel (2002) further identified two specific difficulties experienced by students. First, students tended to consider mathematical induction as a case of circular reasoning as they believe that the proof assumes $S(n)$ is true for all positive integers. Second, students demonstrated a belief that the general argument for mathematical induction can be derived from a number of particular cases, rather than proving for all cases.

Divisibility

An alternative method that can be used to prove induction divisibility problems (such as Worked example 2) requires the use of two assumptions. Because the strength of a mathematical argument relies on the extent to which assumptions are minimised, the method shown below should be treated cautiously and avoided. If we recommence Worked Example 2 at the inductive step, it could be written that:

$$3^{2k} - 1 = 8A \text{ for some integer } A,\ A \geq 1$$

We can rearrange this inductive assumption as $3^{2k} = 8A + 1$ (1), which will be used when manipulating the statement $S(k+1)$. Herein:

$$\begin{aligned} 3^{2(k+1)} - 1 &= 3^{2k+2} - 1 \\ &= 3^{2k} \times 3^2 - 1 \\ &= 9 \times 3^{2k} - 1 \\ \text{We now substitute (1):} \quad &= 9(8A + 1) - 1 \\ &= 72A + 9 - 1 \\ &= 72A + 8 \\ &= 8(9A + 1) \end{aligned}$$

which is clearly a multiple of 8.

Having completed the necessary algebraic manipulations to reach a final statement which is divisible by 8, we are able to conclude that the conjecture is indeed true. However, looking back at the Inductive Step, we assumed that not only was the conjecture true for $k \geq 1$ but we also assumed that it was equal to a 8A (a multiple of 8) for $A \geq 1$. As such, the inductive assumption itself rested upon an assumption, which is a practice that should be avoided. Rather, to fulfil the logical steps of the proof we need to actually use the inductive assumption of the proof (i.e., $3^{2k} - 1$) in the final stages and not a substitute.

Conclusion

The purpose of this paper was to offer insight to educators about proof by mathematical induction as it pertains to the Australian Curriculum: Mathematics. In particular, this method of proof has been outlined in a step-by-step fashion, and some worked examples have been offered to amplify these steps and the theoretical approach overall.

Additionally, a cursory review of literature has revealed how scholars have championed the place of proof in a mathematics curriculum. In a study where mathematics professors were asked to evaluate and score undergraduate university students' completion of proofs (an example of mathematical induction was Task 4), these professors acknowledged that the most important characteristics of a well-written proof are logical correctness, clarity, fluency, and demonstration of understanding of the proof (Moore, 2016). It is the author's hope that this paper will be useful to mathematics educators within Australia—and perhaps internationally—as they model to secondary students how to apply the principles of mathematical induction to statements. Moreover, it is hoped that as students strive to master those characteristics of well-written proofs, their efforts will be underscored by a demonstration of procedural understanding.

Examples to try with secondary students

Use mathematical induction to prove that for all positive integers n:

1. $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$
2. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$
3. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

Use mathematical induction to prove that for all positive integers n:

4. $5^n + 3$ is divisible by 4
5. $3^{4n} - 1$ is divisible by 80
6. $4^n - 1$ is divisible by 3

Use mathematical induction to prove the following statements for all natural numbers $n \geq 5$:

7. $2^n > n^2$
8. $4n < 2^n$
9. $1 \times 2 \times 3 \times \ldots \times (n-1) > 2^n$

References

Ashkenzi, Y. & Itzkovitch, E. (2014). Proof by mathematical induction. *International Journal of Innovation and Research in Educational Sciences*, *1*(3), 2349–5219.

Ball, D.L., Hoyles, C., Jahnke, H.N. & Movshovitz-Hadar, N. (2002). The teaching of proof. In *Proceedings for the International Congress of Mathematicians* (pp. 907–920). Beijing, China: Higher Education Press.

Baştürk, S. (2010). First-year secondary school mathematics students' conceptions of mathematics proofs and proving. *Educational Studies*, *36*(3), 283–298.

Garnier, R. & Taylor, J. (2010). *Discrete mathematics: Proofs, structures and applications* (3rd ed.). Boca Raton, FA: Taylor & Francis.

Guler, G. (2016). The difficulties experiences in teaching proof to prospective mathematics teachers: Academician views. *Higher Education Studies*, *6*(1), 145–158.

Hanna, G. (1995). Challenges to the importance of proof. *For the Learning of Mathematics*, *15*(3), 42–50.

Harel, G. (2002). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campbell & R. Zaskis (Eds). *Learning and Teaching Number Theory: Research in Cognition and Instruction* (pp. 185–212). New Jersey: Ablex Publishing Corporation.

Moore, R.C. (2016). Mathematics professors' evaluation of students' proofs: A complex teaching practice. *International Journal of Research in Undergraduate Mathematics Education*, *2*, 246–278.

Peressini, A.P., McConnell, J.W., Usiskin, Z., Epp, S.S., Ahbel, N.P., Witonsky, D., Hollowell, K.A., Brown, S., Ellis Jnr, W., Sorteberg, J. (1998). *Precalculus and discrete mathematics* (The University of Chicago School Mathematics Project). Glenview, Illinois: Scott Foresman Addison Wesley.

Siemon, D., Beswick, K., Brady, K., Clark, J., Faragher, R., Warren, E. (2015). *Teaching mathematics: Foundations to middle years* (2nd ed.). South Melbourne: Oxford University Press.

Styliandes, G.J., Styliandes, A.J., Phillippou, G.N. (2007). Preservice teachers' knowledge of proof by mathematical induction. *Journal of Mathematics Teacher Education*, *10*, 145–166.

Reid, D. (2011). Understanding proof and transforming teaching. In Wiest, L.R. & Lamberg, T. (Eds). *Proceedings of the 33rd Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*. Reno, NV: University of Nevada: Reno.

Tall, D. (1998). *The cognitive development of proof: Is mathematical proof for all or for some?* Conference of the University of Chicago School Mathematics Project, USA.

Wilkerson-Jerde, M.H. & Wilensky, U.J. (2011). How do mathematicians learn math? Resources and acts for constructing and understanding mathematics. *Educational Studies in Mathematics*, *78*, 21–43.

QUESTIONS TO REVEAL AND GUIDE FRACTIONAL THINKING

CATHERINE PEARN, ROBYN PIERCE & MAX STEPHENS

The University of Melbourne

cpearn@unimelb.edu.au

Fractional thinking with its interconnected constructs is key to understanding number and multiplicative thinking. Identifying students' thinking and choosing questions to develop understanding and proficiency is a challenge for teachers. This paper examines the results of six tasks completed by 570 middle years' students from Victorian schools. These tasks, administered as part of a Fraction Screening Test, not only reveal students' thinking but also provide examples for teachers of the type of questions that will guide students forward in their thinking. In particular, questions that prompt 'reverse thinking' are often overlooked but are powerful in challenging students to generalise.

Introduction

Kieren (1980) suggested that there are five interconnected sub-constructs or interpretations of fractions that are both mathematically and psychologically dependent. These interpretations include part/whole, measure, operator, quotient and ratio. Conceptual understanding of fractions incorporates the ability to make connections within, and between, these different interpretations. Kieren (1980) suggested that difficulties experienced by children solving rational number tasks arise because rational number ideas are sophisticated and different from natural number ideas and that children have to develop the appropriate images, actions, and language to precede the formal work with fractions, decimals, and rational algebraic forms. Empson, Levi and Carpenter (2010) suggest that students should develop and use computational procedures using relational thinking to integrate their learning of whole numbers and fractions.

Many researchers believe that much of the basis for algebraic thought rests on a clear understanding of rational number concepts (Kieren, 1980; Wu, 2001) and the ability to manipulate common fractions. According to Wu (2001) the ability to efficiently manipulate fractions is "vital to a dynamic understanding of algebra" (p. 17). The National Mathematics Advisory Panel (NMAP, 2008) stated that the conceptual understanding of fractions and fluency in using procedures to solve fraction problems are central goals of students' mathematical development and are the critical foundations for algebra learning.

Siegler et al.'s (2012) analysis of longitudinal data from both the United States and United Kingdom showed that competence with fractions and division in fifth or sixth grade is a uniquely accurate predictor of their attainment in algebra and overall mathematics performance five or six years later when other factors were controlled. They controlled for factors such as whole number arithmetic, intelligence, working memory, and family background.

Pearn and Stephens (2016) identified Year 8 proficient fractional thinkers as students who demonstrated a capacity to represent fractions in various ways, and to use reverse thinking with fractions to solve problems. Their research also suggested that effective reverse thinking depends on a capacity to apply multiplicative operations to transform the value of known fractions to make a whole. Pearn and Stephens (2007) used a Fraction Screening Test and Fraction Interview using number lines to probe students' understanding of fractions as numbers. Results from these assessments showed that successful students demonstrated easily accessible and correct whole number knowledge and knew relationships between whole and parts. A number line representation of number quantity has been shown in cognitive studies to be particularly important for the development of numerical knowledge and a precursor of children's academic success (Siegler & Booth, 2004).

This paper is about the sort of questions that teachers need to ask in order to prompt structural thinking about fractions. It will focus on question sequences that drive the need for generalisation - not just finding an answer to one particular question but developing patterns of thinking that apply to similar questions.

The Australian context

According to the rationale given for the *Australian Curriculum: Mathematics* (ACARA, 2016) the mathematics curriculum "focuses on developing increasingly sophisticated and refined mathematical understanding, fluency, reasoning, and problem-solving skills. These proficiencies enable students to respond to familiar and unfamiliar situations by employing mathematical strategies to make informed decisions and solve problems efficiently."

In Table 1 are examples of Content Descriptors from The Australian Curriculum: Mathematics (ACARA, 2016) for Years 5–8. The Content Descriptors for Years 5–7 include those that refer to fraction calculations and the use of number lines to locate and represent fractions. The Content Descriptors for Year 8 include solving problems involving rates and ratios.

The quote from the rationale of the Australian Curriculum: Mathematics (ACARA, 2016) above suggests that students need "to respond to familiar and unfamiliar situations by employing mathematical strategies"; however, there are no references in the Content Descriptors about starting with a known fractional part and requiring students to find the whole (see Pearn & Stephens, 2007). In fact there is very little attention in these Content Descriptors to helping students to understand the structural properties of fractions that will be necessary to deal successfully with "a range of problems involving rates and ratios" as stated in ACMNA188. Essentially this requires students to be able to deal multiplicatively with fractions.

Table 1. Examples of Content Descriptors
(Australian Curriculum: Mathematics, ACARA, 2016).

Year	Content Descriptors
5	Compare and order common unit fractions and locate and represent them on a number line (ACMNA102)
	Investigate strategies to solve problems involving addition and subtraction of fractions with the same denominator (ACMNA103)
6	Compare fractions with related denominators and locate and represent them on a number line (ACMNA125)
	Solve problems involving addition and subtraction of fractions with the same or related denominators (ACMNA126)
7	Compare fractions using equivalence. Locate and represent positive and negative fractions and mixed numbers on a number line (ACMNA152)
	Solve problems involving addition and subtraction of fractions, including those with unrelated denominators (ACMNA153)
	Multiply and divide fractions and decimals using efficient written strategies and digital technologies (ACMNA154)
8	Solve a range of problems involving rates and ratios, with and without digital technologies (ACMNA188)

Pearn & Stephens (2016) argue that effective reverse thinking depends on a capacity to apply multiplicative operations to find the whole when details of a fractional part are known. This position guided the selection of fraction tasks from the Fraction Screening Test that forms the basis for this study.

This study

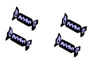
This paper examines the results of 570 students for five fraction tasks from the Fraction Screening Test (Pearn & Stephens, 2014). These students came from ten Victorian schools. Eight of the schools were from metropolitan Melbourne, whereas the other two schools were from regional Victoria. Table 2 shows the number of students from each year level.

Table 2. Number of students from each year level.

Year 5	Year 6	Year 8	Total
187	265	118	570

The Fraction Screening Test (Pearn & Stephens, 2016) is divided into three parts. Part A includes 12 routine fraction tasks that include topics such as equivalent fractions, ordering fractions and recognising simple representations. For example, Figure 1 includes two of the 12 tasks from Part A. Task A4 includes the familiar fraction of one-half and a picture of four lollies. Task A12 includes an unfamiliar fraction of one-seventh and has no diagram. Both these tasks deal with unit fractions where students are expected to find the whole. The first could be solved additively by adding another four lollies or multiplicatively by doubling. The second task is less attractive to solve additively and was intended to be solved multiplicatively.

A4.
This is one -half of the lollies I started with.



How many did I start with? ___

A12.
To buy a new workbook I spent $4.
This is $\frac{1}{7}$ of what I saved up.
How much did I save up? ____

Figure 1. Two examples of Part A tasks.

Part B includes five number line tasks. Task B2 in Figure 2 requires the students to mark where the number 75 would be given the distance from zero to 25. In order to solve this task successfully, students need to recognise 75 as a multiple of 25. They are expected to explain that in their answer to part b.

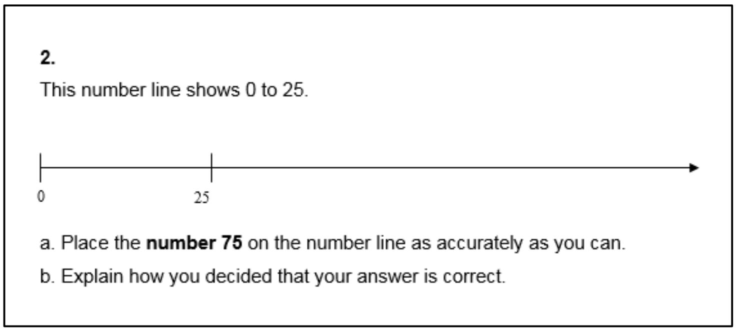


Figure 2. Task B2 from Part B.

Task B4 from Part B shown in Figure 3 is similar to Task B2 as students are required to mark where the number one would be given the distance from zero to one-third. In order to solve this question additively or multiplicatively students need to recognise the relationship between one-third and the whole.

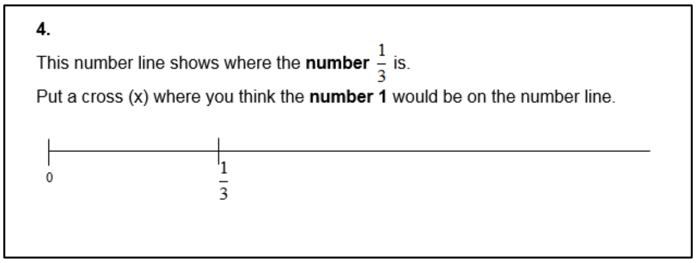


Figure 3. Task B4 from Part B.

Figure 4 shows Task B5 from Part B and tests whether students believe three-quarters is 'nearly one' or can determine that the correct response is C and give an appropriate and reasoned answer as to why Response C is the correct one. This question also requires students to understand the structural relationship between three-quarters and one whole. That is, one-quarter more needs to be added; and to find a measure for one-quarter students need to partition three-quarters into three parts. Alternatively, having found one-quarter, students could multiply that by four.

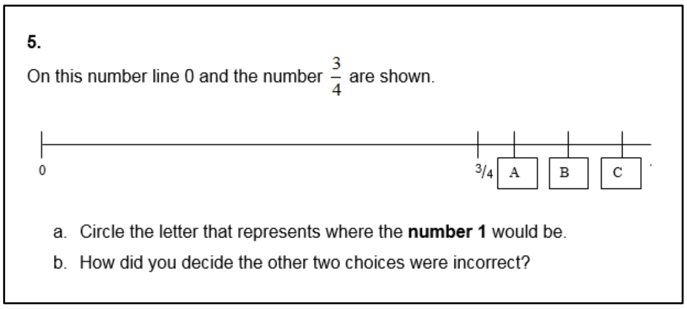


Figure 4. Task B5 from Part B.

Part C included three questions that explicitly required students to use reverse thinking using less familiar fractions (see Figure 5). Student performance on these three tasks was reported by Pearn and Stephens (2016). Two student performances on the task shown below in Figure 5, where no diagram is given, are discussed later (see Figures 6 and 7). Whether finding the number of CDs additively or multiplicatively, it is essential to find the equivalent number of CDs represented by one-seventh and then scale up to find the number of CDs in Kaye's collection, which is 21 CDs.

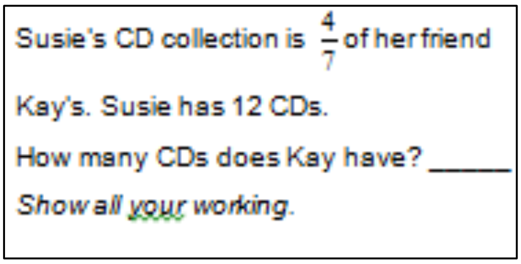


Figure 5. Example of Part C task.

The six tasks shown in Figures 1–5 either start with the part and require students to find the whole as in Figures 1 and Figure 5; or have number lines marked with a given fraction as in Figures 3 and 4, where students are asked to show where the number one would be marked.

Results

The two tasks shown in Table 3 were successfully answered by at least 80% of the students. These two tasks involve familiar unit fractions and small numbers representing quantities. The same group of students were far less successful with the three tasks shown in Figures 2, 3 and 4 using number lines where students needed to understand structural relationships as ratios. The reverse fraction task shown in Figure 5 was the most difficult for students in Year 6 and Year 8.

Most students confidently and correctly answered Task A4 in which were included a familiar fraction, one-half, and a picture of four lollies. Table 3 shows that approximately 95% of Year 5 students, 96% of Year 6 students and 98% of Year 8 students correctly identified the whole was eight lollies given that one-half of the whole was represented by four lollies (Task A4, Figure 1).

Overall 66% of the total number of students correctly answered Task A12 with another 16.3% giving the correct answer without the dollar sign. This means that overall about 82% of the students could actually complete the calculation with this fraction task without a diagram. In Table 3 the numbers in brackets are the percentages of students who correctly calculated the cost of the workbook for task A12 but did not include the dollar sign.

Table 3. Percentage of correct responses to Part A from Fraction Screening Test tasks.

Task	Year 5 (n = 187)	Year 6 (n = 265)	Year 8 (n = 118)	Total (n = 570)
A4. This is one-half of the lollies I started with. How many did I start with? ___ (Shown in Figure 1)	94.7	95.5	97.5	96
A12. To buy a new workbook I spent $4. This is $\frac{1}{7}$ of what I saved up. How much did I save up? (Shown in Figure 1)	60.4 (+ 22)	67.9 (+ 14)	72 (+13)	66 (+ 16.3)

Table 4 shows the percentage of students who successfully completed the three number line tasks. These tasks do not require students to mark a fraction of the number line but instead need them to mark in a larger whole number (Task B2, Figure 2) or the number one when given the number one-third (Task B4, Figure 3), or when given the fraction three-quarters (Task B5, Figure 4).

Table 4 shows that nearly 57% of the students were able to place the number 75 accurately by marking three times the distance from zero to 25. However, nearly 35% of the total number of students placed number 75 three-quarters of the length of the whole line as though the line were drawn from zero to 100. Their justifications incorrectly implied that the end point of the line was 100.

Table 4. Percentage of correct responses to Part B Fraction Screening Test tasks.

Task	Year 5 (n = 187)	Year 6 (n = 265)	Year 8 (n = 118)	Total (n = 570)
B2a. This number line shows 0 to 25. Place the number 75 on the number line as accurately as you can. (See Figure 2)	52.9	56.2	64.4	56.8
B4. This number line shows where the number $\frac{1}{3}$ is. Put a cross (x) where you think the number 1 would be on the number line. (See Figure 3)	54.4	63	76.3	63
B5. On this number line 0 and the number $\frac{3}{4}$ are shown. Circle the letter that represents where the number 1 would be. (See Figure 4)	70	76.2	85.6	76

Overall 63% of the total number of students accurately placed the number one on the number line when they were given the distance from zero to one-third. Nearly 16% of all students marked the number one at the end of the number line. For Task B5 76% of the total number of students chose the correct letter C to indicate where the number one would be. Approximately ten percent of the students chose each of A or B.

Table 5 below examines the performance of students from two of the ten schools referred to earlier. At the time of writing, Year 5 data for this question were not available. However, the data below are from intact classes in the two schools concerned, and show how difficult this task is for nearly half of the Year 8 students.

Table 5. Percentage of correct responses to reverse fraction task (Figure 5).

Task (Figure 5)	Year 6 (n = 67)	Year 8 (n = 118)	Total (n = 185)
Susie's CD collection is $\frac{4}{7}$ of her friend Kay's. Susie has 12 CDs. How many CDs does Kay have? _____ *Show all your working.*	57	54	54.9

Figure 6 shows a typical response using additive thinking. In the first place the student has deduced that one-seventh of the collection must be equivalent to three CDs. This student then shows that three-sevenths is equivalent to 9 CDs, so adds the nine to the 12 CDs that comprised four-sevenths to get a total of 21. This shows clear understanding of the relationship between four-sevenths and one-seventh and how the whole can be thought of as four-sevenths and three-sevenths combined.

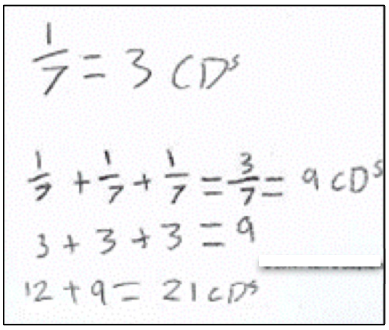


Figure 6. Additive solution to reverse fraction task in Figure 5.

Figure 7 illustrates a more complete multiplicative solution. In the first and second lines the student divides 12 by four to show that three CDs is equivalent to one-seventh. This is stated (idiosyncratically?) in the second line as an equivalence statement between a fraction and its related numerical quantity. In the third line the student scales up from one-seventh to a whole by multiplying the equivalent quantity by seven.

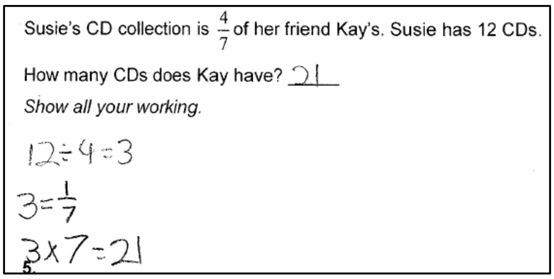


Figure 7. Multiplicative solution to reverse fraction task in Figure 5.

Implications for teaching

For teachers these six questions are important as representing the kind of questions that need to be asked. First, they reveal students' thinking and show where students are performing. Second and equally important, they help to guide students' thinking. These questions are intended to act as prompts to guide teachers to move students towards generalised understanding and proficiency. Developing and guiding students' structural understanding of fractions requires them to be able to use both multiplicative and additive methods. Later work on ratios and proportion will clearly preference the ability to use multiplicative thinking. Teachers need to recognise that additive thinking, whilst effective, should be seen as a bridge to more confident multiplicative thinking.

This study shows the importance of including work on number lines, either involving fractions where the whole is unknown, or whole numbers that are multiples or fractional parts. Prior work on locating and representing fractions on a number line involves partitioning a length to find fractional parts. But a structural understanding of fractions also requires students to be able to move from a known fraction on a number line to find an unknown whole.

Explicit inclusion of reverse fraction tasks similar to that shown in Figure 5 are necessary to strengthen and guide understanding of ratios and proportion. In later work on solving linear equations with rational coefficients students need to be fluent in simplifying fractional values of an unknown using procedures similar to those discussed in this paper.

References

Australian Curriculum, Assessment and Reporting Authority [ACARA] (2016). *Australian Curriculum: Mathematics. V8.3*. Retrieved from http://www.australiancurriculum.edu.au/mathematics/curriculum/f-10?layout=1

Empson, S., Levi, L. & Carpenter, T. (2011). The algebraic nature of fractions: Developing relational thinking in elementary school. In J. Cai & E. Knuth (Eds), *Early algebraization: A global dialogue from multiple perspectives* (pp. 409–428). Heidelberg: Springer.

Kieren, T. E. (1980). The rational number construct: Its elements and mechanisms. In T. E. Kieren (Ed.), *Recent research on number learning* (pp. 125–149). Columbus: Ohio State University. (ERIC Document Reproduction Service No. ED 212 463).

National Mathematics Advisory Panel [NMAP] (2008). *Foundations for success: The final report of the National Mathematics Advisory Panel*. US Department of Education. Retrieved from http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf

Pearn, C. & Stephens, M. (2007). Whole number knowledge and number lines help develop fraction concepts. In J. Watson & K. Beswick (Eds), *Mathematics: Essential research, essential practice. Proceedings of the 30th annual conference of the Mathematics Education Research Group of Australasia*, pp. 601–610. Adelaide: MERGA.

Pearn, C. & Stephens (2016). Competence with fractions in fifth or sixth grade as a unique predictor of algebraic thinking? In B. White, M. Chinnappan & S.Trenholm (Eds), *Opening up mathematics education research. Proceedings of the 39th Annual Conference of the Mathematics Education Research Group of Australasia*, pp. 519–526. Adelaide: MERGA.

Siegler, R., Duncan, G., Davis-Kean, P., Duckworth, K., Claessens, A., Engel, M., Susperreguy, M. & Chen, M. (2012). *Early Predictors of High School Mathematics Achievement*. Retrieved from http://pss.sagepub.com/content/early/2012/06/13/0956797612440101

Siegler, R. S. & Booth, J. L. (2004). Development of Numerical Estimation in Young Children. *Child Development*, *75*(2), 428–444. doi:10.1111/j.1467-8624.2004.00684.

Wu, H. (2001). How to prepare students for algebra. *American Educator*, *25*(2), 10–17.

A CLASSROOM SCENARIO FOR PROFESSIONAL LEARNING IN STATISTICS

JANE WATSON
University of Tasmania
jane.watson@utas.edu.au

The aim of this workshop is to trial a professional learning activity developed as part of the Reframing Mathematical Futures II Project (RMIT and AAMT). The activity models a classroom lesson based on three of the Big Ideas underpinning statistics: Variation, Expectation, and Distribution. The lesson is created with cartoon characters in PowerPoint. Using student responses from research, participating teachers view the classroom interaction and discuss how the classroom teacher could scaffold students' progress. This is followed by the use of an Australian open data set to illustrate the Big Ideas in a related authentic context.

Introduction

The Reframing Mathematical Futures II project (RMFII; Siemon, 2016) is extending previous work that used rich assessment tasks to identify an evidence-based learning and assessment framework for multiplicative thinking in the middle years (Siemon, Breed, Dole, Izard & Virgona, 2006). The RMFII project is using a similar approach to develop evidence-based learning progressions in mathematical reasoning, specifically, algebraic, spatial and statistical reasoning. Based on literature reviews and the big ideas in each area, a range of rich assessment tasks have been developed and trialled, with the results used to identify learning progressions in each area. Targeted teaching advice and resources are being developed for each learning progression for teachers to use. Ultimately, the materials and professional learning associated with the project outcomes will be disseminated through the AAMT's Dimensions portal.

This presentation is a result of the work in statistics based on the big ideas of variation, expectation, and distribution. Variation is the foundation big idea that underpins all of statistics: without variation to characterise differences there would be no statistics (Moore, 1990). Expectation arises from variation when an attempt is made for example to describe a typical value or a chance. Distribution is the lens through which variation is viewed, identifying and describing it in order to look for and confirm expectation. Distribution underlies data representation for samples and populations. These are the big ideas used in conjunction with randomness and informal inference for decision-making (see http://topdrawer.aamt.edu.au/Statistics/Big-Ideas).

Learning progression

Following the work of Watson and Callingham (2003) and Callingham and Watson (2005), and follow-up analysis from the RMFII data, an eight-zone hypothetical learning progression was developed (Watson & Callingham, 2017). This is shown in Table 1. In a classroom setting, a teacher is likely to have students displaying understanding across a range, if not all, of these zones. In individual encounters with students, it may be possible to detect the student's current learning zone and choose a question or response that will help scaffold the student to reach a higher zone. Some examples of scaffolding questions are given in Watson (2016). In a whole-class discussion, however, with students operating in many zones, the teacher is faced with on-the-spot decisions on how to proceed.

Table 1. Hypothetical learning progression for statistical reasoning with selected examples (from Watson & Callingham, 2017).

Big idea	**Variation in expectation**	**Variation in distribution**	**Variation in inference**
	Idiosyncratic response or single procedural focus		
Zone 1	Uncertainty expressed as 50%	Reads single value on graph	Ignores context
	Considers aggregated information but without recognising value		
Zone 2	Anything can happen	Describes isolated features of a graph	One characteristic of a sample
	Emerging statistical appreciation but without explanation		
Zone 3	Claims for average without justification	Elaborated physical description of graphs	Choose "all" for sample
	Recognises influence of variation but interprets inappropriately		
Zone 4	Rejects "luck"; suggests unlikely	Does not distinguish scale in graph reading	Recognises sample but not its bias
	Straightforward explanation and simple numerical justification		
Zone 5	Orders chance phrases correctly	Appropriate attention to graph details	Partial recognition of sample requirements
	Informal appreciation of uncertainty and variation in choices		
Zone 6	Recognises outlier	Recognises correct variation in graphs	Suggests random sampling
	Makes inferences across ideas using proportional reasoning		
Zone 7	Creates appropriate probability distribution	Creates hypothesis based on data	Criticises sample size and bias
	Integrates proportional, statistical, and contextual reasoning		
Zone 8	Correct association in 2-way tables	Conclusion with both positives & negatives	Includes human/ psychological component

Scaffolding in classroom discussion

Based on the previous research and data from RMFII students, one of the professional learning (PL) resources developed envisages a classroom context, reviewing the three big ideas of statistics, setting a task based on the weather, and moving to Australian national data to challenge students' critical thinking further. The students' responses in reviewing the ideas and in task discussion are all taken from actual student responses. The leader of the PL stops throughout the presentation of the classroom scenario asking participating teachers to suggest (i) what their students might say and (ii) how they would respond to their classes at this point. Some suggestions are then provided from the teacher or other students in the scenario. The final section of the PL illustrates the possibilities for continuing to reinforce the big ideas through the introduction of weather data from Australia's capital cities (see www.bom.gov.au/climate/data). Although not linked directly to the hypothetical learning progression during the presentation, further discussion can help teachers to place their students' on the progression with respect to the scenario context.

The presentation used here is created in PowerPoint with cartoon characters telling the classroom story. The reader needs to imagine the dialogue and responses being presented on the forward click of a mouse (or key), not all at once as shown in the figures here. They can be managed at the discretion of the presenter depending on what the teachers in the audience wish to contribute. The pace depends on the context of the PL session, as does further work linking teachers' beliefs about their students' abilities to progress to higher zones of the hypothetical learning progression.

Part 1: Review of the big ideas

In the classroom a teacher may start a review of the big ideas, for example, by asking students about "things that vary" leading to asking for a definition (Watson, 2016) or start with a reminder of the definition and ask students for examples. The latter is shown in Figure 2 as an introduction to the context for the lesson, focussing on the weather.

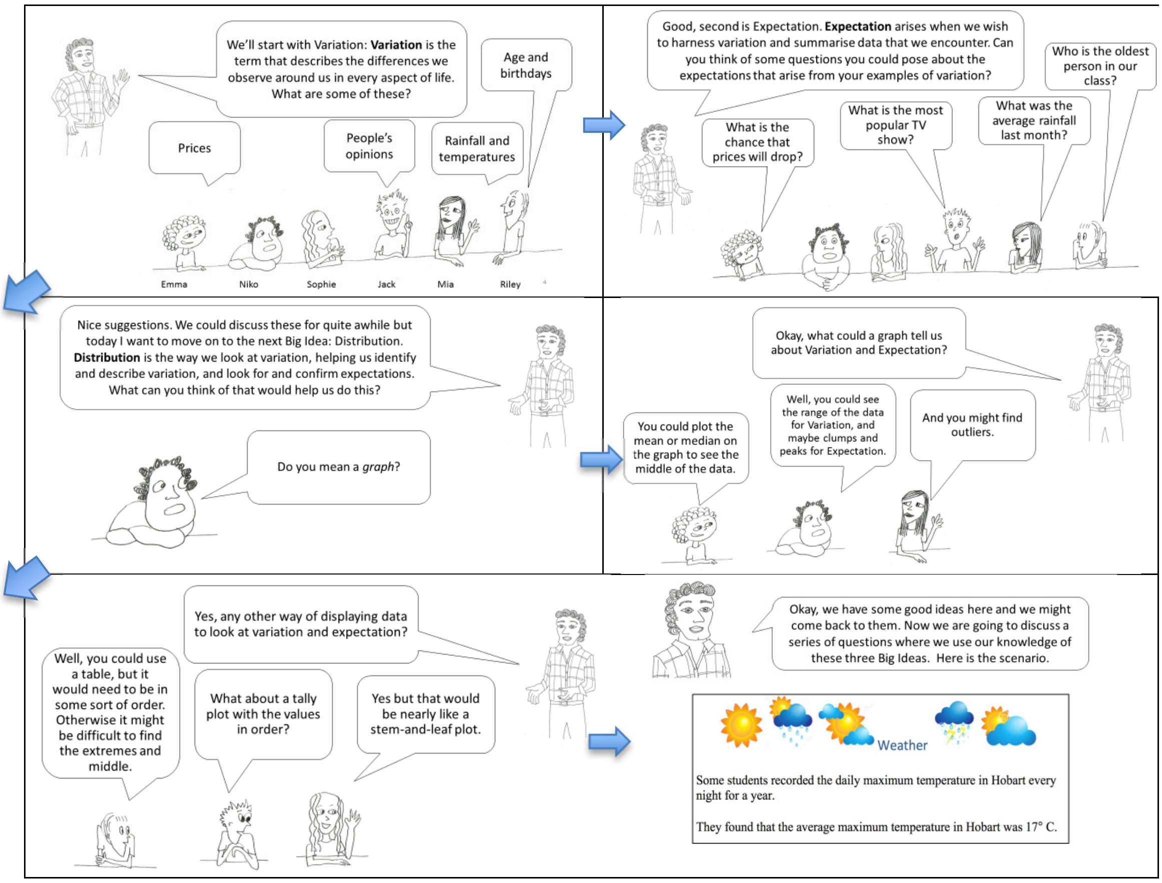


Figure 2. Reviewing the big ideas.

After each initial question by the cartoon teacher, participating teachers can be asked what their own students would say before seeing the responses in the

PowerPoint. After the second slide in Figure 2 asking about *expectation*, teachers could be asked *how* the student responses illustrate different types of expectation (e.g., probabilities, modes, means, or ranges). They could also be asked to think about other examples of data sets that would elicit responses not given by the students, such as house prices and the median.

Part 2: Task based on the weather

There are three stages to the class interaction that follows the last slide in Figure 2. Figure 3 shows the teacher asking how the value of 17°C was found. After the question is asked, before showing the responses from the cartoon students, the participating teachers can have a discussion about what their students would say.

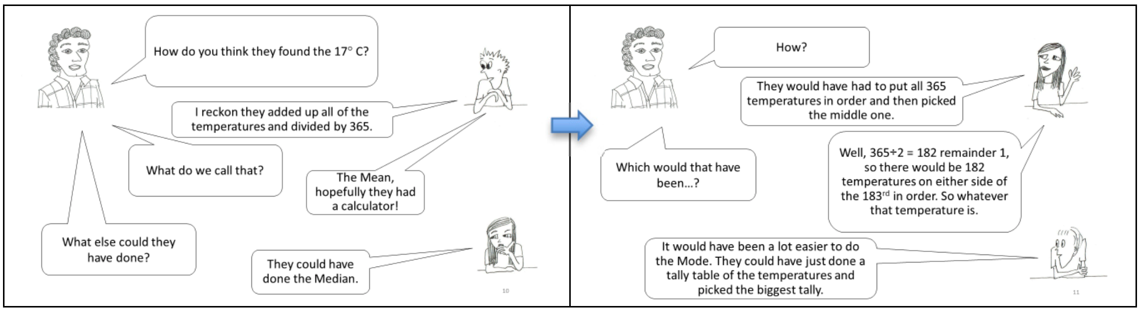


Figure 3. How to find the 'average' temperature.

Next, the teacher asks for description of Hobart's weather for an exchange student focussed on variation and expectation, as shown in Figure 4.

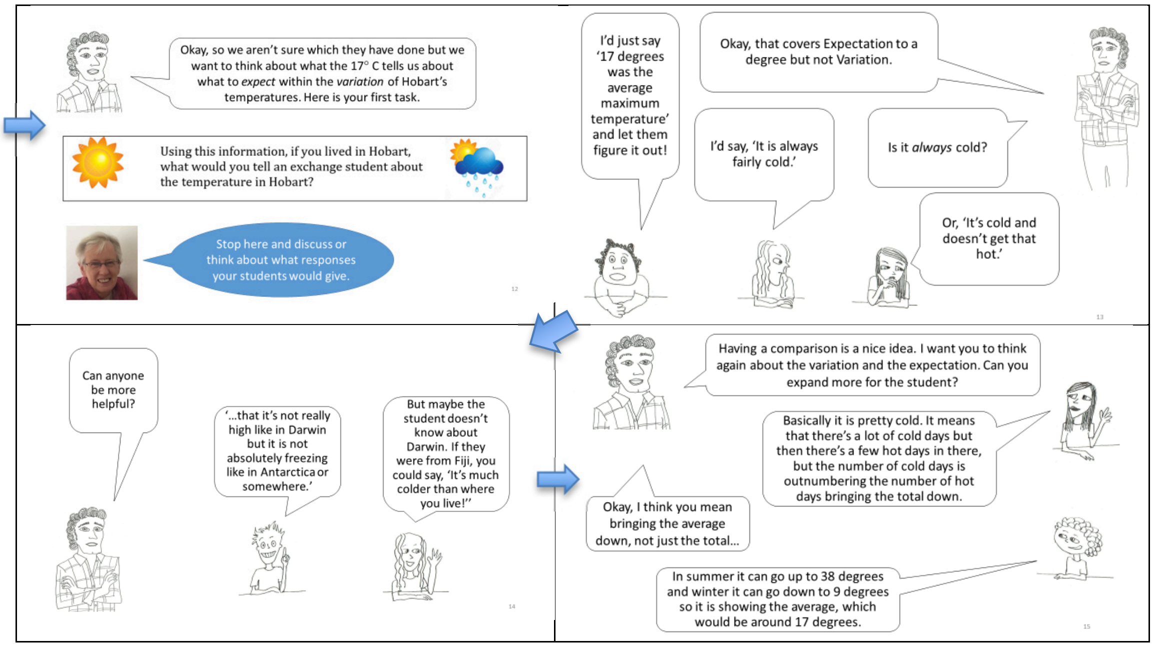


Figure 4. Focussing on variation and expectation.

The cartoon teacher then moves to the big idea of distribution asking his students to draw a sketch that would show the variation and expectation for the maximum temperatures in Hobart throughout the year. Again teachers are asked what their students would draw and perhaps to draw how they would represent the scenario themselves. Several responses from research are shown in Figure 5 with class interaction.

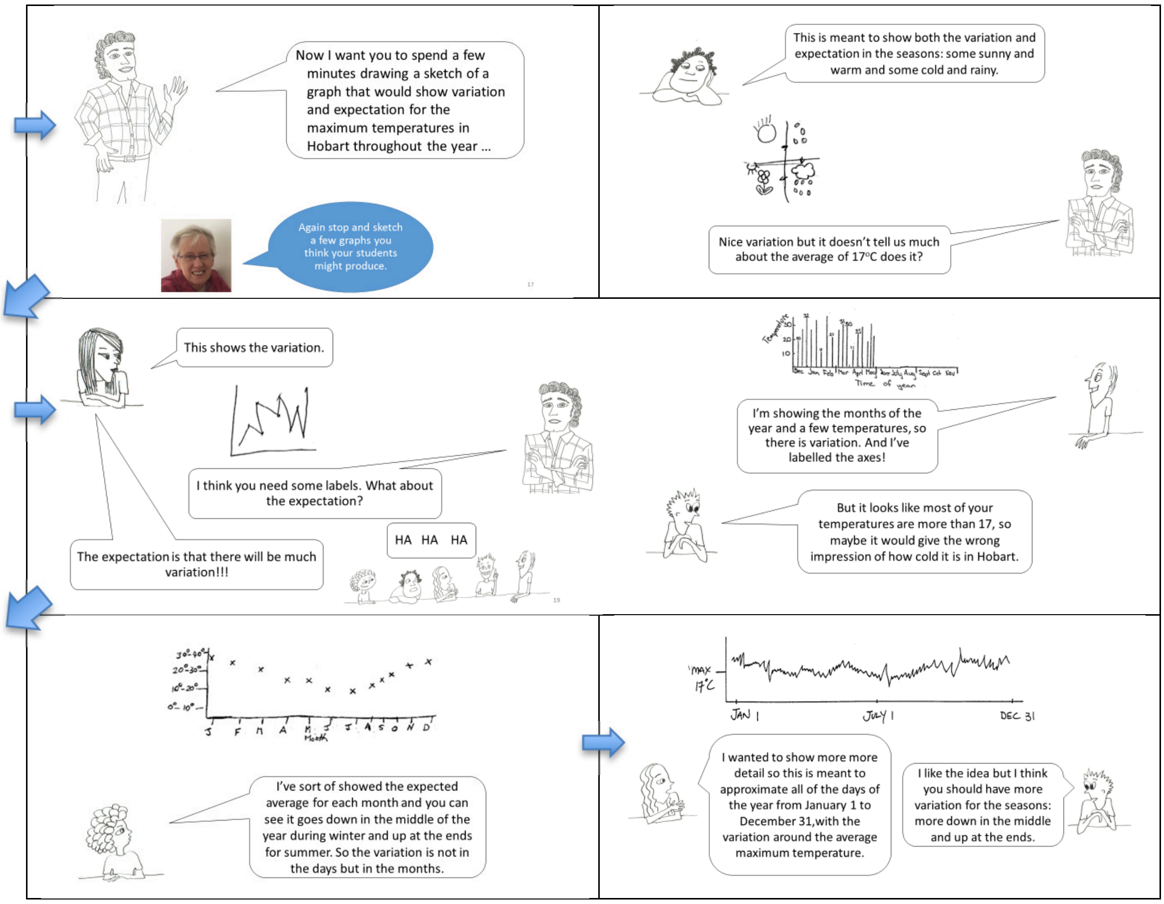


Figure 5. Distributions produced by students to show the weather over a year.

Finally the teacher presents a different representation to challenge the students' thinking in representing data like those based on temperatures. This is shown in Figure 6.

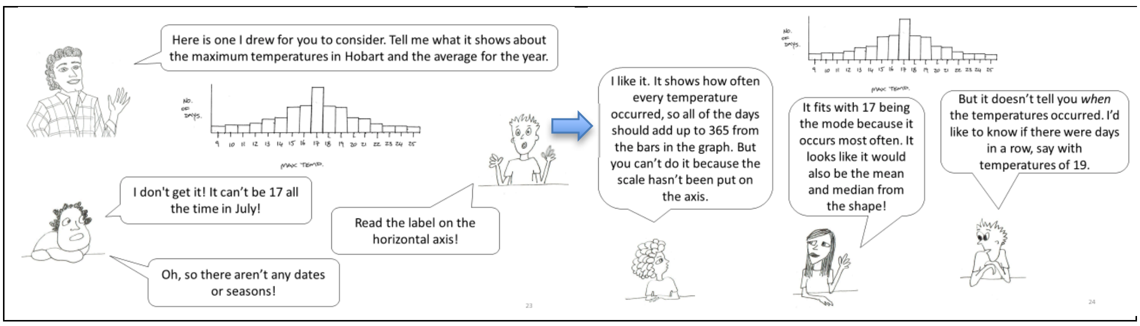


Figure 6. A different representation for temperature.

Part 3: Introduction of a large data set

The lesson scenario then moves on to introduce students to a national data set of average monthly maximum temperatures for 54 years from Australia's nine capital cities and asks for speculation on differences and similarities in the cities based on students' knowledge of Australia's geography. This moves the students into an authentic context where the data do not fit simple explanations. The initial dialogue begins with speculation.

Riley: I reckon that Hobart will be the coldest and Darwin will be the hottest!
Mia: I don't know, maybe Canberra is colder because it is inland and at a higher elevation.
Teacher: So what do you mean by colder?
Riley & Mia: We're thinking of the mean maximum temperature for the year.
Teacher: You are talking expectation!
Sophie: What about the different seasons?
Niko: I think that Canberra will be hotter in the summer and colder in the winter.
Teacher: So why would you want to consider seasons?
Sophie: Because of the *different variation and expectation*!
Teacher: Well done. We can have a look at Hobart and Canberra across the seasons, as well as overall …So, what do you want to look at first?
Niko: Summer and winter.

The teacher produces the distributions of average monthly maximum temperatures for Hobart and Canberra in summer and scaffolds the discussion as seen in Figure 7.

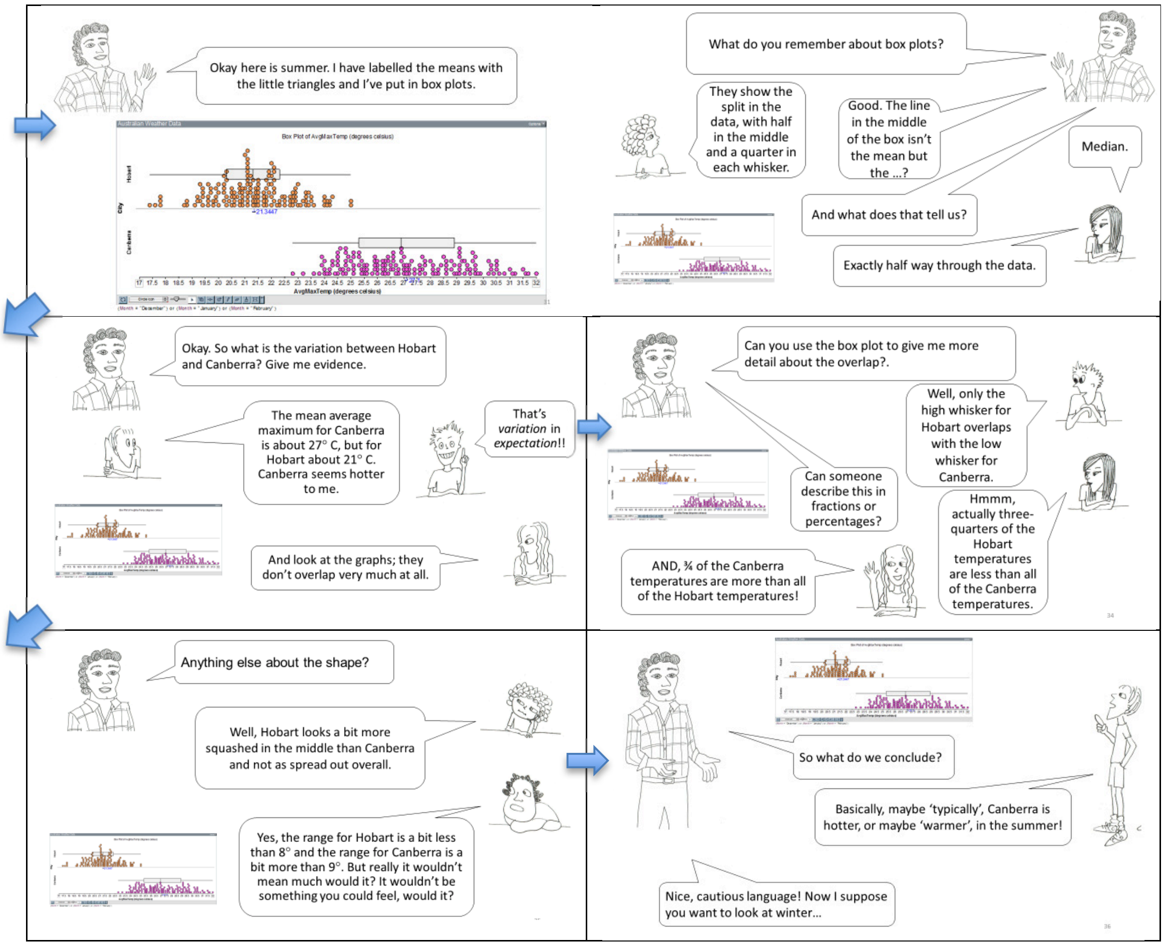


Figure 7. Comparing summer data for Hobart and Canberra.

In looking at the data for winter for the two cities, the mean values (expectation) are nearly identical but there is more variation in the data for Canberra, which is discussed by students. Students are then given the task to write a paragraph comparing spring and autumn between the two cities. Four examples are shown in Figure 8.

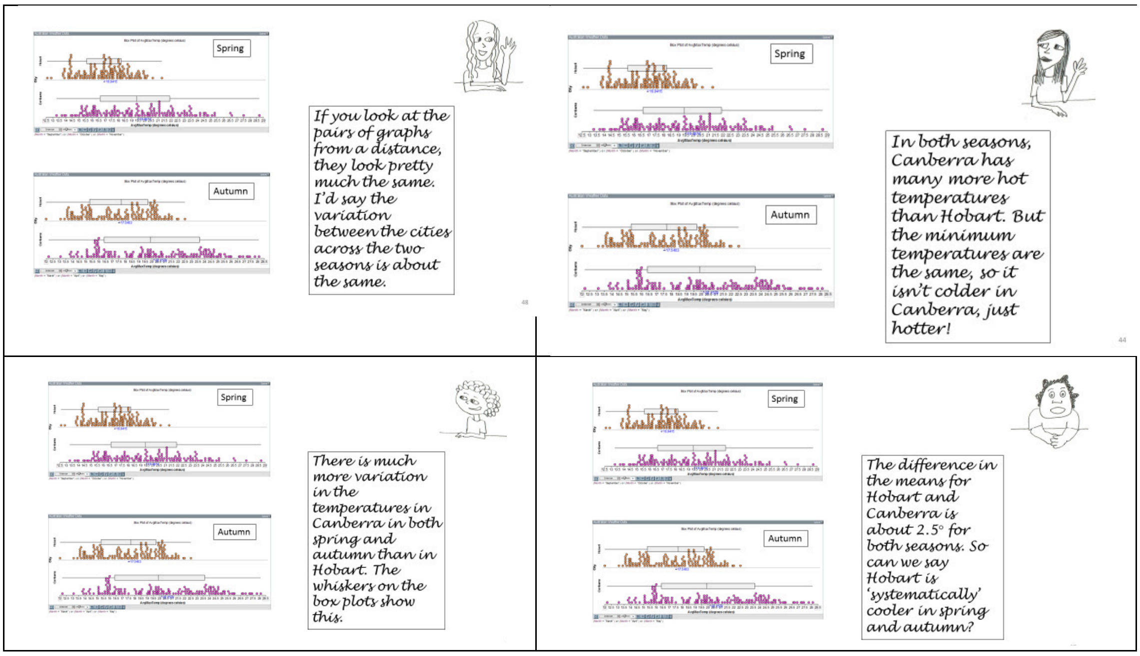


Figure 8. Describing the difference in the two cities for spring and autumn.

The teacher does a quick review of the three big ideas in relation to the discussion of the capital city temperatures.

Teacher: Let's review the big ideas we have used. Which did we use the most?
Class: VARIATION!
Teacher: And it occurred in many ways, didn't it? What next?
Class: Distribution.
Emma: Because we had to have distributions to look at the variation!
Teacher: Definitely. What is left?
Class: Expectation.
Jack: But where was it?
Sophie: I think it is in the averages, because they summarised what was happening in the variation.
Teacher: Right, but anywhere else?
Sophie: Maybe with the box plots, because they summarised the middle, and the top and bottom 25%, easily.
Teacher: Definitely, well done. We needed all three Big Ideas to tell the story. To finish, would you like to see all of the capital cities together?
Class: YES!

Finally, the data for all years and months for all cities are displayed and students ask for the distributions to be ordered by expectation, that is, by their increasing mean values, as seen in Figure 9.

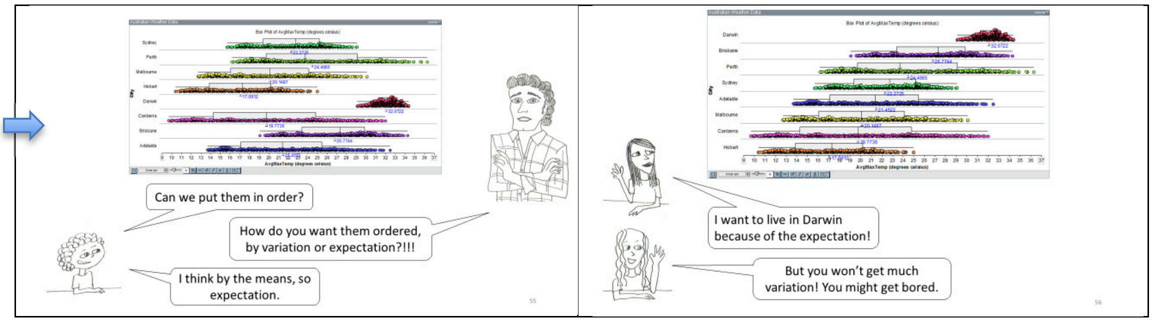


Figure 9. Data from all Australian capital cities.

Using weather and temperatures of Australian capital cities links well to Science and Geography and shows the power of 'interpreting statistical information' as a part of numeracy in the *General Capabilities* of the Australian Curriculum (Australian Curriculum, Assessment and Reporting Authority, 2013). Hobart was chosen as a starting point because as the southern-most capital, intuition may suggest it is the coldest without further consideration. Choosing Darwin as a starting point would be similar but offers much less internal variation and greater variation in expectation across cities.

Although formal analysis of difference and similarity can be quite complex and not appropriate at middle or high school, introducing the concepts in context with graphical representations alerts students to possibilities and the value of the Big Ideas. It builds intuitions that are useful when checking out the validity of formal calculations in later studies.

Conclusion

Weather offers many possibilities for activities employing and emphasising three of the big ideas of statistics—variation, expectation, and distribution—and how they contribute to drawing inferences. Some of the original material from which examples were chosen is found in Watson and Kelly (2005) and Watson, Callingham, and Kelly (2007). The classroom scenario described illustrates possibilities for teacher interaction and students. Further uses of the weather data are detailed in Watson et al. (2011) and more data are available at the Bureau of Meteorology website. Other topics and scaffolding related to the big ideas of statistics are discussed in Watson (in press). This suggested method of providing interactive PL for teachers is just part of the RMFII's aim to improve mathematical futures. The provision of tests, which provide teachers with responses like illustrated here, and specific learning progressions, such as seen in Figure 1, are part of the vision of the project. Teachers in the project are encouraged to share their experiences and those of their students with the wider education community (e.g., presentations at this AAMT conference).

Acknowledgement

Thanks to Sue Stack who created the cartoon characters and to Di Siemon for her continued support.

References

Australian Curriculum, Assessment and Reporting Authority. (2013). *General capabilities in the Australian Curriculum, January, 2013 (updated September 2014)*. Sydney, NSW: ACARA.

Callingham, R. & Watson, J. M. (2005). Measuring statistical literacy. *Journal of Applied Measurement, 6*(1), 19–47.

Moore, D.S. (1990). Uncertainty. In L. A. Steen (Ed.), *On the shoulders of giants: New approaches to numeracy* (pp. 95–137). Washington, DC: National Academy Press.

Siemon, D. (2016). *Reframing Mathematical Futures: Building a learning and teaching resource to enhance mathematical reasoning in Years 7 to 10. Second Progress Report. Australian Maths and Science Partnerships Programme*. Canberra: Australian Government Department of Education and Training.

Siemon, D., Breed, M., Dole, S., Izard, J. & Virgona, J. (2006). *Scaffolding Numeracy in the Middle Years: Project Findings, Materials, and Resources. Final Report submitted to Victorian Department of Education and Training and the Tasmanian Department of Education, October*. Retrieved from www.eduweb.vic.gov.au/edulibrary/public/teachlearn/student/snmy.ppt

Watson, J. (2016). Scaffolding statistics understanding in the middle school. In W. Widjaja, E. Y-K. Loong & L.A. Bragg (Eds), *2016 Mathematical Association of Victoria's Annual Conference Proceedings* (pp. 132–139). Melbourne: MAV.

Watson, J. M. (in press). Reasoning and sense making in statistics and probability. In M. Battista (Ed.), Reasoning and sense making in grades 6–8. Reston, VA: National Council of Teachers of Mathematics.

Watson, J., Beswick, K., Brown, N., Callingham, R., Muir, T. & Wright, S. (2011). Lesson 7.5: Using Filters to study weather data. In *Digging into Australian data with TinkerPlots* (pp. 318–323). Melbourne: Objective Learning Materials.

Watson, J.M. & Callingham, R. (2003). Statistical literacy: A complex hierarchical construct. *Statistics Education Research Journal, 2*(2), 3–46.

Watson, J. & Callingham, R. (2017). Developing learning progressions to support mathematical reasoning in the middle years – Statistical reasoning. In Reframing Mathematical Futures: Developing Learning Progressions to Support Mathematical Thinking in the Middle Years, Symposium at the 40th annual conference of the Mathematics Education Research Group of Australasia, Melbourne.

Watson, J. M., Callingham, R. A. & Kelly, B. A. (2007). Students' appreciation of expectation and variation as a foundation for statistical understanding. *Mathematical Thinking and Learning, 9*, 83–130.

Watson, J. M. & Kelly, B. A. (2005). The winds are variable: Student intuitions about variation. *School Science and Mathematics, 105*, 252–269.