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CHAPTER 2.

ONE-TO-ONE STUDENT INTERVIEWS PROVIDE POWERFUL INSIGHTS AND CLEAR FOCUS FOR THE TEACHING OF FRACTIONS IN THE MIDDLE YEARS*

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Why use a one-to-one task-based interview?

One-to-one, task-based, assessment interviews can give teachers powerful insights into children's understanding of key mathematical ideas, and their preferred strategies in solving problems. Combining formative assessment with an understanding of the "big ideas" of fractions informs the classroom teacher's planning, enabling a clear focus for the teaching of fractions in the middle years. The use of a student assessment interview, embedded within an extensive and appropriate in-service or pre-service program, can be a powerful tool for teacher professional learning (Clarke, Mitchell, & Roche, 2005). While one-to-one interviews take time, our research has shown that the use of interviews enhances teachers' knowledge of how mathematics learning develops differently for different children, improves teachers' *content knowledge* (the actual mathematics needed for the task), and develops teachers' *pedagogical content knowledge* (the intersection of pedagogical knowledge and content knowledge). The protocols of the interview stipulate no "teaching", and the teacher's interest in the *student's* thinking is affirmed in the question, "How did you work that out?", with no feedback as to whether the student is correct or incorrect. This is a very powerful way of establishing the classroom norms of students reflecting on their own mathematical thinking and considering multiple approaches to tasks.

* This chapter is based on research findings presented in Clarke, D., Roche, A. & Mitchell, A. (2007). Year six fraction understanding: A part of the whole story. In J. Watson & K. Beswick (Eds), *Mathematics: Essential research, essential practice* (Proceedings of the 30th annual conference of the Mathematics Education Group of Australasia, pp. 207–216). Adelaide: MERGA.

Teachers in Victorian schools have been using the Early Numeracy Interview, now called the Mathematics Online Interview, since 2001, following its development in the Early Numeracy Research Project in 1999. In 2008, the Fractions Online Interview was introduced as a suggested formative assessment tool. All of the interview scripts, task templates, supporting information, and a booklet of associated classroom activities are available for public download on the website of the Victorian Department of Education and Early Childhood Development. The four fraction tasks that we discuss in this chapter can also be found there.

What is it that makes fractions difficult to teach and to learn?

Much of the confusion in teaching and learning fractions is evident when students cannot synthesise the many different interpretations (sub-constructs) of fractions, and are not familiar with a variety of representations (models). Also, generalisations that have occurred during instruction on whole numbers are frequently misapplied to fractions. There also appears to be a gap between students' procedural and conceptual understanding of fractions. Students also find it difficult to link intuitive knowledge (or familiar contexts) with symbols (or formal classroom instruction). The dilemma for both teachers and students is how to make all the appropriate connections so that a mature, holistic and flexible understanding of fractions and the wider domain of rational numbers can be obtained, because fractions form an important part of middle years mathematics curriculum, underpinning the development of proportional reasoning, and are important for later topics including algebra and probability.

What are the key interpretations that teachers and students need to understand?

Kieren (1980) identified five different interpretations (or sub-constructs) of *rational numbers.* These are often summarised as part–whole, measure, quotient (division), operator, and ratio. Decimal or fraction models or notation can be used in all of these five contexts.

The *part-whole* interpretation depends on the ability to partition either a continuous quantity (such as area, length and volume models) or a set of discrete objects into equal-sized subparts or sets. The part-whole construct is the most common interpretation of fractions and likely to be the first interpretation that students meet at school. On its own, it is not a sufficient foundation for a conceptual understanding of rational numbers.

A fraction can represent a *measure* of a quantity relative to one unit of that quantity. The measure interpretation is different from the other constructs in that the number of equal parts in a unit can vary depending on how many times you partition. This successive partitioning allows you to "measure" with precision. We speak of these measurements as "points" and the number line provides a model to demonstrate this (see Figures 2.1 and 2.2 for examples of this, and also Mitchell and Horne, this volume).

A fraction (*a*/*b*) may also represent the operation of *division* or the result of a division such that

$$3 \div 5 = \frac{3}{5}.$$

The division interpretation may be explored in the context of equal sharing, such as determining how much of a pizza a child would get if three pizzas are shared between five children.

A fraction can be used as an *operator* to shrink or stretch a number, for example,

$$\frac{3}{4} \times 12 = 9$$

and

$$\frac{5}{4} \times 8 = 10.$$

The misconception that multiplication always makes bigger and division always makes smaller is common. It could also be suggested that students' lack of experience in using fractions as operators may contribute to this misconception.

Fractions can be used as a method for comparing the sizes of two sets or two measurements such as: the number of girls in the class is $\frac{3}{5}$ the number of boys, i.e., a *ratio*.

While these constructs can be considered separately, they have some unifying elements or "big ideas" that connect them. Carpenter, Fennema and Romberg (1993) identified three unifying elements to these interpretations and they are: identification of the unit, partitioning and the notion of quantity (how "big" a fraction is).

Students' performance on key interview tasks

For the four tasks following, the task is outlined, the mathematical idea it was designed to address is stated in the subheading, the percentage of student success is given, and common strategies and solutions, including misconceptions are outlined. In the interests of space, we are focusing particularly on tasks relating to constructs of fraction as operator, fraction as measure, and fraction as division, and the "big idea" of the relative size of fractions.

The success rates come from a study conducted by the authors in 2005. The participants were 323 Grade 6 students who were interviewed at the end of the school year. The interviews were conducted individually over a 30- to 40-minute period in students' own schools, with interviews following a strict script for consistency, and using a standard record sheet to record students' answers, methods and any written calculations or sketches.

Fraction as an operator

Students were asked four questions, with no visual prompts, which required them to work out the answers in their heads. They were as follows:

- "One-half of six?" (97.2% success);
- "One-fifth of ten?" (73.4%);
- "Two-thirds of nine?" (69.7%); and
- "One third of a half?" (17.6%).

These data confirm the difficulty of these kinds of tasks, the first three of which most teachers would predict to be quite straightforward. There was a pattern evident in incorrect responses, with the most common errors usually involving the denominator of the operator (e.g., for $\frac{1}{5}$ of 10, the most common error was 5; for $\frac{2}{3}$ of 9, the most common error was 3). There were 52 different incorrect answers to $\frac{1}{3}$ of a $\frac{1}{2}$, with $\frac{1}{4}$ being the most common error, and 47.1% of students giving no response. It is important to emphasise that for these tasks, students were not asked to explain how they arrived at their answers.

Fractions as measure

Students were asked to "please draw a number line and put two thirds on it." If students did not choose to indicate where 0 and 1 were on their drawing, they were asked by the interviewer, "Where does zero go? … Where does 1 go?" Only 51.1% of students were successful in correctly locating $\frac{2}{3}$ on the number line. Common errors were placing $\frac{2}{3}$ after 1 (see Figure 2.1), or placing $\frac{2}{3}$ two-thirds along some line, e.g., at 4 on a number line from 0 to 6, or two-thirds of the way from 0 to 100 (see Figure 2.2). Mitchell and Horne (this volume) describe this latter response as demonstrating *limited part-whole understanding*, where students are placing $\frac{2}{3}$ two-thirds of the way along the line, rather than adopting the conventions of the number line.

Figure 2.1. A student's incorrect solution for placing $\frac{2}{3}$ on a number line.

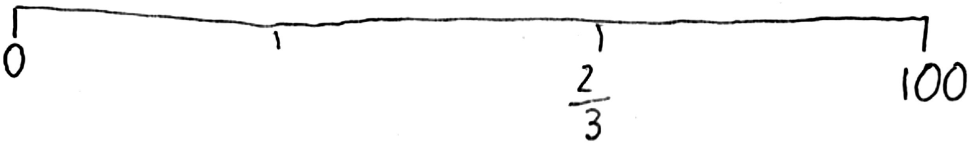

Figure 2.2. Another student's incorrect solution for placing $\frac{2}{3}$ on a number line.

Fractions as division

Children were shown a picture (see Figure 2.3), and told: "Three pizzas were shared equally between five girls. ... How much does each girl get?" Students were invited to use pen and paper if they appeared to require it.

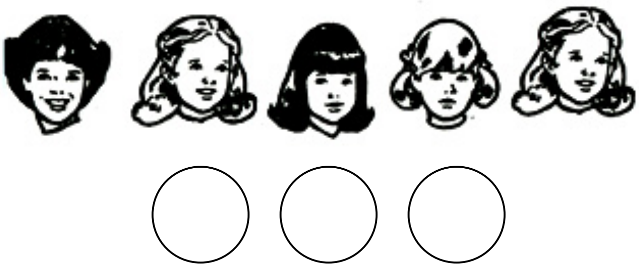

Figure 2.3. Pizza task.

Although 30.3% of Grade 6 students responded with a correct answer, it was apparent that most either drew a picture or mentally divided the pizzas to calculate the equal share. A concerning result was that 11.8% of students were unable to make a start. Greater exposure to division problems and explicit discussion connecting division with their fractional answers (e.g., $3 \div 5 = \frac{3}{5}$) may help lead students to the generalisation that

$$a \div b = \frac{a}{b}.$$

It is reasonable that only a small percentage of students by the end of Grade 6 are aware of the notion of fraction as division, but the 11.8% figure is particularly concerning. We believe students require more experience in doing their own partitioning in sharing situations. Extensive experience of partitioning is likely to increase the chances of appropriate generalisations.

Understanding the size of fractions

In a task designed to get at students' understanding of the "size" of fractions, we used the *Construct a Sum* task (Behr, Wachsmuth, & Post, 1985). The student is asked to place number cards in the boxes to make fractions so that when you add them, the answer is as close to one as possible, but not equal to one. The number cards included 1, 3, 4, 5, 6, and 7 (see Figure 2.4).

Each card could be used only once. The capacity for students to move cards around as they consider possibilities is a strong feature of this task. Only 25.4% of students produced a solution within $\frac{1}{5}$ of 1, the most common response being $\frac{1}{5} + \frac{3}{4}$ (5.3% of the total group). Sixty-seven per cent of students chose fractions that when added were greater than $\frac{1}{10}$ away from 1. Thirty-three percent of students chose an improper fraction as one of their two, making the task of obtaining a sum close to 1 extremely difficult, and possibly indicating they have a misunderstanding about the size of improper fractions The answer closest to one ($\frac{1}{7} + \frac{5}{6}$) was chosen by only four students.

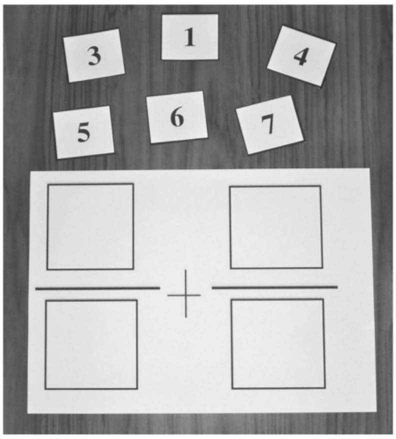

Figure 2.4. Construct a Sum task.

So what did we learn from these interviews?

Despite the strong recommendations from researchers that school mathematics should present students with experiences of all key constructs of fractions, and the many useful models that illustrate these constructs, it is clear that a large, representative group of Victorian Grade 6 students do not generally have a confident understanding of these and their use. Although simple fraction as an *operator* tasks were straightforward for some students, only around one-sixth of students were able to find one-third of a half. Students may need more encouragement to form mental pictures when doing such calculations.

Given that only around half of the students could draw an appropriate number line which showed $\frac{2}{3}$, it is clear that fraction as a *measure* requires greater emphasis in curriculum documents and professional development programs, as many students are not viewing fractions as numbers in their own right. The experience of the authors is that Australian students spend relatively little time working with number lines in comparison to countries such as The Netherlands.

The Construct a Sum task revealed similar difficulties with understanding the size of fractions, particularly improper fractions, and a lack of use of *benchmarks* in student thinking. By benchmarks, we mean the capacity of students to relate the size of a given fraction to the size of other numbers such as 0, $\frac{1}{2}$ or 1. Possibly, the urge to move students to formal algorithms at the expense of developing such strategies as benchmarking may limit students to a procedural understanding rather than the conceptual understanding that is more appropriate for this task.

From our experience, few Australian primary school and middle school teachers and even fewer students at these levels are aware of the notion of *fraction as division*. Most students who concluded that 3 pizzas shared between 5 people would result in $\frac{3}{5}$ of a pizza each, either drew a picture or mentally divided the pizzas to calculate the equal share. A very small percentage knew the relationship automatically.

Practical hints for the classroom teacher arising from our work

Elsewhere (Clarke, Roche, & Mitchell, 2008), we offered ten tips for classroom teachers arising from our work. They are summarised below:

1. Give a greater emphasis to the meaning of fractions than on procedures for manipulating them.

Sometimes, curriculum documents give the sense that the ultimate goal of teaching fractions is that students will be able to carry out the four operations with them. We believe that students need to be given time to understand what fractions are about (rather than moving quickly to computations) and that the ultimate goal should be to develop students who can reason proportionally.

2. Develop a general rule for explaining the numerator and denominator of a fraction.

When students are first trying to make sense of common fractions, teachers have typically defined them as follows: "The denominator tells you how many parts the whole has been broken up into, and the numerator tells you how many of these parts to take, count or shade in." Now, this works reasonably well for fractions between 0 and 1, but not well for improper fractions. We prefer this explanation for students: “In the fraction a/b, b is the name or size of the part (e.g., fifths have this name because 5 equal parts can fill a whole) and a is the number of parts of that name or size.” So if we have $\frac{7}{3}$, the three tells the name or size of the parts (thirds), and the 7 tells us that there are 7 of those thirds (or $2\frac{1}{3}$).

We believe this alternative explanation may assist students to use more appropriate language when labelling fractions. For example, we have noticed some students referring to three-quarters as “three-fours” and “four-threes”. This use of whole number rather than fractional language appeared to be an indicator that the students do not yet understand which digit refers to the number of parts or the size of the parts.

3. Emphasise that fractions are numbers, making extensive use of number lines in representing fractions and decimals.

Number lines have many advantages. They help students to see how whole numbers, fractions and decimals relate, they provide a way of understanding why $\frac{5}{3}$ is the same as $1\frac{2}{3}$, and $\frac{6}{3}$ is the same as 2, and they make it easier for students to understand the density of rational numbers, i.e., between any two distinct numbers, there is an infinite number of fractions and decimals.

4. Take opportunities early to focus on improper fractions and equivalences.

If our language for fractions is in good shape (see hint 2) and we are making use of number lines extensively, then improper fractions seem to fit in quite naturally. The activity *Colour in fractions* (see Clarke, Roche, & Mitchell, 2008 and the Resource section at the back of this book) develops both an understanding of equivalence and of the meaning of improper fractions.

5. Provide a variety of models to represent fractions.

Posing problems involving equal-sharing tasks, similar to the Sharing Pizza assessment task described earlier, provides opportunities for students to generate their own models that embody many crucial aspects of fractions. “Ready-made” fractions materials may not enable children to construct important concepts regarding fractions. For example, when comparing two fractional parts, the unit must be the same size, but this is not always noticed by students if the diagrams are always provided. Expecting students to reflect on their own partitioning can strengthen their conceptual understanding that equal shares are not necessarily *congruent* (the same shape and size).

6. Link fractions to key benchmarks, and encourage estimation.

The most successful students on tasks involving comparing the relative size of fractions used two strategies that they may well not have been taught at school. The first was what we call *benchmarking*, where the students compared fractions, by relating the size of fractions of interest to that of 0, $\frac{1}{2}$, or 1. Another innovative strategy was what we call *residual thinking*. The term residual refers to the amount which is required to build up to the whole. For example, when comparing $\frac{5}{6}$ and $\frac{7}{8}$, students may conclude that the first fraction requires $\frac{1}{6}$ more to make the whole ("the residual"), while the second requires only $\frac{1}{8}$ to make the whole (a smaller piece or number), so $\frac{7}{8}$ is larger. We believe that when students share their strategies for comparing fractions during class, other students may be convinced to take on board both benchmarking and residual thinking in tackling relative size and ordering problems.

7. Give emphasis to fractions as division.

We believe that the notion of "fraction as division" is not a common construct in most people's minds. The sharing chocolate activity (see Clarke, 2006, which also appears as Chapter 3 in this book) is a helpful introduction to this notion. Students need lots of opportunities to partition objects in sharing and other contexts to help build this notion.

8. Link fractions, decimals and percentages wherever possible.

We have noticed that many middle school students, when given a problem to solve involving fractions, will choose to convert it to decimals or percentages, in order to make sense of it. This flexible thinking is to be encouraged, as percentages particularly seem to make sense to many students intuitively.

9. Take the opportunity to interview several students one-to-one on the kinds of tasks discussed in this chapter, to gain awareness of their thinking and strategies.

We have discussed several tasks which we have used in a one-to-one interview setting as part of our research, and encourage teachers to try a number of these with their students. In almost every case, teachers report back to us that the interviews were particularly useful in gaining insights into their students' thinking, and in many cases, the interview process developed a new respect for students' attempts to make sense of fractions. They were keen to share students' methods with their other students, and to encourage them to try these strategies with a range of other problems.

10. Look for examples and activities which can engage students in thinking about fractions in particular and rational number ideas in general.

One of our favourite activities involves Cuisenaire rods, with an emphasis on moving from part to whole, whole to part, and part to part. Each group of students has a box of Cuisenaire rods. They work through problems like the following:

- What fraction of the brown rod is the red rod?
- If the purple is $\frac{2}{3}$, which rod is the whole?
- If the brown rod is $\frac{4}{3}$, which rod is one?
- If the blue rod is $1\frac{1}{2}$, which rod is $\frac{2}{3}$?
- Your brilliant question for another group!

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Victorian Department of Education and Early Childhood Development, Fractions online interview, www.education.vic.gov.au/studentlearning/teachingresources/maths/interview/fractions.htm[1]

[1] Please note: the facility of using the interview online and submitting data is only available to Victorian government school teachers, but others can print all the relevant materials for their own use manually.